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PAPERS

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THE INVARIANT THEORY OF A GENERAL BILINEAR FORM

By H. W. Turnbull.

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Although as far back as the year 1869, Clebsch and Gordan* discussed the ternary bilinear form from the point of view of the invariant theory, obtaining incidentally a complete system of eight concomitants, yet I cannot find anywhere in the extensive literature on such forms an account of the corresponding problem for the general n-ary case. References are given in the article in the *Encyk. der Math. Wiss.*, III 3, 10, by R. Weitzenböck, including the cases of the quaternary collineation and the correlation, when the variables x, u in the bilinear form $\sum_{i,j=1}^{n} a_{ij}x_i u_j$ are cogredient. There is also an interesting discussion by E. Study in his *Vectorenrechnung*† where the bilinear form is treated as a *collineation*, the variables x, u being contragredient; but the Clebsch-Gordan problem is not attacked. In what follows I give a solution of this problem, analogous to that already given for the case of quadratics, *i.e. symmetrical* bilinear forms‡. The symbolic treatment is founded upon the method of Study, and some of the results in §§ 1–3 are naturally quite well known: and the formula § 3 (6) is closely related to that given by Weitzenböck in

* *Math. Annalen*, 1 (1869), 359–400.

† Braunschweig (1923), 170–183. This is based on an earlier work: Study, "Recurrirende Reihen und bilineare Formen", *Monatshefte für Math. und Phys.*, 2 (1891), 23–54.

‡ *Proc. London Math. Soc.* (2), 30 (1930), 469–486.

a discussion of the quadratic complex*. The treatment here is, however, different, and incidentally links the theory with that of matrix differentiation (§ 6). In § 7 a complete Clebsch-system of concomitants for a bilinear form is found, in the case when the variables x and u are entirely independent, followed in § 8 by a complete system for the collineation, or contragredient case. This latter permits any number of variables, cogredient with either u or x, to enter, but does not include irresoluble compound variables (such as line coordinates p_{ij} in quaternary forms). This more elaborate problem, giving the Clebsch-system of a collineation, has not been attempted here.

The same methods apply to the case of the *correlation* also, but produce a result somewhat alien from the present instances.

A remark should perhaps be added to indicate the connection between this theory and that of elementary divisors and invariant factors. That such a connection exists is clear from the investigations of Study; and there are several places in the present work which suggest possibilities of carrying the matter further. But from the point of view adopted in the invariant theory, the general bilinear form affords the primary interest; and this general form is not well adapted for illustrating the alternative theory.

1. *The bilinear form.*

Let the general square matrix A of order n, composed of ordinary complex numbers a_{ij}, be written

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \tag{1}$$

and be further symbolized, by use of the ordinary Clebsch-Aronhold symbols, as

$$[a_i a_j] = \begin{bmatrix} a_1 a_1 & a_1 a_2 & \dots & a_1 a_n \\ a_2 a_1 & a_2 a_2 & \dots & a_2 a_n \\ \dots & \dots & & \dots \\ a_n a_1 & a_n a_2 & \dots & a_n a_n \end{bmatrix}. \tag{2}$$

Thus $a_{ij} = a_i a_j$, and the $2n$ symbols a_i, a_j obey the usual laws of algebra, except that in any product neither symbol may be repeated. If two

* "Komplex-Symbolik", Sammlung Schubert, 57 (1908), 59.

distinctive bracket notations are introduced, we may save space by writing $[a]$ and $\{a\}$ to denote respectively the single row and the single column of symbols

$$a_1, \quad a_2, \quad \ldots, \quad a_n,$$

in this order, and $[a]$, $\{a\}$ to denote a similar row and column for a. Further, if an accent denotes the process of transposition, so that $A' = [a_{ji}]$, we may write

$$A = [a_{ij}] = \{a\}[a], \qquad A' = [a_{ji}] = \{a\}[a], \tag{3}$$

on using the ordinary rule of matrix multiplication $[b_{ij}][c_{ij}] = [\Sigma b_{ik} c_{kj}]$. This rule applies whenever the number q of elements in a row of the left factor is equal to that in a column of the right factor. This number may be any positive integer: and in (3) the extreme case occurs, with $q = 1$. It expresses a matrix A, and also the transposed A', as symbolic products of single row matrices. If, however, the order of factors is inverted, we obtain *scalar* quantities, again by the same rule of matrix multiplication, this time with $q = n$. Thus

$$[a]\{a\} = a_1 a_1 + a_2 a_2 + \ldots + a_n a_n = [a]\{a\}.$$

Other alternative forms for this are

$$a_a = \sum_{i=1}^{n} a_{ii} = (a\,|\,a). \tag{4}$$

To denote second and higher powers of the matrix symbolically, we shall need equivalent symbols given by the relations

$$a_i a_j = \beta_i b_j = \gamma_i c_j = \ldots. \tag{5}$$

In applying the process of interchange, whenever a is interchanged with b, so also is α with β. Thus the general *unsymmetrical* matrix (and bilinear form) is distinguished symbolically from the *symmetrical* (and quadratic form), by not admitting the interchange of italic with Greek letters. In general, therefore, $a_i a_j$ and $a_j a_i$ differ; but for the symmetrical case they do not; and consequently the usual simpler notation $a_i a_j$ is available, without the use of further symbols α, β, γ.

By definition, the typical ij-th term in A^2, the square of the matrix A, is $\Sigma_k a_{ik} a_{kj}$, and in A^3 is $\Sigma a_{ik} a_{kl} a_{lj}$, the summation extending over $1, 2, \ldots, n$ for each repeated suffix. Symbolically, we immediately obtain

$$A = [a_i a_j], \quad A^2 = [a_i a_\beta b_j], \quad A^3 = [a_i a_\beta b_\gamma c_j], \tag{6}$$

and so on. As leader to this sequence may be attached the unit matrix $A^0 = I = [\delta_{ij}]$, where $\delta_{ij} = 0$ or 1, according as $i \neq j$ or $i = j$.

If now we introduce two sets of n variables, a row $[u_1, u_2, \ldots, u_n]$ denoted by u, and a column $\{x_1, x_2, \ldots, x_n\}$ denoted by x, we obtain a sequence of bilinear forms, on multiplying each power of A, fore and aft, by u and x respectively. As in (4), such products are scalar quantities,

$$\phi_0 = ux, \quad \phi_1 = uAx, \quad \phi_2 = uA^2x, \quad \ldots. \tag{7}$$

Here $\phi_0 = u_1x_1 + u_2x_2 + \ldots + u_nx_n$, and $\phi_1 = \sum_{i,j=1}^{n} u_i a_{ij} x_j$. If each ϕ is expressed in symbols, as in (6), the first two become u_x, u_α, a_x respectively; and in general we have

$$\left.\begin{aligned} \phi_0 = ux &= \Sigma u_i x_i &&= u_x, \\ \phi_1 = uAx &= \Sigma u_i a_{ij} x_j &&= u_\alpha a_x, \\ \phi_2 = uA^2x &= \Sigma u_i a_{ij} a_{jk} x_k &&= u_\alpha a_\beta b_x, \\ \phi_3 = uA^3x &= \Sigma u_i a_{ij} a_{jk} a_{kl} x_l &&= u_\alpha a_\beta b_\gamma c_x, \end{aligned}\right\} \tag{8}$$

and so on.

If the set of n variables x undergoes a general linear transformation of non-singular matrix H, to another set ξ of n variables, say,

$$x_i = \Sigma h_{ij} \xi_j \quad \text{or} \quad x = H\xi, \quad |H| \neq 0,$$

and u undergoes a contragredient transformation

$$u = \theta H^{-1},$$

then ϕ_0 remains an absolute invariant ($\phi_0 = ux = \theta\xi$), while the symbols a, b, c behave as if cogredient to u, and α, β, γ as if cogredient to x. The final column in (8), in the light of the fundamental theorem of symbolic methods, shows at once that each ϕ_i ($i \geq 1$) is a concomitant of the ground form ϕ_1. In this case the expression a_α given in (4) is manifestly an invariant. Geometrically speaking, such a form ϕ_1 gives rise to a *collineation*, because it transforms a point x, of space S_{n-1}, whose homogeneous coordinates are x_i, to a point ξ whose coordinates are $a_x a_i$.

It is the complete system of this bilinear form, or collineation, together with any number of points x, y, z, ..., and primes u, v, w, ..., which is to be found in § 8.

If, on the other hand, the variables x and u undergo *independent* linear transformations

$$x = H\xi, \quad u = \theta K,$$

with non-singular matrices H and K, the bilinear form ϕ_1 no longer gives rise to a collineation; nor are the forms ϕ_0, ϕ_2, ... concomitants. This case is dealt with in § 7.

Once more, if $H = K'$, so that x and u undergo cogredient transformations, then ϕ_1 gives rise to a *correlation*. Again the system ϕ_i does not furnish concomitants. The necessary and sufficient condition for the ϕ's to be concomitants is that a_α, the sum of the diagonal elements of the matrix A, should be an invariant of the transformation; and this requires that $K = H^{-1}$, giving the case of the collineation.

The ring of invariants for the collineation.

2. In the case of the collineation, if we operate with $\sum_{i=1}^{n} \frac{\partial}{\partial u_i}\frac{\partial}{\partial x_i}$ upon each of these forms ϕ in turn, we obtain the following functions of the coefficients a_{ij} in ϕ_1:

$$\left.\begin{aligned} P_1 &= \Sigma a_{ii} &&= a_\alpha, \\ P_2 &= \Sigma a_{ij}a_{ji} &&= a_\beta b_\alpha, \\ P_3 &= \Sigma a_{ij}a_{jk}a_{ki} &&= a_\beta b_\gamma c_\alpha, \end{aligned}\right\} \tag{1}$$

and so on. The middle column expressions show that P_s is the sum of the principal diagonal elements in the matrix A^s, and the final column shows that each P_s is an invariant. These invariants are closely connected with the *characteristic equation* of the matrix A, namely

$$|\lambda I - A| \equiv \begin{vmatrix} \lambda - a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \dots & -a_{2n} \\ \dots & \dots & \dots & \dots \\ -a_{n1} & -a_{n2} & \dots & \lambda - a_{nn} \end{vmatrix} = 0. \tag{2}$$

When this is expanded in powers of λ, let it be written

$$f(\lambda) \equiv \lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} + \dots + p_n = 0. \tag{3}$$

It will now be shown that an alternative symbolic form of this polynomial

$f(\lambda)$ is

$$\lambda^n - a_\alpha \lambda^{n-1} + \frac{1}{2!}(ab\,|\,a\beta)\lambda^{n-2} - \ldots (-)^n \frac{1}{n!}(abc\ldots m)(a\beta\gamma\ldots\mu), \qquad (4)$$

which involves *compound inner products* defined as follows :

$$(ab\,|\,a\beta) = \Sigma(ab)_{ij}(a\beta)_{ij} = \begin{vmatrix} a_\alpha & b_\alpha \\ a_\beta & b_\beta \end{vmatrix} \quad (i \neq j),$$

$$(abc\,|\,a\beta\gamma) = \Sigma(abc)_{ijk}(a\beta\gamma)_{ijk} = \begin{vmatrix} a_\alpha & b_\alpha & c_\alpha \\ a_\beta & b_\beta & c_\beta \\ a_\gamma & b_\gamma & c_\gamma \end{vmatrix}, \qquad (5)$$

and so on. The expression $(ab\ldots d\,|\,a\beta\ldots\delta)$, involving ν symbols before and after the vertical line, is called a ν-th compound inner product. If $\nu = 1$ it is the ordinary simple inner product as in § 1 (4). If $\nu = n$ it factorizes into $(ab\ldots m)(a\beta\ldots\mu)$; and if $\nu > n$ it vanishes identically.

To prove (4), consider any minor D of the determinant $|a_{ij}|$, when it is expressed symbolically. Let D have ν rows and columns, so requiring ν pairs of equivalent symbols, α, a; β, b; ...; δ, d. Thus

$$D = \begin{vmatrix} a_{ir} & a_{is} & \ldots & a_{it} \\ \ldots & \ldots & & \ldots \\ a_{kr} & a_{ks} & \ldots & a_{kt} \end{vmatrix} = \begin{vmatrix} a_i a_r & \beta_i b_s & \ldots & \delta_i d_t \\ \ldots & \ldots & \ldots & \ldots \\ a_k a_r & \beta_k b_s & \ldots & \delta_k d_t \end{vmatrix}$$

$$= (a\beta\ldots\delta)_{ij\ldots k}\, a_r b_s \ldots d_t$$

$$= \frac{1}{\nu!}(ab\ldots d)_{rs\ldots t}(a\beta\ldots\delta)_{ij\ldots k},$$

after interchanging the ν equivalent symbols in every possible way, adding the $\nu!$ equal results, and then dividing by $\nu!$. This minor has been selected from elements common to rows $i, j, \ldots, k$, and columns $r, s, \ldots, t$ of the original determinant. If now $r, s, \ldots, t$ are respectively equal to $i, j, \ldots, k$, then D is a *principal* minor; and if, further, we sum for all $\binom{n}{\nu}$ different combinations $i, j, \ldots, k$, we obtain

$$\frac{1}{\nu!}(ab\ldots d\,|\,a\beta\ldots\delta).$$

But, by a well known theorem in determinants, this sum of all principal minors of order ν gives the requisite coefficient of $\lambda^{n-\nu}$ in (2), with the sign $(-)^\nu$, so proving the theorem.

The M invariants.

3. The expression of each compound inner product, like $(abc\,|\,\alpha\beta\gamma)$ in § 2 (5), as a determinant $\begin{vmatrix} a_\alpha b_\alpha c_\alpha \\ a_\beta b_\beta c_\beta \\ a_\gamma b_\gamma c_\gamma \end{vmatrix}$, follows from the theorem of corresponding matrices—the Binet–Cauchy theorem. On expanding the determinant and using § 2 (1), these expressions are seen to be polynomials in the P's, and are therefore invariants. Let them be denoted by

$$\left.\begin{aligned} M_1 &= a_\alpha, \\ M_2 &= (ab\,|\,\alpha\beta) = a_\alpha b_\beta - a_\beta b_\alpha, \\ M_3 &= (abc\,|\,\alpha\beta\gamma) = \Sigma \pm a_\alpha b_\beta c_\gamma, \end{aligned}\right\}, \tag{1}$$

and so on. This sequence may be continued until

$$M_n = (abc\ldots m)(\alpha\beta\gamma\ldots\mu) = n!\,|\,A\,| \tag{2}$$

is reached, giving the determinant $|a_{ij}|$ of the matrix A. The whole set constitutes a complete system of invariants of the bilinear form, as will appear in § 8. In particular, it may be used to express the invariants P_1, P_2, P_3, ... already found.

In fact, since equivalent symbols, § 1 (5), are involved, we have $a_\alpha = b_\beta = c_\gamma = P_1$. Hence, by § 2 (1),

$$M_2 = P_1^2 - P_2.$$

Again, by a fundamental identity,

$$M_3 = (abc\,|\,\alpha\beta\gamma) = (ab\,|\,\alpha\beta)\,c_\gamma + (ab\,|\,\beta\gamma)\,c_\alpha + (ab\,|\,\gamma\alpha)\,c_\beta. \tag{3}$$

But, by interchange of symbols a, α with b, β,

$$(ab\,|\,\gamma\alpha)\,c_\beta = (ba\,|\,\gamma\beta)\,c_\alpha = (ab\,|\,\beta\gamma)\,c_\alpha.$$

Hence

$$\begin{aligned} M_3 &= (ab\,|\,\alpha\beta)\,c_\gamma + 2(ab\,|\,\beta\gamma)\,c_\alpha \\ &= M_2 P_1 + 2(a_\beta b_\gamma - a_\gamma b_\beta)\,c_\alpha \\ &= M_2 P_1 - 2M_1 P_2 + 2P_3. \end{aligned} \tag{4}$$

Similarly, by a fundamental identity,

$$\begin{aligned} M_4 &= (abcd\,|\,\alpha\beta\gamma\delta) \\ &= (abc\,|\,\alpha\beta\gamma)\,d_\delta - (abc\,|\,\delta\beta\gamma)\,d_\alpha - (abc\,|\,\alpha\delta\gamma)\,d_\beta - (abc\,|\,\alpha\beta\delta)\,d_\gamma. \end{aligned}$$

The final three terms are equal on interchange among α, β, γ. Thus

$$\begin{aligned} M_4 &= M_3 d_\delta - 3(abc\,|\,\alpha\beta\delta)\, d_\gamma \\ &= M_3 P_1 - 3\{(ab\,|\,\alpha\beta)\, c_\delta - 2(ab\,|\,\alpha\delta)\, c_\beta\}\, d_\gamma, \end{aligned}$$

on using the previous case. Or

$$M_4 = M_3 P_1 - 3\{M_2 P_2 - 2(M_1 P_3 - P_4)\}. \tag{5}$$

Similar reasoning leads to

$$M_{\nu+1} = (ab\ldots cd\,|\,\alpha\beta\ldots\gamma\delta) = M_\nu d_\delta - \nu(ab\ldots c\,|\,\alpha\beta\ldots\delta)\, d_\gamma,$$

and finally

$$\begin{aligned} \frac{1}{\nu!} M_{\nu+1} = \frac{1}{\nu!} M_\nu P_1 - \frac{1}{(\nu-1)!} M_{\nu-1} P_2 \\ + \ldots + (-)^i \frac{1}{(\nu-i)!} M_{\nu-i} P_{i+1} + \ldots + (-)^\nu P_{\nu+1}, \end{aligned} \tag{6}$$

where $0 < \nu < n$. For all suffixes $\nu \geq n$, M_ν vanishes identically.

These formulae (3)–(6) may be reversed, yielding

$$\left.\begin{aligned} P_1 &= M_1, \\ P_2 &= -M_2 + M_1^2, \\ P_3 &= \frac{1}{2!}(M_3 - 3M_1 M_2 + 2M_1^3), \text{ etc.} \end{aligned}\right\} \tag{7}$$

In fact we have $\sum_{\nu=1}^{n} \nu M_{\nu-1} \frac{\partial P_\mu}{\partial M_\nu} = 0\ (\mu > 1)$; and the coefficients in (7) therefore follow the rule for those of a seminvariant, whose degree and weight are equal, for a binary form $(1, M_1, M_2, \ldots, M_n)(X_1, X_2)^n$.

The prepared bilinear forms.

4. Just as the original invariants P have been replaced by the set M, so it is found convenient to express the bilinear forms $\phi_0 = u_x$, $\phi_1 = u_\alpha a_x$, ... in terms of a set ψ, called the prepared bilinear forms, and defined as follows:

$$\left.\begin{aligned} \psi_0 &= (u\,|\,x) &&= \Sigma u_i x_i = \phi_0, \\ \psi_1 &= (ua\,|\,ax) &&= \Sigma (ua)_{ij} (ax)_{ij}, \\ \psi_2 &= (uab\,|\,\alpha\beta x) &&= \Sigma (uab)_{ijk} (\alpha\beta x)_{ijk}, \\ \ldots &\quad \ldots \quad \ldots \quad \ldots &&\ldots \quad \ldots \quad \ldots \quad \ldots \\ \psi_{n-1} &= (uab\ldots cd\,|\,\alpha\beta\ldots\gamma\delta x) &&= (uab\ldots cd)(\alpha\beta\ldots\gamma\delta x). \end{aligned}\right\} \tag{1}$$

The typical ψ_ν has $\nu-1$ symbols and one variable, of each kind. Since these forms are similar in structure to the M's, the process of the preceding paragraph is available, and we obtain

$$\left.\begin{aligned}
&\psi_1 = (ua|ax) = u_a a_x - u_x a_a = \phi_1 - M_1\phi_0,\\
&\psi_2 = (uab|a\beta x) = 2\phi_2 - 2M_1\phi_1 + M_2\phi_0,\\
&\psi_3 = (uabc|a\beta\gamma x) = 3\,.\,2\,.\,1\phi_3 - 3\,.\,2M_1\phi_2 + 3M_2\phi_1 - M_3\phi_0,\\
&\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\\
&\frac{1}{\nu!}\psi_\nu = (uab\ldots c|a\beta\ldots\gamma x)\\
&\qquad\qquad = \phi_\nu - M_1\phi_{\nu-1} + \frac{1}{2!}M_2\phi_{\nu-2} - \ldots + (-)^\nu\frac{1}{\nu!}M_\nu\phi_0,
\end{aligned}\right\} \tag{2}$$

where $0 < \nu \leqslant n$.

Since no factor on the right side of equations (1) exists with more than n suffixes $ij\ldots k$, it follows that ψ_n and all succeeding ψ's vanish. Thus

$$0 = \frac{1}{n!}\psi_n = \phi_n - M_1\phi_{n-1} + \frac{1}{2!}M_2\phi_{n-2} - \ldots + (-)^n\frac{1}{n!}M_n\phi_0. \tag{3}$$

It is interesting to rewrite the relations (2) and (3) in matrix notation, noticing that $\phi_\nu = uA^\nu x$, § 1 (8), and that each M is a scalar number. On using § 3 (1) and § 2 (3) and (4) we have the relations

$$M_1 = -p_1,\quad \frac{1}{2!}M_2 = p_2,\quad \frac{1}{3!}M_3 = -p_3,\quad \ldots,\quad \frac{1}{n!}M_n = (-)^n p_n. \tag{4}$$

Substituting in (2) we obtain

$$\left.\begin{aligned}
&\psi_1 = u(A + p_1 I)x,\\
&\psi_2 = 2!\,u(A^2 + p_1 A + p_2 I)x,\\
&\psi_3 = 3!\,u(A^3 + p_1 A^2 + p_2 A + p_3 I)x,\\
&\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\\
&0 = \psi_n = n!\,u(A^n + p_1 A^{n-1} + p_2 A^{n-2} + \ldots + p_{n-1}A + p_n I)x,
\end{aligned}\right\} \tag{5}$$

where I denotes the unit matrix. Since each ψ_ν is a bilinear form, it has a matrix which may be denoted by $\nu!B_\nu$. Thus

$$\psi_\nu = u\,\nu!\,B_\nu x = \nu!\,uB_\nu x, \tag{6}$$

$$\left.\begin{aligned}
&B_1 = A + p_1 I,\\
&B_2 = A^2 + p_1 A + p_2 I,\\
&\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\\
&0 = B_n = A^n + p_1 A^{n-1} + \ldots + p_n I.
\end{aligned}\right\} \tag{7}$$

This last B_n (but not necessarily the others) is equal to zero, since it is the matrix of ψ_n, a vanishing bilinear form. Incidentally, this proves* the Cayley–Hamilton theorem that *a matrix satisfies its own characteristic equation*; for this polynomial B_n is $f(A)$, where $f(\lambda) = |\lambda I - A|$, § 2 (3). Hence

$$f(A) \equiv A^n + p_1 A^{n-1} + p_2 A^{n-2} + \ldots + p_n I = 0, \tag{8}$$

Application of the method of Frobenius.

5. A very simple alternative way of arriving at these matrices B_ν is as follows†. Let the ordinary process of long division be carried out for

$$(\lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} + \ldots + p_n) \div (\lambda - A),$$

treating A for the moment as an ordinary number, and λ as the variable. The quotient is at once seen to be

$$\lambda^{n-1} + B_1\lambda^{n-2} + B_2\lambda^{n-3} + \ldots + B_{n-1}$$

with remainder B_n. Since we are dealing with one matrix A and scalar numbers, the process is still legitimate when A is once more a matrix. On writing the dividend as $f(\lambda)$ we obtain

$$f(\lambda) I \div (\lambda I - A) = (I\lambda^{n-1} + B_1\lambda^{n-2} + \ldots + B_{n-1}) + B_n \div (\lambda I - A),$$

provided, of course, that λ is not a latent root of A, so that the divisor matrix $\lambda I - A$ is non-singular. But the left-hand member of this identity can be written $f(\lambda)\, C$, where C is the matrix reciprocal to $\lambda I - A$. The ij-th element of C is consequently the ij-th first minor of the determinant $f(\lambda) \equiv |\lambda I - A|$, divided by $f(\lambda)$. Hence this left-hand member is a polynomial $C_0\lambda^{n-1} + C_1\lambda^{n-2} + \ldots + C_{n-1}$, with matrix coefficients, of order $n-1$ in λ. By cross multiplication this gives

$$f(\lambda) I = (C_0\lambda^{n-1} + \ldots + C_{n-1})(\lambda I - A).$$

By comparing coefficients of powers of λ on each side of this last identity, the undetermined coefficients C_ν are found to be the polynomials in A given in § 3 (7), terminating with the condition that $f(A)$ vanishes.

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* This is noticed by Study, *loc. cit.* (1923), 172.

† This is based on the classical thesis of Frobenius: "Über lineare Substitutionen und bilineare Formen", *Journal für Math.*, 84 (1878), 1–63 (12). *Cf.* Baker, *Proc. London Math. Soc.* (1), 30 (1899), 196.

Hence

$$\frac{1}{\lambda I - A} = \frac{I\lambda^{n-1} + B_1\lambda^{n-2} + \ldots + B_{n-1}}{f(\lambda)}.$$

If A is non-singular we may put $\lambda = 0$ in this identity, and obtain the result $A^{-1} = -B_{n-1}/f(0) = (-)^{n-1}B_{n-1}/|A|$. But B_{n-1} is the matrix of the bilinear form ψ_{n-1}: hence $\psi_{n-1} \equiv (uab \ldots d)(\alpha\beta \ldots \delta x)$ defines a collineation which is the reciprocal of the given collineation $u_a a_x$. Negative powers $A^{-2}, A^{-3}, \ldots$, of A are connected with $\psi_{n-2}, \psi_{n-3}, \ldots$ just as positive powers are with $\psi_2, \psi_3, \ldots$.

Matrix differential operators.

6. Still another independent approach to these matrices B_ν is provided by a theory of matrix differentiation formulated by the present writer. In fact, it has been shown that if Ω is a matrix differential operator defined by

$$\Omega \equiv \begin{bmatrix} \frac{\partial}{\partial a_{11}} & \frac{\partial}{\partial a_{21}} & \cdots & \frac{\partial}{\partial a_{n1}} \\ \cdots & \cdots & & \cdots \\ \frac{\partial}{\partial a_{1n}} & \frac{\partial}{\partial a_{2n}} & \cdots & \frac{\partial}{\partial a_{nn}} \end{bmatrix} \equiv \left[\frac{\partial}{\partial a_{ji}}\right], \tag{1}$$

so that the operation of Ω upon a scalar function f would be the matrix $[\partial f/\partial a_{ji}]$ whose ij-th element is $\partial f/\partial a_{ji}$, then the following relations are true* :

$$\left.\begin{aligned} \Omega p_1 + I &= 0, \\ \Omega p_2 + p_1 I + A &= 0, \\ \Omega p_3 + p_2 I + p_1 A + A^2 &= 0, \quad \text{etc.} \end{aligned}\right\} \tag{2}$$

Hence

$$B_1 = -\Omega p_2, \quad B_2 = -\Omega p_3, \quad \ldots, \quad B_{n-1} = -\Omega p_n, \tag{3}$$

and, by § 4 (4), if $0 < \nu \leqslant n$,

$$\Omega M_\nu = (-)^\nu \nu!\, \Omega p_\nu = (-)^{\nu-1} \nu!\, B_{\nu-1}. \tag{4}$$

In other words, the operation of Ω upon the scalar quantity M_ν produces the matrix of the bilinear form $\psi_{\nu-1}$, modified by the sign $(-)^{\nu-1}$. But

$$\psi_{\nu-1} = (uab \ldots c \,|\, \alpha\beta \ldots \gamma x) = (-)^{\nu-1}(ab \ldots cu \,|\, \alpha\beta \ldots \gamma x).$$

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* *Proc. Edinburgh Math. Soc.* (2), 1 (1928), 111–128 (119).

Hence ΩM_ν is ν times the matrix of the bilinear form

$$(ab \dots cu \,|\, a\beta \dots \gamma x). \tag{5}$$

Regarding M_ν as a function of n^2 independent variables a_{ij}, we may call ΩM_ν its matrix derivate. So we have the theorem :

The matrices of each of the bilinear forms $(au \,|\, ax)$, $2(abu \,|\, a\beta x)$, $3(abcu \,|\, a\beta\gamma x)$, *etc., are the respective matrix derivates of the prepared invariants* $(ab \,|\, a\beta)$, $(abc \,|\, a\beta\gamma)$, $(abcd \,|\, a\beta\gamma\delta)$, *etc.*

Conversely, it is clear that, if (4) is true, then (2) immediately follows. To prove (4) directly, consider the partial derivative $\partial\phi / \partial a_{ji}$ of a polynomial ϕ of order ν in the arguments a_{ji}. If ϕ is thrown into symbolic form, it requires ν pairs of equivalent symbols $a, a; \dots; \delta, d$. If, further, we treat these 2ν symbols as independent until after differentiation has been carried out, we obtain a symbolic form of the partial derivative, namely

$$\frac{\partial \phi}{\partial a_{ji}} = \left(\frac{\partial^2}{\partial a_j \partial a_i} + \frac{\partial^2}{\partial \beta_j \partial b_i} + \dots + \frac{\partial^2}{\partial \delta_j \partial d_i} \right) \phi = \Sigma \frac{\partial^2 \phi}{\partial a_j \partial a_i}, \tag{6}$$

where the summation has ν terms. This enables us to express the matrix $\Omega\phi = [\partial\phi / \partial a_{ji}]$ as a sum of ν terms

$$\Sigma \left\{ \frac{\partial}{\partial a} \right\} \left[\frac{\partial}{\partial a} \right] \phi,$$

where $\{\partial / \partial a\}$ denotes a single *column* and $[\partial / \partial a]$ a single *row*, each of n differential operators, as in § 1 (3). On multiplying fore and aft by the *row* u and the *column* x respectively, we obtain

$$u \,.\, \Omega\phi \,.\, x = \Sigma u \left\{ \frac{\partial}{\partial a} \right\} \left[\frac{\partial}{\partial a} \right] \phi \,.\, x. \tag{7}$$

Since ϕ is scalar, it may be placed as final factor on the right of x. By the associative law of multiplication, the factors $u\{\partial / \partial a\}[\partial / \partial a]x$ may be taken in pairs. But, by § 1 (4),

$$u \left\{ \frac{\partial}{\partial a} \right\} = u_1 \frac{\partial}{\partial a_1} + u_2 \frac{\partial}{\partial a_2} + \dots + u_n \frac{\partial}{\partial a_n} = \left(u \,\middle|\, \frac{\partial}{\partial a} \right),$$

and similarly

$$\left[\frac{\partial}{\partial a} \right] x = \left(x \,\middle|\, \frac{\partial}{\partial a} \right).$$

Hence

$$u \,.\, \Omega\phi \,.\, x = \Sigma \left(u \,\middle|\, \frac{\partial}{\partial a} \right) \left(x \,\middle|\, \frac{\partial}{\partial a} \right) \phi, \tag{8}$$

where ϕ is any polynomial in the coefficients a_{ij}. These right-hand operations are the ordinary polarizing processes; and if, for example, $\phi = (abc\,|\,\alpha\beta\gamma) = M_3$, we have $\nu = 3$ and

$$\Sigma\left(u\,\Big|\,\frac{\partial}{\partial a}\right)\left(x\,\Big|\,\frac{\partial}{\partial a}\right)\phi = (ubc\,|\,x\beta\gamma)+(auc\,|\,ax\gamma)+(abu\,|\,a\beta x),$$

where u, x are substituted* for each pair of symbols a, α in turn. But these three terms are equal, since the symbols are equivalent. Hence

$$u\,.\,\Omega M_3\,.\,x = 3(abu\,|\,a\beta x) = 3\psi_2.$$

Similarly, in general,

$$u\,.\,\Omega M_\nu\,.\,x = \nu(ab\,\ldots\,cu\,|\,a\beta\,\ldots\,\gamma x) = (-)^{\nu-1}\nu(uab\,\ldots\,c\,|\,a\beta\,\ldots\,\gamma x)$$
$$= (-)^{\nu-1}\nu\psi_{\nu-1}.$$

Using § 3 (6), we eliminate $\psi_{\nu-1}$ and obtain

$$u\,.\,\Omega M_\nu\,.\,x = (-)^{\nu-1}\nu!\;uB_{\nu-1}x, \tag{9}$$

which at once gives $\Omega M_\nu = (-)^{\nu-1}\nu!\;B_{\nu-1}$, the required result (4).

Complete Clebsch system of the bilinear form for the case of independent transformations of x and u.

6. When the variables x and u undergo independent linear transformations to ξ and θ respectively, as mentioned at the end of § 1, it is a comparatively simple matter to establish a complete system of irreducible concomitants of a bilinear form $\phi_1 = u_\alpha a_x$. Adopting the matrix notation already employed, let the transformations be given by the equations

$$x = H\xi, \quad u = \theta K, \quad |H| \neq 0, \quad |K| \neq 0,$$

where, as always, x and ξ denote single column, and u, θ single row matrices. The bilinear form ϕ_1, with matrix A becomes a bilinear form in ξ, θ with matrix KAH, in the following way,

$$\phi_1 = uAx = \theta KAH\xi;$$

and the case is characterized by assuming K and H to be arbitrary

* *Cf.* Turnbull, *The theory of determinants, matrices, and invariants* (1928), 112.

independent transformation matrices. From the point of view of the invariant theory this case provides an example of multiple fields*, where the symbols and variables fall into two groups, convolution only taking place within a group and never between members of distinct groups. In the first, called the x group, the following elements occur :

$$a, b, c, \ldots, x, y, \ldots, z,$$

consisting of all symbols equivalent to a and all variables $y, \ldots, z$ cogredient with x. In the second, or u group, there are

$$\alpha, \beta, \gamma, \ldots, u, v, \ldots, w,$$

consisting of all symbols equivalent to α and all variables $v, \ldots, w$ cogredient with u. A polynomial concomitant, composed from such groups, can always be symbolized by an aggregate of factors of two kinds,

$$\left.\begin{aligned} &\text{(i)} && (abc \ldots \,|\, xy \ldots z) \equiv (A_r \,|\, X_r) \\ &\text{(ii)} && (\alpha\beta\gamma \ldots \,|\, uv \ldots w) \equiv (A'_r \,|\, U_r) \end{aligned}\right\} \quad (0 \leqslant r \leqslant n),$$

where A_r denotes the r equivalent $a, b, c, \ldots$, convolved together, and X_r denotes r cogredient variables $x, y, \ldots, z$, also convolved. Similarly for A' and U. Simplicity is gained here, in contrast to the collineation case of § 8, since there is no convolution, such as a_α, possible between elements of distinct groups.

By a Clebsch system is meant a system wherein the variables of each group enter as compound coordinates,

$$\pi_1 = x, \quad \pi_2 = xy, \quad \ldots, \quad \pi_{n-1} = xy \ldots z,$$

$$p_1 = u, \quad p_2 = uv, \quad \ldots, \quad p_{n-1} = uv \ldots w.$$

Here π_2 indicates the set of $\binom{n}{2}$ determinants of the array

$$\left\| \begin{matrix} x_1 & x_2 & \ldots & x_n \\ y_1 & y_2 & \ldots & y_n \end{matrix} \right\|,$$

and so on for the others. Accordingly, π_r and p_r are called r-th com-

* *Cf.* Weitzenböck, *Invariantentheorie* (1923), 169, and Turnbull, *Invariants* (1928), 240, 297.

pounds of the x and u groups respectively, and each consists of a set of $\binom{n}{r}$ variables. Since p and π belong to distinct groups, they themselves are independent of each other. In the $n-1$ different compounds π_r the variables x, y, ... overlap, π_{r+1} containing these same variables as π_r, together with a new one. Similarly for p_r. To these may be appended the n-th compounds π_n and p_n which are concomitants of the field, each being a determinant of order n, whose rows consist of n sets of variables cogredient to x and u respectively. This leads to the following theorem :

THEOREM. *The $n+2$ symbolic expressions*

$$\pi_n, \quad p_n, \quad P_r = (A_r|\pi_r)(A'_r|p_r), \quad 0 < r < n, \quad P_n = (A_n)(A'_n)$$

constitute a complete system of concomitants of a single bilinear form $u_a a_x$, when x and u are independent of each other, but admit of compound variables π and p in their respective fields.

Proof. This is virtually the same as that for a single quadratic (quoted on page 297 of the work cited in the last foot-note). Any polynomial concomitant can be expressed as a sum of terms

$$P = (A_r|\pi_r)\ldots(A'|p_s)\ldots,$$

consisting of a product of factors entirely of type (i), followed by a product entirely of type (ii). Corresponding to each of the r equivalent symbols a, b, ..., contained in A_r, there are r equivalent symbols α, β, ... situated in the second part of the whole product, possibly all convolved in one, or possibly distributed in several factors. Without loss of generality we may take r and s to be highest values of suffixes in the two parts of P, and may suppose, further, that $r \geqslant s$. For if $r < s$ we merely interchange the rôles of the two groups. Now let the r equivalent symbols a, b, ..., contained in A_r, be permuted in all $r!$ ways, and the results added together. This expresses $r!P$ as a sum of terms to which a fundamental identity, dual to the Sylvester type, applies* [*cf.* § 3 (3)]. We may then convolve the corresponding r symbols α, β, ... entirely into the first factor $(A'_s|p_s)$ of the second group, obtaining a sum of terms in which this factor now takes the form

$$(A'_r A'_{s_1}|p_s q_t) \quad (0 \leqslant t = r+s_1-s),$$

where A'_{s_1} denotes some or all of the original s symbols of A', and q

* Turnbull, *Invariants* (1928), 93, § 13.

denotes t of the variables $u, v, \ldots$, diverted from the several factors in which the parts of A'_r originally stood. Since, by hypothesis, p_s is the highest compound of its type existing in P, it already contains all these variables $u, v, \ldots$; and hence the convolution $p_s q_t$ contains duplicates (and therefore vanishes) unless $t = 0$. But $r \geqslant s$. Hence the form vanishes unless $r = s$, $s_1 = 0$; and in this case the factor

$$P_r = (A_r | \pi_r)(A'_r | p_r)$$

is apparent. If $r = n$, the compound factors resolve into

$$(A_n | \pi_n) = (ab \ldots m)(xy \ldots),$$

$$(A'_n | p_n) = (\alpha\beta \ldots \mu)(uv \ldots),$$

and P_n may be taken as the invariant $(ab \ldots m)(\alpha\beta \ldots \mu)$. This proves the theorem.

COROLLARY 1. *The bilinear form has one and only one irreducible invariant for general linear transformations.*

It is the determinant of the bilinear form

$$|a_{ij}| = \frac{1}{n!}(ab \ldots m)(\alpha\beta \ldots \mu).$$

COROLLARY 2. *The complete system is given by the bordered determinants, obtained by using new variables contragredient to x and u respectively.*

In fact, to use a ternary illustration, the bordered determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \xi_1 & \xi'_1 \\ a_{21} & a_{22} & a_{23} & \xi_2 & \xi'_2 \\ a_{31} & a_{32} & a_{33} & \xi_3 & \xi'_3 \\ \theta_1 & \theta_2 & \theta_3 & . & . \\ \theta'_1 & \theta'_2 & \theta'_3 & . & . \end{vmatrix} \equiv \begin{vmatrix} a_{ij} & \xi & \xi' \\ \theta & . & . \\ \theta' & . & . \end{vmatrix}$$

is equivalent to the form $P_1 \equiv \Sigma u_i a_{ij} x_j \equiv u_a a_x$, if $u_1 = (\xi\xi')_{23}, \ldots$, $x_1 = (\theta\theta')_{23}, \ldots$, by resolution of u into contragredient variables ξ, ξ' and x into θ, θ'. Symbolically

$$P_1 = (a\xi\xi')(a\theta\theta');$$

and similarly for n variables and for borders of any size less than n.

Complete system for the collineation.

8. For a ternary form $\phi_1 = u_\alpha a_x = u_\beta b_x = \ldots$ $(n = 3)$, Clebsch and Gordan gave* the complete system of eight forms, which can be written as

$$\left.\begin{aligned} &u_x = \psi_0, \quad u_\alpha a_x = \phi_1, \quad (abu)(a\beta x) = \psi_2, \\ &a_\alpha = M_1, \quad (ab\,|\,a\beta) = M_2, \quad (abc)(a\beta\gamma) = M_3, \\ &Q = (abu)(a\beta\,|\,cu)\,u_\gamma, \quad Q' = (a\beta x)(ab\,|\,\gamma x)\,c_x. \end{aligned}\right\} \tag{1}$$

In the last section the variables u, x were independent, but in this case they are contragredient. When $n > 3$ the Clebsch system requires compound coordinates p_r, π_r, as in § 7, but with the additional condition that p_r, π_{n-r} are equal. Although the problem of establishing such a complete Clebsch system is very elaborate, and is not attempted here, it is possible to give a complete system, involving any number of variables y, z, ..., and u, v, ... cogredient to x and u respectively. This general system, which is very similar to the ternary system (1), consists of the n bilinear forms ψ_ν, the n invariants M_ν, together with a rather complicated set of forms, denoted by Q, including one covariant and one contravariant, both of order n. These last two forms, in fact, are each composed of n linear factors, and geometrically they give the simplex whose vertices are the n latent points of the general collineation. For instance, when $n = 3$, Q and Q' give the same triangle alternatively in line and point coordinates.

By the fundamental theorem, any polynomial concomitant of the bilinear form $\phi_1 = u_\alpha a_x$ can be expressed as a sum of terms P, composed of factors of three kinds :

(i) inner products a_x, a_β, u_α, u_x;

(ii) outer a-products $(ab \ldots uv \ldots w) \equiv (A_r U_{n-r})$;

(iii) outer α-products $(a\beta \ldots xy \ldots z) \equiv (A'_r X_{n-r})$.

In (ii) and (iii) r may be any integer from 0 to n inclusive, and the cases when $0 < r < n$ are new, not having arisen in § 7. The types (i) and (ii) of § 7 are directly related to the present type (i), as in § 3 (1).

To every symbol a in P there corresponds a symbol α; accordingly, if P first consists entirely of inner products involving u, we can obviously

* *Math. Annalen*, 1 (1869), 359.

place symbols a, α of any pair of factors in juxtaposition. Possible irreducible products must therefore be included in the set

$$\phi_0 = u_x, \quad \phi_1 = u_\alpha a_x, \quad \phi_2 = u_\alpha a_\beta b_x, \quad \ldots. \tag{2}$$

But, by reversing the relations § 4 (2), each ϕ can be expressed in terms of the set

$$\psi_0 = u_x, \quad \psi_1 = -(au\,|\,\alpha x), \quad \psi_2 = (abu\,|\,\alpha\beta x), \quad \ldots, \tag{3}$$

which can therefore take their place. Incidentally, this settles the case whenever P is composed entirely of factors of type (i) and involves x. Similarly, if P consists of inner products a_β with no variable, then possible irreducible forms can only be

$$P_1 = a_\alpha, \quad P_2 = a_\beta b_\alpha, \quad P_3 = a_\beta b_\gamma c_\alpha, \quad \ldots,$$

which, in turn, can be replaced by

$$M_1 = a_\alpha, \quad M_2 = (ab\,|\,\alpha\beta), \quad M_3 = (abc\,|\,\alpha\beta\gamma), \quad \ldots, \tag{4}$$

as in § 3 (12). For brevity, the typical ψ and M are written

$$\left.\begin{aligned} \psi_\nu &= (uA_\nu\,|\,A'_\nu x) \quad (0 \leqslant \nu < n-1), \\ \psi_{n-1} &= (ab \ldots du)(\alpha\beta \ldots \delta x) = (A_{n-1}u)(A'_{n-1}x), \\ M_\nu &= (A_\nu\,|\,A'_\nu) \quad (0 < \nu \leqslant n-1), \\ M_n &= (ab \ldots m)(\alpha\beta \ldots \mu) = (A_n)(A'_n). \end{aligned}\right\} \tag{5}$$

For higher values of the suffix, each ψ and M vanishes identically. This exhausts the possible irreducible forms originally composed entirely of inner products.

Next, let P contain factors of type (ii) or (iii). Since a product $(d_1 d_2 \ldots d_n)\,(\delta_1 \delta_2 \ldots \delta_n)$, involving both these types, can be resolved into $\Sigma \pm d_{1\delta_1} d_{2\delta_2} \ldots d_{n\delta_n}$ by the multiplication theorem of determinants, while this last expression consists entirely of inner products (i), we may exclude the case when P contains types (ii) and (iii) simultaneously. Let P then contain factor types (i) and (ii), but excluding (iii).

Either P contains a factor $(uv \ldots w)$, with n cogredient variables, or else

$$P = (ab \ldots cuv\ldots)N = (A_r U_{n-r})N \quad (r > 0).$$

Since r symbols α, β, ..., γ, matching a, b, ..., c, must occur in P, while the factor type (iii) is absent, they present themselves entirely among inner products d_α. Using N and M to denote remaining factors of P,

not written out explicitly, we obtain, after $r!$ interchanges of equivalent symbols,

$$r!\, P = (ab \ldots cuv \ldots)\, \Sigma \pm d_\alpha e_\beta \ldots M$$

$$= (A_r U_{n-r})(\alpha\beta \ldots \gamma\,|\,de \ldots)\, M$$

$$= (A_r U_{n-r})(A'_r\,|\,de \ldots)\, M.$$

Here, the second factor is an r-th compound inner product, where the r symbols $de \ldots$ denote symbols a or variables u; so we write

$$r!\, P = (A_r U_{n-r})(A'_r\,|\,B_s U_{r-s})\, M \quad (0 \leqslant s \leqslant r).$$

If $s = 0$, P contains the concomitant $(A_r U_{n-r})(A'_r U_r)$, where U_r consists of r variables of type u, which are not necessarily distinct from those occurring in U_{n-r}.

If, however, $s > 0$, B_s consists of s new symbols a, b, ..., not already found in A_r. They are accordingly matched by s symbols α, β, ..., somewhere among the inner products of M. By $s!$ interchanges of these equivalent symbols we have

$$r!\, s!\, P = (A_r U_{n-r})(A'_r\,|\,B_s U_{r-s})(B'_s\,|\,C_t U_{s-t})\, N \quad (0 \leqslant t \leqslant s).$$

Such a process, sufficiently repeated, groups *all* the symbols into pairs of matrices A_r, A'_r; B_s, B'_s; and so on. For it certainly exhausts all symbols a, and therefore all the symbols α. Ultimately an actual concomitant is reached,

$$Q = (A_r U_{n-r})(A'_r\,|\,B_s U_{r-s})(B'_s\,|\,\ldots) \ldots (\ldots\,|\,D_t \ldots)(D'_t\,|\,U_t).$$

Hence, if P contains an outer product of type (ii), P is reducible in terms of previous forms and of the form Q. This Q contains an even number of compound inner products, led by one outer product.

We may impose the further condition that

$$r \geqslant s \geqslant \ldots \geqslant t.$$

For, if $r = s$, a dual Sylvester identity* gives

$$(A'_r\,|\,B_s U_{r-s})(B'_s\,|\,\ldots) - (B'_s\,|\,B_s U_{r-s})(A'_r\,|\,\ldots) = \sum_{i>0} (A_{r+i}\,|\,\ldots)(\ldots),$$

whence Q is expressed in terms of forms with increased r and reducible terms. Likewise for any two adjacent factors with equal suffixes r, s.

* Turnbull, *Invariants*, 95.

These considerations show that P is reduced to specified forms, provided that they are considered in the following order :

(1) In ascending degree and weight.

(2) For the same degree and weight, in ascending lexicographical order of the suffixes. By this is meant that the form Q is considered before Q_1 if the first of the differences $r-r_1$, $s-s_1$, ..., which is not zero, is positive.

Finally, P may contain factors of type (iii), but not (ii). Exactly the same methods apply reciprocally to show that the only possible irreducibles are the duals of the above forms. Hence the complete system is as follows.

Absolute concomitants.

$$(uv \dots w), \quad u_x, \quad (xy \dots z).$$

Bilinear forms.

$$\psi_0 = u_x, \quad \psi_1 = (ua|ax) = \psi_1 - a_\alpha u_x, \quad \psi_2 = (uab|a\beta x) = (uA_2|A_2' x),$$
$$\dots, \quad \psi_{n-2} = (uA_{n-2}|A_{n-2}' x), \quad \psi_{n-1} = (uA_{n-1})(A_{n-1}' x).$$

Invariants.

$$M_1 = a_\alpha, \quad M_2 = (ab|a\beta) = (A_2|A_2'), \quad \dots, \quad M_n = (A_n)(A_n').$$

Rank concomitants.

$$(A_r U_{n-r})(A_r'|U_r), \quad (A_r' X_{n-r})(A_r|X_r) \quad (r = 1, 2. \dots, n-1).$$

Q concomitants.

$$Q = (A_{q_1} U_{n-q_1})(A_{q_1}'|A_{q_2} U_{q_1-q_2})(A_{q_2}'|\dots)\dots(\dots|A_{q_\nu} U_{q_{\nu-1}-q_\nu})(A_{q_\nu}'|U_{q_\nu}),$$
$$Q' = (A_{q_1}' X_{n-q_1})(A_{q_1}|A_{q_2}' X_{q_1-q_2})(A_{q_2}|\dots)\dots(\dots|A_{q_\nu}' X_{q_{\nu-1}-q_\nu})(A_{q_\nu}|X_{q_\nu})$$
$$(n > q_1 > q_2 \dots > q_\nu > 0).$$

If $r_1, r_2, \dots, r_\nu$ denote the first differences of the descending sequence of integers $n, q_1, \dots, q_\nu$, then Q may be denoted by the characteristic

$$\{r_1 r_2 \dots r_\nu\}.$$

If $\nu = n$, each r must be unity, and Q is therefore a contravariant (or polar of a contravariant) of order n in u, while Q' is a covariant in x. These are seen to be the Jacobians $\partial(\psi)/\partial(u)$ and $\partial(\psi)/\partial(x)$, respectively,

of the n bilinear forms ψ. When all the latent roots of the original matrix A are distinct, these Jacobians can be shown to represent the latent simplex of the collineation.

Suppose that $(A_r U_{n-r})(A'_r | U_r)$ vanishes identically for all values of the variables. Then, by the methods of § 2, it follows that every minor of order r, in the determinant $|A|$, vanishes. Hence, if r is the *rank* of the matrix A, so that all minors of order greater than r vanish, while some of order r do not, the form in question does not vanish, whereas all such forms with suffix exceeding r do so. Similar remarks apply to the dual form $(A'_r X_{n-r})(A_r X_r)$; and this is the reason for calling these the rank concomitants. They can be represented, nonsymbolically, by bordered determinants analogous to those considered in § 7, and are, in fact, virtually the same concomitants. The like property possessed by the sets of ψ and M, namely that, when r is the rank,

$$\psi_r \neq 0, \quad M_r \neq 0, \quad \psi_{r+i} = M_{r+i} = 0 \quad (i > 0),$$

is, however, peculiar to the case of the collineation, and is therefore not so fundamental.

ASYMPTOTIC LINES IN A V_m IN A V_n*

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1. A curve in a V_m, which is immersed in a V_n, has $m-1$ curvatures and $m-1$ normals relative to V_m, and $n-1$ curvatures and $n-1$ normals relative to V_n, since it may be regarded as contained either in V_m or in V_n. (In the case of a V_2 in an E_3, the first curvature and first normal, relative to V_2, are the geodesic curvature and a normal to the curve tangential to V_2.)

We shall denote coordinates in V_n and V_m by x^α ($\alpha = 1, \ldots, n$) and x'^i ($i = 1, \ldots, m$) respectively†; the metric tensors by $g_{\alpha\beta}$ and g'_{ij}; Christoffel symbols by $\Gamma^\alpha_{\beta\gamma}$ and Γ'^i_{jk}; the curvatures and unit normals relative to V_n by $\underset{r}{\kappa}$ ($r = 1, \ldots, n-1$) and $\underset{r}{\xi^\alpha}$ ($r = 2, \ldots, n$), and those relative to V_m by $\underset{r}{\kappa'}$ ($r = 1, \ldots, m-1$) and $\underset{r}{\xi'^i}$ ($r = 2, \ldots, m$). The unit tangent will be denoted by $\underset{1}{\xi^\alpha}$ or $\underset{1}{\xi'^i}$, according as we regard it as a vector in V_n or in V_m.

Every vector in V_m is also a vector in V_n, and if v'^i are the components of such a vector in the x'-coordinates its components in the x-coordinates are

$$v'^i \frac{\partial x^\alpha}{\partial x'^i}.$$

It is clear that

$$\underset{1}{\xi^\alpha} = \underset{1}{\xi'^i} \frac{\partial x^\alpha}{\partial x'^i}$$

all along every curve in V_m; but, in general,

$$\underset{r}{\xi^\alpha} \neq \underset{r}{\xi'^i} \frac{\partial x^\alpha}{\partial x'^i}$$

for $r > 1$.

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* V_n is used throughout to denote a Riemannian n-space; S_n to denote a V_n of constant curvature; E_n a Euclidean n-space.

† Greek indices take the values 1, 2, ..., n, Latin indices the values 1, 2, ..., m, throughout this paper, except where otherwise stated. Indices written vertically above or below a letter have a purely ordinal significance.

2. A curve in V_m, in V_n, is defined* as an "asymptotic line of order p" of V_m if, at every point of the curve, its first p normals relative to V_n are all tangential to V_m; p is essentially less than m.

In this paper we prove that each of the sets of conditions $[A\,;\,p]$, $[B\,;\,p]$, given below, is necessary and sufficient for an asymptotic line of order p.

$$[A\,;\,p] \qquad \underset{r+1}{\xi^a} = \underset{r+1}{\xi^{\prime i}} \frac{\partial x^a}{\partial x^{\prime i}} \qquad (r = 1, \ldots, p\,;\ a = 1, \ldots, n),$$

i.e. the first p normals relative to V_n coincide, respectively, with the first p normals relative to V_m;

$$[B\,;\,p] \qquad \underset{r}{\kappa} = \underset{r}{\kappa'} \quad (r = 1, \ldots, p).$$

Further, we obtain an expression for $\underset{m}{\kappa}$ at any point of an asymptotic line of highest order (viz. $m-1$), which reduces, in the simple case of a V_2 in an E_3, to the well-known formula of Beltrami and Enneper for the torsion of an asymptotic line, in the ordinary theory of surfaces.

3. In order to avoid using two types of absolute derivation, one with respect to V_n and the other with respect to V_m, it is convenient to use the following method†

Let $T^{\alpha_1 \ldots \alpha_\rho\, i_1 \ldots i_r}_{\beta_1 \ldots \beta_\sigma\, j_1 \ldots j_s}$ be a set of quantities, defined at points of V_m, having the tensor character indicated by the Greek indices for transformations of the V_n-coordinates, and that indicated by the Latin indices for transformations of the V_m-coordinates. Then we define the "absolute differential" $DT^{\alpha_1 \ldots i_r}_{\beta_1 \ldots j_s}$ as follows:

$$DT^{\cdots}_{\cdots} = dT^{\cdots}_{\cdots} + \sum_{\tau=1}^{\rho} \Gamma^{\alpha_\tau}_{\lambda\gamma} \binom{\lambda}{\alpha_\tau} T^{\cdots}_{\cdots} dx^\gamma - \sum_{\tau=1}^{\sigma} \Gamma^{\lambda}_{\beta_\tau \gamma} \binom{\beta_\tau}{\lambda} T^{\cdots}_{\cdots} dx^\gamma$$

$$+ \sum_{t=1}^{r} \Gamma'^{\,it}_{lk} \binom{l}{i_t} T^{\cdots}_{\cdots} dx'^k - \sum_{t=1}^{s} \Gamma'^{\,l}_{j_t k} \binom{j_t}{l} T^{\cdots}_{\cdots} dx'^k\,;$$

where $\binom{\lambda}{\alpha_\tau} T^{\cdots}_{\cdots}$ denotes the result of replacing α_τ by λ in $T^{\cdots}_{\cdots}$, with similar meanings for $\binom{\beta_\tau}{\lambda}$, $\binom{l}{i_t}$ and $\binom{j_t}{l}$.

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* D. J. Struik, *Grundzüge der mehrdimensionalen Differentialgeometrie* (1922), 79–80, 110.

† *Cf.* E. Bortolotti, "Sulle varietà subordinate negli spazi a connessione affine . . . ", *Boll dell' Unione Matem. Italiana* (17), (1928), 87–94; R. Lagrange, *Calcul différentiel absolu* (1926). I am indebted to a referee for this suggestion.

This expression is a tensor of the same type as $T^{\cdots}_{\cdots}$, with respect to both V_n and V_m.

If ϕ is any scalar, defined at points of V_m,

$$D\phi = d\phi;$$

hence, along any curve in V_m,

$$Ds = ds$$

where s is the arc-length from a fixed point. The tensor $DT^{\cdots}_{\cdots}/Ds$ is accordingly defined along any curve. When the Latin indices are absent, the operators D and D/Ds give the ordinary absolute differential and derivative in V_n; when the Greek indices are absent they give the ordinary absolute differential and derivative in V_m.

It is evident that $\partial x^\alpha/\partial x'^i$ is a mixed tensor of the type above, being contravariant with respect to transformations in V_n and covariant with respect to transformations in V_m. We shall denote it by $\theta_i^{\ \alpha}$.

Then $$DT^{\cdots}_{\cdots} = \nabla_k T^{\cdots}_{\cdots} dx'^k,$$

where the operator ∇_k is defined as follows:

$$\nabla_k = \frac{\partial}{\partial x'^k} + \sum_{\tau=1}^{\rho} \theta_k^{\ \gamma} \Gamma^{\alpha_\tau}_{\lambda\gamma} \binom{\lambda}{\alpha_\tau} - \sum_{\tau=1}^{\sigma} \theta_k^{\ \gamma} \Gamma^{\lambda}_{\beta_\tau\gamma} \binom{\beta_\tau}{\lambda}$$

$$+ \sum_{t=1}^{\tau} \Gamma'^{it}_{lk} \binom{l}{i_t} - \sum_{t=1}^{s} \Gamma'^{l}_{jtk} \binom{j_t}{l}.$$

$\nabla_k T^{\cdots}_{\cdots}$ is a tensor having one more index of covariance in V_m than the original tensor $T^{\cdots}_{\cdots}$.

4. The tensor $\nabla_i \theta_j^{\ \alpha}$ is the "curvature tensor of V_m in V_n"*; we denote it by $\Omega_{ij}^{\ \ \alpha}$. Thus

$$\Omega_{ij}^{\ \ \alpha} = \frac{\partial \theta_j^{\ \alpha}}{\partial x'^i} + \Gamma^{\alpha}_{\beta\gamma} \theta_j^{\ \beta} \theta_i^{\ \gamma} - \Gamma'^{k}_{ji} \theta_k^{\ \alpha}. \tag{1}$$

From the relations

$$g_{\alpha\beta} \theta_j^{\ \alpha} \theta_k^{\ \beta} = g'_{jk}, \tag{2}$$

we have, operating with ∇_i,

$$g_{\alpha\beta} (\Omega_{ij}^{\ \ \alpha} \theta_k^{\ \beta} + \Omega_{ik}^{\ \ \beta} \theta_j^{\ \alpha}) = 0,$$

* Bortolotti, *loc. cit.* *Cf.* also J. A. Schouten, *Der Ricci-Kalkül* (1924), 159, 183; Struik, *loc. cit.* 92.

and, since $\Omega_{ij}{}^{\alpha}$ is symmetric in i, j, it is easily verified that this reduces to

$$g_{\alpha\beta}\Omega_{ij}{}^{\alpha}\theta_k{}^{\beta} = 0. \tag{3}$$

The m vectors $\theta_i{}^{\alpha}$ (for fixed i; $i = 1, \ldots, m$) are tangential to V_m, their components in the x'-coordinates being, in fact, δ_i^j; this is evident from the definition of $\theta_i{}^{\alpha}$, or from the relations

$$\theta_i{}^{\alpha} = \delta_i^j \frac{\partial x^{\alpha}}{\partial x'^j}.$$

Hence, from (3), the $\frac{1}{2}m(m+1)$ vectors $\Omega_{ij}{}^{\alpha}$ (for fixed i, j; $i, j = 1, \ldots, m$) are orthogonal to V_m.

Putting $n-m \equiv q$, let $\underset{r}{\chi}{}^{\alpha}$ $(r = 1, \ldots, q)$ be any set of q mutually orthogonal unit vector fields, defined at all points of V_m, each orthogonal to V_m. Then

$$\Omega_{ij}{}^{\alpha} = -\sum_{r=1}^{q} \overset{r}{\omega}_{ij} \underset{r}{\chi}{}^{\alpha}, \tag{4}$$

where the tensor fields $\overset{r}{\omega}_{ij}$ $(r = 1, \ldots, q)$ are so defined, viz.

$$\overset{r}{\omega}_{ij} = -\Omega_{ij}{}^{\alpha}\underset{r}{\chi}_{\alpha}. \tag{5}$$

Differentiating the relations

$$\theta_j{}^{\alpha}\underset{r}{\chi}_{\alpha} = 0,$$

we see also that

$$\overset{r}{\omega}_{ij} = \theta_j{}^{\alpha}\nabla_i\underset{r}{\chi}_{\alpha}. \tag{6}$$

5. An asymptotic line of order p is defined by the equations

$$[D;\ p] \qquad \underset{r+1}{\xi}{}^{\alpha}\underset{s}{\chi}_{\alpha} = 0 \qquad (r = 1, \ldots, p;\ s = 1, \ldots, q).$$

We can now prove that each of the sets of conditions $[A;\ p]$, $[B;\ p]$ is necessary and sufficient for such a curve:

$$[A;\ p] \qquad \underset{r+1}{\xi}{}^{\alpha} = \underset{r+1}{\xi}{}^{\prime i}\theta_i{}^{\alpha} \quad (r = 1, \ldots, p;\ \alpha = 1, \ldots, n).$$

$$[B;\ p] \qquad \underset{r}{\kappa} = \underset{r}{\kappa}{}' \qquad (r = 1, \ldots, p).$$

(i) *To show that* $[A;\ p]$ *and* $[B;\ p)$ *are necessary*:

We proceed by induction. Suppose that $[A;\ p]$ and $[B;\ p]$ are satisfied along an asymptotic line of order p; then we shall prove that

$[A\,;\ p+1]$ and $[B\,;\ p+1]$ are satisfied along an asymptotic line of order $p+1$.

Since an asymptotic line of order $p+1$ is, *ipso facto*, asymptotic of every lower order, we have, from $[A\,;\ p]$,

$$\underset{p+1}{\xi^{\alpha}} = \underset{p+1}{\xi^{\prime i}}\theta_i^{\ \alpha}.$$

Differentiating this along the curve, we get

$$\underset{p+1}{\kappa}\ \underset{p+2}{\xi^{\alpha}} - \underset{p}{\kappa}\ \underset{p}{\xi^{\alpha}} = \theta_i^{\ \alpha}\left[\underset{p+1}{\kappa'}\ \underset{p+2}{\xi^{\prime i}} - \underset{p}{\kappa'}\ \underset{p}{\xi^{\prime i}}\right] + \Omega_{ij}^{\ \ \alpha}\ \underset{1}{\xi^{\prime i}}\ \underset{p+1}{\xi^{\prime j}},$$

which, in virtue of $[A\,;\ p]$ and $[B\,;\ p]$, reduces to

$$\underset{p+1}{\kappa}\ \underset{p+2}{\xi^{\alpha}} = \underset{p+1}{\kappa'}\ \underset{p+2}{\xi^{\prime i}}\theta_i^{\ \alpha} + \Omega_{ij}^{\ \ \alpha}\ \underset{1}{\xi^{\prime i}}\ \underset{p+1}{\xi^{\prime j}}. \tag{7}$$

Multiplying this by $\underset{s}{\chi_{\alpha}}$ $(s = 1, \ldots, q)$ and using $[D\,;\ p+1]$, which define an asymptotic line of order $p+1$, we have

$$\overset{s}{\omega}_{ij}\ \underset{1}{\xi^{\prime i}}\ \underset{p+1}{\xi^{\prime j}} = 0 \quad (s = 1, \ldots, q).$$

Substituting in (7) we get

$$\underset{p+1}{\kappa}\ \underset{p+2}{\xi^{\alpha}} = \underset{p+1}{\kappa'}\ \underset{p+2}{\xi^{\prime i}}\theta_i^{\ \alpha},$$

which give

$$\underset{p+2}{\xi^{\alpha}} = \underset{p+2}{\xi^{\prime i}}\theta_i^{\ \alpha}, \qquad \underset{p+1}{\kappa} = \underset{p+1}{\kappa'}.$$

These, together with $[A\,;\ p]$ and $[B\,;\ p]$, constitute $[A\,;\ p+1]$ and $[B\,;\ p+1]$.

Now along any curve in V_m

$$\underset{1}{\xi^{\alpha}} = \underset{1}{\xi^{\prime i}}\theta_i^{\ \alpha},$$

whence

$$\underset{1}{\kappa}\ \underset{2}{\xi^{\alpha}} = \underset{1}{\kappa'}\ \underset{2}{\xi^{\prime i}}\theta_i^{\ \alpha} + \Omega_{ij}^{\ \ \alpha}\ \underset{1}{\xi^{\prime i}}\ \underset{1}{\xi^{\prime j}}, \tag{8}$$

and it follows that $[A\,;\ 1]$ and $[B\,;\ 1]$ are necessary for an asymptotic line of order 1. Thus the theorem is true for $p = 1$, and hence for $p = 2, 3, \ldots, m-1$.

(ii) *To show that* $[A\,;\ p]$ *is sufficient*:

This is obvious since $\underset{r+1}{\xi^{\prime i}}\theta_i^{\ \alpha}$ are all tangential to V_m, and thus $[A\,;\ p]$ implies that $\underset{r+1}{\xi^{\alpha}}$ $(r = 1, \ldots, p)$ are all tangential to V_m.

(iii) *To show that* $[B\,;\ p]$ *is sufficient*:

Suppose that $[B\,;\ p]$ is sufficient to define an asymptotic line of order p. Then $[B\,;\ p+1]$ defines an asymptotic line of order p, with the additional property

$$\underset{p+1}{\kappa} = \underset{p+1}{\kappa'}. \tag{9}$$

Hence $[A\,;\ p]$ and $[B\,;\ p]$ are satisfied, and therefore also (7). Thus

$$\underset{p+1}{\kappa^2} = \underset{p+1}{\kappa'^2} + \sum_{s=1}^{q} (\omega_{ij} \overset{s}{\underset{1}{\xi}}{}^{\prime i} \underset{p+1}{\xi}{}^{\prime j})^2,$$

which, in virtue of (9), gives

$$\overset{s}{\omega}_{ij} \overset{s}{\underset{1}{\xi}}{}^{\prime i} \underset{p+1}{\xi}{}^{\prime j} = 0 \quad (s = 1, \ldots, q).$$

Substituting in (7) we get

$$\underset{p+2}{\xi}{}^{a} = \underset{p+2}{\xi}{}^{\prime i}\, \theta_i^{\,a},$$

which, together with $[A\,;\ p]$, gives $[A\,;\ p+1]$.

Hence, from (ii), the curve is an asymptotic line of order $p+1$. But it is seen from (8) that $[B\,;\ 1]$ is sufficient to define an asymptotic line of order 1; so that the theorem is true for $p = 1$, and hence for $p = 2, 3, \ldots, m-1$.

6. The same inductive argument shows that the conditions $[C\,;\ p]$, below, are also necessary and sufficient for an asymptotic line of order p.

$[C\,;\ p]$

$$\boxed{\Omega_{ij}^{\ a} \overset{1}{\xi}{}^{\prime i} \overset{r}{\xi}{}^{\prime j} = 0} \qquad (r = 1, \ldots, p\,;\ a = 1, \ldots, n). \tag{10}$$

This can be written in the equivalent form

$[C'\,;\ p]$

$$\overset{s}{\omega}_{ij} \overset{1}{\xi}{}^{\prime i} \overset{r}{\xi}{}^{\prime j} = 0 \quad (r = 1, \ldots, p\,;\ s = 1, \ldots, q).$$

In the case of a V_{n-1} in a V_n there is a unique normal field to V_{n-1} and a single tensor ω_{ij} (the "second fundamental tensor"). In that case the equations take the simple form

$$\omega_{ij} \overset{1}{\xi}{}^{\prime i} \overset{r}{\xi}{}^{\prime j} = 0 \quad (r = 1, \ldots, p).$$

7. We now consider any asymptotic line of highest order (viz. $m-1$). Since, in this case, the vectors $\overset{r}{\xi}{}^{a}$ $(r = 1, \ldots, m)$ are all tangential to V_m,

it follows that the remaining normals $\underset{r}{\xi}^{\alpha}$ $(r = m+1, \ldots, n)$ are all orthogonal to V_m.

The first $m-1$ curvatures in V_n are given by

$$\underset{r}{\kappa} = \underset{r}{\kappa}' \quad (r = 1, \ldots, m-1).$$

We can also find $\underset{m}{\kappa}$ at any point. For, differentiating the relation

$$\underset{m}{\xi}^{\alpha} = \underset{m}{\xi}^{\prime i}\theta_i^{\ \alpha}$$

along the curve, we find

$$\underset{m}{\kappa}\ \underset{m+1}{\xi}^{\alpha} = \Omega_{ij}^{\ \ \alpha}\underset{1}{\xi}^{\prime i}\underset{m}{\xi}^{\prime j}; \tag{11}$$

whence

$$\underset{m}{\kappa} = \underset{m+1}{\xi}_{\alpha}\ \Omega_{ij}^{\ \ \alpha}\underset{1}{\xi}^{\prime i}\underset{m}{\xi}^{\prime j}. \tag{12}$$

We can eliminate $\underset{m}{\xi}^{\prime i}$ and $\underset{m+1}{\xi}_{\alpha}$ from this expression, and get a form which involves only a knowledge of the point and the direction of the curve at the point.

Since, at any point, $\underset{r}{\xi}^{\alpha}$ $(r = 1, \ldots, n)$ are n mutually orthogonal unit vectors in V_n,

$$\sum_{r=1}^{n} \underset{r}{\xi}_{\alpha}\underset{r}{\xi}_{\beta} = g_{\alpha\beta}; \tag{13}$$

and, since $\underset{s}{\xi}^{\prime i}$ $(s = 1, \ldots, m)$ are m mutually orthogonal unit vectors in V_m,

$$\sum_{s=1}^{m} \underset{s}{\xi}^{\prime i}\underset{s}{\xi}^{\prime k} = g^{\prime ik}. \tag{14}$$

Now, from (11),

$$\underset{r}{\xi}_{\alpha}\Omega_{ij}^{\ \ \alpha}\underset{1}{\xi}^{\prime i}\underset{m}{\xi}^{\prime j} = 0 \quad (r = 1, \ldots, m, m+2, \ldots, n).$$

Hence

$$\begin{aligned}
\underset{m}{\kappa}^2 &= \underset{m+1}{\xi}_{\alpha}\ \underset{m+1}{\xi}_{\beta}\ \Omega_{ij}^{\ \ \alpha}\Omega_{kl}^{\ \ \beta}\underset{1}{\xi}^{\prime i}\underset{1}{\xi}^{\prime k}\underset{m}{\xi}^{\prime j}\underset{m}{\xi}^{\prime l} \\
&= \sum_{r=1}^{n} \underset{r}{\xi}_{\alpha}\underset{r}{\xi}_{\beta}\Omega_{ij}^{\ \ \alpha}\Omega_{kl}^{\ \ \beta}\underset{1}{\xi}^{\prime i}\underset{1}{\xi}^{\prime k}\underset{m}{\xi}^{\prime j}\underset{m}{\xi}^{\prime l} \\
&= g_{\alpha\beta}\Omega_{ij}^{\ \ \alpha}\Omega_{kl}^{\ \ \beta}\underset{1}{\xi}^{\prime i}\underset{1}{\xi}^{\prime k}\underset{m}{\xi}^{\prime j}\underset{m}{\xi}^{\prime l} \\
&= \Omega_{ij}^{\ \ \alpha}\Omega_{kl\alpha}\underset{1}{\xi}^{\prime i}\underset{1}{\xi}^{\prime k}\left(g^{\prime jl} - \sum_{s=1}^{m-1}\underset{s}{\xi}^{\prime j}\underset{s}{\xi}^{\prime l}\right),
\end{aligned}$$

which*, in virtue of $[C;\ m-1]$, reduces to

$$\underset{m}{\kappa^2} = g'_{ij}\Omega_{\underset{1}{i}}{}^{\alpha}\Omega_{\underset{1}{j}\alpha}\xi'^{i}\xi'^{k},$$

or

$$\boxed{\underset{m}{\kappa^2} = \Omega_{\underset{1}{i}}{}^{\alpha}\Omega_{\underset{1}{j}\alpha}\xi'^{i}\xi'^{k}.} \tag{15}$$

In the case of a V_{n-1} in a V_n this takes the form

$$\underset{n-1}{\kappa^2} = \omega_{ij}\omega_{k}{}^{j}\xi'^{i}\xi'^{k}.$$

If $R_{\alpha\beta\gamma\delta}$, R'_{ijkl} are the curvature tensors (Riemann-Christoffel tensors) of V_n and V_m respectively, the Gauss equation is†

$$R'_{ijkl} = \theta_{ij}{}^{\alpha\beta}\theta_{kl}{}^{\gamma\delta}R_{\alpha\beta\gamma\delta} + (\Omega_{ik}{}^{\alpha}\Omega_{jl\alpha} - \Omega_{jk}{}^{\alpha}\Omega_{il\alpha}), \tag{16}$$

where we write $\theta_{i_1 i_2 \cdots}{}^{\alpha_1\alpha_2\cdots}$ to denote $\theta_{i_1}{}^{\alpha_1}\theta_{i_2}{}^{\alpha_2}\ldots$. The term in brackets is the "relative curvature tensor of V_m in V_n"; let us denote it by L_{ijkl}. The relative curvature of V_m for the orientation determined by two orthogonal unit vectors $\underset{1}{\lambda^i}$, $\underset{2}{\lambda^i}$ (in V_m) is

$$L_{ijkl}\underset{1}{\lambda^i}\underset{2}{\lambda^j}\underset{1}{\lambda^k}\underset{2}{\lambda^l};$$

it is the difference of the Riemannian curvature of V_m for that orientation and the V_m-component of the Riemannian curvature of V_n for the orientation determined by the same two vectors, regarded as in V_n.

By analogy with the "mean (absolute) curvature of V_m in a given direction"‡, we shall define the "mean relative curvature of V_m in a direction at a point" as $1/(m-1)$ times the sum of the relative curvatures for the $m-1$ orientations determined by the given direction and $m-1$ mutually orthogonal directions in V_m at the point, each orthogonal to the given direction. If $\underset{1}{\lambda^i}$ is the unit vector in the given direction, and $\underset{r}{\lambda^i}$ $(r = 2, \ldots, m)$ are unit vectors in the other $m-1$ directions, the mean relative curvature in the direction $\underset{1}{\lambda^i}$ is

$$\frac{1}{m-1}\sum_{r=2}^{m} L_{ijkl}\underset{1}{\lambda^i}\underset{r}{\lambda^j}\underset{1}{\lambda^k}\underset{r}{\lambda^l}.$$

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* We use the tensor $g_{\alpha\beta}$ to raise and lower Greek indices, the tensor g'_{ij} to raise and lower Latin indices.

† Schouten, *loc. cit.*, 198.

‡ Struik, *loc. cit.*, 66-67.

Since L_{ijkl} is skew-symmetric in i, j, this is equal to

$$-\frac{1}{m-1}\sum_{r=1}^{m} L_{ijkl}\lambda^j_1\lambda^k_1\lambda^i_r\lambda^l_r = -\frac{1}{m-1}g'^{il}L_{ijkl}\lambda^j_1\lambda^k_1.$$

Thus the mean relative curvature for an asymptotic direction $\underset{1}{\xi}{}'^i$ becomes, on account of $[C;\ 1]$,

$$-\frac{1}{m-1}g'^{il}\Omega_{ik}{}^{\alpha}\Omega_{jl\alpha}\underset{1}{\xi}{}'^j\underset{1}{\xi}{}'^k = -\frac{1}{m-1}\Omega_{ik}{}^{\alpha}\Omega_{j\cdot\alpha}^{\ i}\underset{1}{\xi}{}'^j\underset{1}{\xi}{}'^k. \tag{17}$$

Since $\Omega_{ij}{}^{\alpha}$ is symmetric in i, j, it follows that (15) can be written in the form

$$\boxed{\underset{m}{\kappa}{}^2 = -(m-1)H} \tag{18}$$

where H is the mean relative curvature of V_m, in V_n, in the direction of the curve.

8. The following are some special cases.

(i) *V_m in S_n:*

If K is the (constant) Riemannian curvature of S_n,

$$R_{\alpha\beta\gamma\delta} = K(g_{\alpha\gamma}g_{\beta\delta} - g_{\beta\gamma}g_{\alpha\delta}),$$

whence the mean induced curvature in the direction $\underset{1}{\xi}{}'^i$ is equal to

$$\begin{aligned}-\frac{1}{m-1}g'^{il}\theta^{\cdots\alpha\beta\gamma\delta}_{ijkl}R_{\alpha\beta\gamma\delta}\underset{1}{\xi}{}'^j\underset{1}{\xi}{}'^k &= -\frac{1}{m-1}g'^{il}K(g'_{ik}g'_{jl}-g'_{jk}g'_{il})\underset{1}{\xi}{}'^j\underset{1}{\xi}{}'^k\\ &= Kg'_{jk}\underset{1}{\xi}{}'^j\underset{1}{\xi}{}'^k\\ &= K\end{aligned}$$

for every direction. Thus

$$\underset{m}{\kappa}{}^2 = (m-1)(K-J'), \tag{19}$$

where J' is the mean absolute (*i.e.* Riemannian) curvature of V_m in the direction of the curve.

(ii) *S_m in S_n:*

In this case $J' = K'$ for every direction, where K' is the (constant) Riemannian curvature of S_m, and

$$\underset{m}{\kappa}{}^2 = (m-1)(K-K'). \tag{20}$$

(iii) *V_m in E_n*:

Here $K = 0$, and (19) becomes

$$\underset{m}{\kappa}^2 = -(m-1)J'. \tag{21}$$

(iv) *V_2 in S_n*:

In the case of a V_2, J' is the same for all directions at a given point (*i.e.* every V_2 is an Einstein space). It is the ordinary Gaussian curvature of V_2 at the point. Denoting this by K' (which is *not* constant, of course, except for an S_2),

$$\underset{2}{\kappa}^2 = K - K'. \tag{22}$$

(v) *V_2 in E_n*:

Here $K = 0$, and

$$\underset{2}{\kappa}^2 = -K'. \tag{23}$$

For $n = 3$ this is the well known formula of Beltrami and Enneper.

THE REARRANGEMENT OF POSITIVE FOURIER COEFFICIENTS

By R. M. GABRIEL.

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1. It is convenient to begin this paper with some remarks upon our notation. We define a symmetrical set of numbers c_t to be such that each value of a c, save the largest, occurs an even number of times in the set, while the largest value occurs an odd number of times. A set is said to be strictly symmetrical if $c_0 \geqslant c_t$ for all t, and if $c_t = c_{-t}$ for all t. Next, unless there is a specific indication to the contrary, any summation is subject to the sum of the suffices being zero, thus

$$\Sigma a_r b_s c_t = \sum_{r+s+t=0} a_r b_s c_t.$$

With this notation we can give a result, first proved by Hardy and Littlewood†, in the following form.

THEOREM. *Given any finite number of strictly symmetrical sets of non-negative numbers a_r, b_s, c_t, etc., then*

$$\Sigma a_r b_s c_t \ldots \leqslant \Sigma a_r^* b_s^* c_t^* \ldots,$$

where the a_r^*'s are the a's rearranged so that

$$a_0^* \geqslant a_1^* = a_{-1}^* \geqslant a_2^* = a_{-2}^* \geqslant \ldots$$

and the b_s^*'s, c_t^*'s, etc., are similarly defined.

From this theorem the original authors were able to deduce the following

THEOREM. *If $\gamma_n = \gamma_{-n}$ and $|\gamma_n| = c_n$, and*

$$f(\theta) = \sum_{|n| \leqslant N} \gamma_n e^{n\theta i},$$

† See Hardy and Littlewood, *Journal London Math. Soc.*, 3 (1928), 105, and also R. M. Gabriel, *Journal London Math. Soc.*, 3 (1928), 134.

while

$$f^*(\theta) = \sum_{|n| \leqslant N} c_n^* e^{n\theta i},$$

then, if k is any positive integer,

$$\int_0^{2\pi} |f(\theta)|^{2k} d\theta \leqslant \int_0^{2\pi} |f^*(\theta)|^{2k} d\theta.$$

It will be seen that in this result there is a limitation imposed on the γ's such that the discussion is limited to $f^*(\theta)$ defined by a *cosine* series; the object of this paper is to remove this limitation and to discuss certain converse forms of the theorems. We cease to assume the strictly symmetrical property $a_n = a_{-n}$, nor do we require that a_0, b_0, etc., should be the largest of their respective sets. We introduce two new types of rearranged order. Given any finite set of numbers a_n, we define the ${}^+a_n$'s and a_n^+'s to be the a's rearranged so that

$${}^+a_0 \geqslant {}^+a_1 \geqslant {}^+a_- \geqslant {}^+a_2 \geqslant {}^+a_{-2} \geqslant \ldots,$$

$$a_0^+ \geqslant a_{-1}^+ \geqslant a_1^+ \geqslant a_{-2}^+ \geqslant a_2^+ \geqslant \ldots.$$

It is clear that ${}^+a_r = a_{-r}^+$ in any case and that, if ${}^+a_r = a_r^+$ for all r, then $a_r^+ = a_{-r}^+$ for all r and we write

$$a_r^+ = a_{-r}^+ = {}^+a_r = {}^+a_{-r} = a_r^* = a_{-r}^*.$$

A theorem of the type

$$\Sigma a_r b_s c_t \leqslant \Sigma {}^+a_r {}^+b_s {}^+c_t = \Sigma a_r^+ b_s^+ c_t^+$$

is impossible, for, taking $c_0 = 1$ and $c_t = 0$ if $t \neq 0$, we should have

$$\Sigma a_r b_s \leqslant \Sigma {}^+a_r {}^+b_s,$$

which is false, since

$$\Sigma {}^+a_r {}^+b_s = \sum_{r=s} {}^+a_r {}^+b_{-s} = \sum_{r=s} {}^+a_r b_s^+$$

$$\leqslant \sum_{r=s} {}^+a_r {}^+b_s = \Sigma {}^+a_r b_s^+ \dagger,$$

and the relation *less than* definitely occurs. We may note in passing that, although such an inequality is impossible in the form in which we have stated it, yet, if we insert a constant factor in the right-hand side, we obtain an inequality which holds. Thus, given $2k$ sets of positive numbers

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† Since $\Sigma a_r b_r \leqslant \Sigma \bar{a}_r \bar{b}_r$, where the $\bar{a}$'s and $\bar{b}$'s are the a's and b's rearranged so that, if $\bar{a}_r - \bar{a}_s \geqslant 0$, then $\bar{b}_r - \bar{b}_s \geqslant 0$.

$a, b, c, \ldots,$

$$\Sigma a_r b_s c_t \ldots \leqslant A(k)\, \Sigma^+ a_r {}^+ b_s {}^+ c_t \ldots \dagger.$$

Also a theorem of the type

$$\Sigma a_r b_s c_t \leqslant \Sigma^+ a_r b_s^{+\,+} c_t$$

is impossible, since an interchange of a's and b's does not leave the right-hand side unaltered, and we notice that the right-hand side of any true inequality of the type we are considering must be left unaltered by any interchange of the a's, b's, and c's that is permitted. This indicates a possible type of inequality, and we can in fact prove the following

THEOREM 1. *Given three sets each of a finite number of non-negative numbers a_r, b_s, c_t of which the c's are symmetrical, then*

$$\Sigma a_r b_s c_t \leqslant \Sigma^+ a_r b_s^+ c_t^* = \Sigma a_r^{+\,+} b_s c_t^*.$$

2. Before we proceed to prove this theorem we point out that it includes as a special case a theorem due to Hardy, Littlewood, and Pólya‡. This special case is obtained from our theorem by taking $c_t = c_t^*$ and by adapting our notation to that of the three authors.

The method of proof consists of a reduction of the theorem to the special case in which each a, b, and c is either 0 or 1, followed by a proof of this special case§.

Write $$\phi = \phi(a, b, c) = \frac{\Sigma a_r b_s c_t}{\Sigma^+ a_r b_s^+ c_t^*} = \frac{\Sigma}{\Sigma^*};$$

then we have to show that $\phi(a, b, c) \leqslant 1$. If $a_r = {}^+a_{r'}$, then

$$\phi = \frac{\sum_{r=-N}^{M} a_r F_r}{\sum_{r=-n}^{m} {}^+a_r G_r} = \frac{\sum_{r=-N}^{M} a_r F_r}{\sum_{r=-N}^{M} a_r G_{r'}},$$

where the F's and G's are non-negative throughout. This last expression for ϕ is therefore a monotonic function of a_r in any interval contained in $a_r \geqslant 0$. Suppose that a number of the a_r's are equal to ξ_0, where ξ_0 is not the largest value of an a_r. Let ξ_1, ξ_2 be the nearest values of a_r's below

† See Hardy and Littlewood, *Journal London Math. Soc.*, 6 (1931), 3–9.

‡ See Hardy, Littlewood, and Pólya, *Proc. London Math. Soc.* (2), 25 (1926), 265–282.

§ *Cf. loc. cit.*, § 1, first foot-note.

and above ξ_0 respectively. If ξ_0 is the smallest value of an a_r, take $\xi_1 = 0$. Replace every a_r equal to ξ_0 by ξ, and consider $\phi = \phi(\xi)$ in $\xi_1 \leqslant \xi \leqslant \xi_2$. Since $\phi(\xi)$ is monotonic in this range we may replace ξ_0 by one of ξ_1, ξ_2 without diminishing ϕ. Further, this does not alter the order of magnitude of the a_r's, so that the values of r, r' which correspond in the formula $a_r = {}^+a_{r'}$ are unaffected. The number of distinct values of the a_r's has been reduced by one without diminishing ϕ, and the process can be repeated until only one distinct non-zero value remains. This is the equivalent of taking each of the a's equal either to 0 or 1. This process can be applied similarly to the b's and c's, so that finally each a, b, and c is either 0 or 1. This completes the reduction, and it is now sufficient to deal with this special case.

3. We now assume that

$$\left.\begin{aligned} {}^+a_r &= 1 \text{ if } -R \leqslant r \leqslant R' \\ &= 0 \text{ otherwise} \end{aligned}\right\} \quad \text{where} \quad R' = R \text{ or } R+1,$$

$$\left.\begin{aligned} b_s^+ &= 1 \text{ if } -S' \leqslant s \leqslant S \\ &= 0 \text{ otherwise} \end{aligned}\right\} \quad \text{where} \quad S' = S \text{ or } S+1,$$

$$\left.\begin{aligned} c_t^* &= 1 \text{ if } -T \leqslant t \leqslant T \\ &= 0 \text{ otherwise} \end{aligned}\right\}.$$

There are therefore exactly $R+R'+1$ non-zero a's, $S+S'+1$ non-zero b's, and $2T+1$ non-zero c's, all equal to unity. We have to show that the special arrangement of these given above, and corresponding to Σ^*, gives the maximum number of values of r, s, and t satisfying $r+s+t=0$. It is convenient to think of the problem in terms of the geometry of the plane $x+y+z=0$ in three dimensions, and the theorem will be proved if we can show that the lines† $x=r$ $(-R \leqslant r \leqslant R')$, $y=s$ $(-S' \leqslant s \leqslant S)$, $z=t$ $(-T \leqslant t \leqslant T)$ give the maximum number of triple intersections obtainable from three separate families of $R+R'+1$, $S+S'+1$, $2T+1$ parallel lines.

Since

$$c_t^* = c_{-t}^*, \quad {}^+a_r = a_{-r}^+, \quad \text{and} \quad b_s^+ = {}^+b_{-s},$$

$$\Sigma\, {}^+a_r b_s^+ c_t^* = \Sigma\, a_{-r}^+ {}^+b_{-s} c_{-t}^* = \Sigma\, a_r^+ {}^+b_s c_t^*,$$

† By the line $x=r$ we mean the intersection of the plane $x+y+z=0$ with the plane $x=r$.

and the a's and b's appear on equal terms in the theorem. There is therefore no loss in generality if we assume that $R \leqslant S$. Further, if

$$T \geqslant \text{Max}\,|r+s| \quad \text{where} \quad -R \leqslant r \leqslant R', \quad -S' \leqslant s \leqslant S$$
$$= \text{Max}\,(R+S',\ R'+S),$$

the result is trivial, for then each one of the $(R+R'+1)(S+S'+1)$ intersections of the lines $x=r$ $(-R \leqslant r \leqslant R')$, $y=s$ $(-S' \leqslant s \leqslant S)$ of the arrangement corresponding to Σ^* is a point of triple intersection, and the number of triple intersections cannot in any case be greater than the number of intersections of the lines $x=\text{constant}$, $y=\text{constant}$. Similarly, if $S' \geqslant R'+T$ and $S \geqslant R+T$ both hold, the theorem is trivial. We may suppose therefore that

$$T \geqslant \text{Max}\,(R+S',\ R'+S) \tag{1}$$

is false, and that one of the inequalities

$$S' \geqslant R'+T, \quad S \geqslant R+T \tag{2}$$

is false.

Now in any arrangement of the lines, the points of triple intersection are bounded by a convex polygon whose sides are portions of the lines $x=\rho_1,\ \rho_2,\ y=\sigma_1,\ \sigma_2,\ z=\tau_1,\ \tau_2$, where $\rho_1,\ \rho_2$ are the maximum and minimum values of r, $\sigma_1,\ \sigma_2$ are the maximum and minimum values of s, and $\tau_1,\ \tau_2$ are the maximum and minimum values of t concerned in the arrangement. We shall refer to this polygon as the boundary polygon; the number of its sides is at most six. Since this convex polygon is cut by any line in two points at most, the number of points of triple intersection on the boundary is at most

$$\tfrac{1}{2}\{2(R+R'-1)+2(S+S'-1)+2(2T-1)+6\}.$$

This full number is obtained in the arrangement corresponding to Σ^* provided that the boundary polygon is a hexagon in the strict sense. This is the case if either

$$T < \text{Min}\,(R+S',\ R'+S) \tag{3}$$

or if both the inequalities

$$S < R+T, \quad S' < R'+T \tag{4}$$

are true. Suppose either inequality (3) to be true or inequalities (4) to be true, and consider the effect of deleting a_{ρ_1}, a_{ρ_2}, b_{σ_1}, b_{σ_2}, c_{τ_1}, c_{τ_2} in Σ, and of the corresponding deletion of ${}^{+}a_{-R'}$, ${}^{+}a_{R'}$, $b^{+}_{-S'}$, b^{+}_{S}, $c^{*}_{\pm T}$ in

Σ^*. In each case the effect is to delete all the points of triple intersection on the boundary polygon, and, by what we have stated above, the number of points deleted in the figure corresponding to Σ^* is not less than the number in that corresponding to Σ. This reduces the problem to the case in which R, R', S, S', and T are each diminished by one. Now

$$(R-1)+(S'-1)-(T-1) = (R+S'-T)-1,$$

$$(R'-1)+(S-1)-(T-1) = (R'+S-T)-1,$$

$$(R-1)+(T-1)-(S-1) = (R+T-S)-1,$$

$$(R'-1)+(T-1)-(S'-1) = (R'+T-S')-1.$$

Hence, after the deletion of sufficient boundary polygons, we arrive at a case in which one of the following inequalities is true :

$$T = \text{Min}\,(R+S',\ R'+S), \tag{5}$$

$$S = R+T, \tag{6}$$

$$S' = R'+T. \tag{7}$$

If (5) is true, and, in addition $R+S' = R'+S$, then

$$T = \text{Min}\,(R+S',\ R'+S) = \text{Max}\,(R+S',\ R'+S),$$

and this we have already shown to be a trivial case. Consequently, in discussing (5), it is sufficient to deal with cases in which $R+S' \neq R'+S$. Further, if (6) and (7) both hold, we again have a case which has already been shown to be trivial. It remains therefore to discuss the following cases :

$$R' = R, \quad S' = S+1, \quad T = R+S \tag{5.1}$$

$$R' = R+1, \quad S' = S, \quad T = R+S \tag{5.2}$$

$$R' = R, \quad S' = S+1, \quad S = R+T \tag{6.1}$$

$$R' = R+1, \quad S' = S, \quad S = R+T \tag{6.2}$$

$$R' = R, \quad S' = S+1, \quad S' = R'+T \tag{7.1}$$

$$R' = R+1, \quad S' = S, \quad S' = R'+T \tag{7.2}$$

We are able further to reduce these cases in number. In the first place (5.1) and (5.2) are equivalent; for in the discussion leading to

(5 . 1) and (5 . 2), R and S have been treated on the same footing, and we have not assumed in this discussion that $R \leqslant S$. Also (6 . 2) is equivalent to (5 . 2), for in (6 . 2), since $S' = S$, we have symmetrical b's, and hence we may interchange the b's and c's, *i.e.* S and T. Again (7 . 2) is trivial; for in this case we have $S' = R'+T$ and $S = R+1+T > R+T$, *i.e.* inequalities (2) are satisfied. Similarly in (6 . 1) we have $S = R+T$, $S' = R+1+T$, and (6 . 1) is trivial.

It suffices, therefore, to discuss (5 . 1) and (7 . 1), and of these we consider first (5 . 1) If $-R \leqslant r \leqslant R$, and $-(S+1) \leqslant s \leqslant S$, then $|r+s| \leqslant R+S = T$, except when $s = -(S+1)$ and $r = -R$. Thus in the arrangement corresponding to Σ^* all the intersections of the lines $x = r$ with the lines $y = s$, where $-R \leqslant r \leqslant R$ and $-(S+1) \leqslant s \leqslant S$, are points of triple intersection, except the one point given by $y = -(S+1)$, $x = -R$. Suppose, if possible, that another arrangement of the lines will give more points of triple intersection. Then all the intersections of the $2R+1$ lines $x =$ constant, with the $2S+2$ lines $y =$ constant, must be points of triple intersection, and since there are then $2(2R-1)+4S+4 = 4(R+S)+2 = 4T+2$ points of triple intersection on the boundary parallelogram, there must be $2T+2$ distinct values of t. We have therefore a contradiction, and in case (5 . 1) we must have $\Sigma \leqslant \Sigma^*$. Similarly we can deal with (7 . 1), and the theorem is therefore established.

4. We first extend our result by proving

THEOREM 2. *Given any two finite sets of non-negative numbers a_r, b_s respectively, and any finite number k of finite symmetrical sets of non-negative numbers C_{n_1}, D_{n_2}, ..., K_{n_k}, then*

$$\Sigma = \Sigma a_r b_s C_{n_1} D_{n_2} \dots K_{n_k} \leqslant \Sigma^+ a_r b_s^+ C^*_{n_1} D^*_{n_2} \dots K^*_{n_k}.$$

We shall show how to deduce the result for k from the result for $k-1$, and remarking that the theorem is true for $k = 1$, by Theorem 1, we shall conclude our result by induction. Suppose the result known then for $k-1$. Write

$$P_n = \sum_{r+s=n} a_r b_s,$$

then $$\Sigma = \Sigma P_n C_{n_1} D_{n_2} \dots K_{n_k} \leqslant \Sigma P_n^+ C^*_{n_1} D^*_{n_2} \dots K^*_{n_k},$$

by the theorem for $k-1$ with the P's and C's taking the place of the

a's and b's. Thus if we write

$$Q_m = \sum_{n_1+n_2+\ldots+n_k=-m} C^*_{n_1} D^*_{n_2} \ldots K^*_{n_k} = Q_{-m},$$

$$\Sigma \leqslant \sum_{n=m} P^+_n Q_m = \sum_{n=m} P_n Q_{\phi(m)},$$

where $\phi(m)$ is such that $P_n = P^+_{\phi(n)}$, and is the permutation function appropriate to this relation, *i.e.* ϕ is such that the correspondence between the P's and P^+'s is one-one. Thus

$$\Sigma \leqslant \sum_{r+s=-m} a_r b_s Q_{\phi(m)}.$$

We now show that the Q's form a symmetrical set and are in "star" order. For any set of distinct x's

$$\sum_{n=m} x_n Q_m = \Sigma x_n C^*_{n_1} D^*_{n_2} \ldots K^*_{n_k}$$

$$\leqslant \Sigma^+ x_n C^*_{n_1} D^*_{n_2} \ldots K^*_{n_k}$$

$$= \sum_{n=m} {}^+x_n Q_m,$$

by the theorem for $k-1$, with x's and C's taking the place of a's and b's. Since this is true for all sets of x's, the Q_m's must be arranged in the same order of magnitude as the ${}^+x_m$'s, *i.e.* if ${}^+x_r \geqslant {}^+x_s$ then $Q_r \geqslant Q_s$†. Similarly the Q_m's are in the same order as the x^+_m's, and are therefore in "star" order. $Q_m = Q^*_m$, and in particular Q_0 is the greatest Q. Thus the Q's form a symmetrical set, and

$$\Sigma \leqslant \sum_{r+s=-m} a_r b_s Q_{\phi(m)} \leqslant \Sigma^+ a_r b^+_s Q^*_m$$

$$= \Sigma^+ a_r b^+_s Q_m = \Sigma^+ a_r b^+_s C^*_{n_1} D^*_{n_2} \ldots K^*_{n_k},$$

and the theorem for k is established. The general result for all k follows from this together with the result for $k = 1$, by induction.

5. From the result given above we can now deduce our main result,

THEOREM 3. *Given any finite number k of finite sets a_r, b_s, c_t, etc.,*

$$\sum_{r_1+r_2+s_1+s_2+\ldots=0} a_{r_1} a_{-r_2} b_{s_1} b_{-s_2} c_{t_1} c_{\ldots t_2} \ldots \leqslant \Sigma^+ a_{r_1} a^+_{r_2} {}^+b_{s_1} b^+_{s_2} {}^+c_{t_1} c^+_{t_2} \ldots.$$

† *Cf.* § 1, second foot-note.

We shall content ourselves with proving the particular case in which $k = 3$, from which the method of proof of the general case will be clear.

$$\Sigma = \sum_{r_1+r_2+s_1+s_2+\ldots=0} a_{r_1} a_{-r_2} b_{s_1} b_{-s_2} c_{t_1} c_{-t_2} = \sum_{r_1+r_2+m+n=0} a_{r_1} a_{-r_2} B_m C_n,$$

where $$B_m = \sum_{s_1+s_2=m} b_{s_1} b_{-s_2}, \quad C_n = \sum_{t_1+t_2=n} c_{t_1} c_{-t_2}.$$

Now

$$B_m = \sum_{s_1+s_2=m} b_{s_1} b_{-s_2} = \sum_{s_1-s_2=m} b_{s_1} b_{s_2} = \sum_{s_2-s_1=m} b_{s_2} b_{s_1} = B_{-m},$$

and $$B_m = \sum_{s_1+s_2=m} b_{s_1} b_{-s_2} \leqslant \Sigma b_s^2 = B_0.$$

Hence the B_m's, and similarly the C_n's, form a strictly symmetrical set. Thus, by Theorem 2,

$$\Sigma \leqslant \Sigma^+ a_{r_1} a_{r_2}^+ B_m^* C_n^*$$

$$= \sum_{m=M} B_m^* \mu_M,$$

where $$\mu_{-M} = \sum_{r_1+r_2+n=M} {}^+a_{r_1} a_{r_2}^+ C_n^*.$$

Hence $$\Sigma \leqslant \sum_{m=M} B_m \mu_{\phi(M)},$$

where ϕ is such that $B_m = B_{\phi(m)}^*$, and is a permutation function appropriate to this relation. In particular, we take $\phi(-m) = -\phi(m)$, since the B's form a strictly symmetrical set. Write

$$a_r = \sum_{r_1+r_2=r} {}^+a_{r_1} a_{r_2}^+.$$

Since $${}^+a_r = a_{-r}^+,$$

$$a_r = \sum_{r_1+r_2=r} {}^+a_{r_1} a_{r_2}^+ = \sum_{r_1+r_2=-r} a_{-r_1}^+ {}^+a_{-r_2}$$

$$= \sum_{r_1+r_2=-r} a_{r_1}^+ {}^+a_{r_2} = a_{-r},$$

and $$a_r = \sum_{r_1+r_2=r} {}^+a_{r_1} a_{r_2}^+ \leqslant \Sigma a_r^2 = \sum_{r_1+r_2=0} {}^+a_{r_1} a_{r_2}^+ = a_0.$$

Hence the a's form a strictly symmetrical set. Consider any strictly symmetrical set of q's.

$$q_0 a_0 + 2 \sum_{r=\rho>0} q_\rho a_r = \sum_{r=\rho} q_\rho a_r = \Sigma q_\rho {}^+a_{r_1} a_{r_2}^+$$

$$\leqslant \Sigma q_\rho^* {}^+a_{r_1} a_{r_2}^+ = \sum_{r=\rho} q^* a_r$$

$$= q_0^* a_0 + 2 \sum_{r=\rho>0} q_\rho^* a_r.$$

This is true for any symmetrical set of q's. Hence the a_r's $(r>0)$ must be in the same order of magnitude as the q_ρ^*'s $(\rho>0)$, and thus the a's are in "star" order, since $a_0 \geqslant a_r$ for all r. Repeating the argument by which we proved the Q's to be in "star" order in § 4, we can show that the μ_m's are in "star" order, and that $\mu_m = \mu_m^*$. Since $\phi(M) = -\phi(-M)$ and $\phi(0) = 0$, the $\mu_{\phi(M)}$'s regarded as of suffix M form a strictly symmetrical set. Hence

$$\begin{aligned}
\Sigma &\leqslant \sum_{s_1+s_2+M=0} b_{s_1} b_{-s_2} \mu_{\phi(M)} \\
&\leqslant \Sigma^+ b_{s_1} b_{s_2}^+ \mu_M^* \\
&= \Sigma^+ b_{s_1} b_{s_2}^+ a_r C_n^* \\
&= \Sigma a_r \beta_s C_n^*,
\end{aligned}$$

where

$$\beta_s = \sum_{s_1+s_2=s} {}^+ b_{s_1} b_{s_2}^+.$$

As in the case of the a's, the β's are in "star" order. Write

$$X_{-N} = \sum_{r+s=N} a_r \beta_s.$$

As in the case of the μ_m's the X_N's are in "star" order, and if $\psi(n)$ be the permutation function appropriate to the relation $C_n = C_{\psi(n)}^*$, we have

$$\begin{aligned}
\Sigma &\leqslant \sum_{n=N} X_N C_n^* = \sum_{n=N} X_{\psi(N)} C_n \\
&= \sum_{t_1+t_2+N=0} X_{\psi(N)} c_{t_1} c_{t_2} \leqslant \Sigma X_N^* c_{t_1}^+ {}^+ c_{t_2} \\
&= \Sigma^+ a_{r_1} a_{r_2}^+ {}^+ b_{s_1} b_{s_2}^+ {}^+ c_{t_1} c_{t_2}^+.
\end{aligned}$$

This proves the theorem in the case of three sets of numbers, and indicates the line along which the proof of the most general case will run.

We are able to deduce at once

THEOREM 4. *If* $|a_r| = a_r$, $|\beta_s| = b_s$, *etc., and*

$$A(\theta) = \sum_{r=-R}^{R} a_r e^{r\theta i}, \qquad A^*(\theta) = \sum_{r=-R}^{R} a_r^+ e^{r\theta i},$$

$$B(\theta) = \sum_{s=-S}^{S} \beta_s e^{s\theta i}, \qquad B^*(\theta) = \sum_{s=-S}^{S} b_s^+ e^{s\theta i}.$$

etc., then

$$\int_0^{2\pi} |A^2(\theta)||B^2(\theta)|\dots d\theta \leqslant \int_0^{2\pi} |A^{*2}(\theta)||B^{*2}(\theta)|\dots d\theta$$

for any finite number of functions $A(\theta)$, $B(\theta)$, *etc.*

For if $\bar{a}$ is the conjugate of a, etc.,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} |A^2(\theta)||B^2(\theta)|\dots d\theta &= \sum_{r_1+r_2+s_1+s_2+\dots=0} a_{r_1}\bar{a}_{-r_2}\beta_{s_1}\bar{\beta}_{-s_2}\dots \\ &\leqslant \sum_{r_1+r_2+s_1+s_2+\dots=0} a_{r_1}a_{-r_2}b_{s_1}b_{-s_2}\dots \\ &\leqslant \Sigma^{+} a_{r_1}a^{+}_{r_2}b_{s_1}b^{+}_{s_2}\dots \\ &= \sum_{r_1+r_2+s_1+s_2+\dots=0} a^{+}_{-r_1}a^{+}_{r_2}b^{+}_{-s_1}b^{+}_{s_2}\dots \\ &= \frac{1}{2\pi}\int_0^{2\pi} |A^{*2}(\theta)||B^{*2}(\theta)|\dots d\theta.\end{aligned}$$

As a particular case, we have

$$\int_0^{2\pi} |A(\theta)|^{2k} d\theta \leqslant \int_0^{2\pi} |A^{*}(\theta)|^{2k} d\theta.$$

7. We proceed to discuss the implications of $\Sigma = \Sigma^*$; in this section of the paper we shall restrict our attention to strictly symmetrical sets. We cannot assert that, for a finite strictly symmetrical set of non-negative a's,

$$\Sigma = \Sigma a_r a_s a_t = \Sigma a_r^* a_s^* a_t^* = \Sigma^*$$

implies that the a's are in "star" order. For if

$$\begin{aligned} a_r &= a_\rho^* \quad \text{if} \quad r = \lambda\rho, \\ &= 0 \quad \text{otherwise,}\end{aligned}$$

where λ is a positive integer, it is clear that $\Sigma = \Sigma^*$ and the a's are not in "star" order. The rearrangement of this type, which converts a's into a^*'s, we shall term a "trivial" rearrangement. We shall need three preliminary lemmas in order to obtain our first theorem with $\Sigma = \Sigma^*$ as hypothesis.

LEMMA 1. *Given a finite set of non-negative* a_r*'s such that the maximum value of a quotient* $E = (\Sigma a_r^* F_r)/(\Sigma a_r^* G_r)$ *is attained for the given set of a's, the F's and G's being fixed non-negative numbers independent of the a's, then we can replace each* a_r^* *by* a_r^* *without altering*

the value of E, where a_r^* is defined by

$$a_r^* = 1 \quad \text{if} \quad a_r^* \geqslant A,$$
$$= 0 \quad \text{if} \quad a_r^* < A,$$

where A is any strictly positive number.

Let ξ_0 be a value other than the largest of any number of the a_r^*'s and let ξ_1 and ξ_2 be the values of a_r^*'s immediately below and above ξ_0; if ξ_0 is the smallest value of a positive a_r^* take $\xi_1 = 0$. Replace every a_r^* equal to ξ_0 by ξ, and write the resulting quotient, replacing the quotient E, as $\phi(\xi)$. Now $\phi(\xi)$ is a linear fraction of the form $(p_1\xi+q_1)/(p_2\xi+q_2)$, where p_1, p_2, q_1, q_2 are non-negative. Hence $\partial\phi/\partial\xi$ is either positive throughout the interval $\langle\xi_1, \xi_2\rangle$, or negative throughout, or zero throughout. If we allow all the a^*'s equal to ξ_0 to take any common value in $\langle\xi_1, \xi_2\rangle$ the "star" order is unaltered; thus the maximum value of ϕ is attained for $\xi = \xi_0$, and we have $\partial\phi/\partial\xi = 0$ for $\xi = \xi_0$, and therefore, by what we have just said, for all ξ in $\xi_1 \leqslant \xi \leqslant \xi_2$. We may therefore replace ξ_0 by either ξ_1 or ξ_2 without altering the value of the quotient. If $\xi_0 \geqslant A$, replace ξ_0 by the larger value ξ_2, and, if $\xi_0 < A$, replace ξ_0 by the smaller value ξ_1. It will be seen that this does not disturb the "star" order of the a's, and that the number of distinct values of the a's has been reduced by one. Further, the process can be repeated until finally all the ξ's greater than or equal to A have been replaced by the largest ξ, and all the ξ's less than A by zero. This will have been done without altering the value of the quotient, and is equivalent to the required result.

LEMMA 2. *Given a finite set of non-negative a_r^*'s positive if and only if $|r| \leqslant R$, write*

$$P_n = \sum_{r+s=-n} a_r^* a_s^*.$$

Then $P_{|n|} > P_{|n|+1}$ if $P_{|n|} > 0$, i.e. if $|n| \leqslant 2R$.

Since $P_n = P_{-n} > 0$ if and only if $|n| \leqslant 2R$, we shall suppose that $0 \leqslant n \leqslant 2R$. In § 4 we have already shown that certain Q's are in "star" order; as a special case we have

$$E_n = \sum_{r+s=-(n+1)} x_r^* y_s^* \Big/ \sum_{r+s=-n} x_r^* y_s^* \leqslant 1$$

for any symmetrical sets of x's and y's. Hence $P_{n+1} \leqslant P_n$. If possible, suppose that $P_{n+1} = P_n$; then it is clear that the maximum value of E_n is attained for $x_r^* = a_r^*$, $y_s^* = a_s^*$. We can now apply Lemma 1, and we

take A to be equal to one-half of the smallest positive a^*. Applying the lemma first to the a^*'s regarded as x^*'s and then as y^*'s we replace every positive a_r^* by $a_r^* = 1$, without disturbing the equation $P_{n+1} = P_n$. Now this is impossible, for we shall have

$$a_r^* = 1 \quad \text{if} \quad |r| \leqslant R,$$

$$= 0 \quad \text{otherwise,}$$

and $$P_n(a) = \sum_{r+s=-n} a_r^* a_s^* = \sum_{r=-R}^{-n+R} 1 = -n+2R+1,$$

since $0 \leqslant n \leqslant 2R$. Thus $P_n(a) = P_{n+1}(a)$ is impossible, and $P_n(a) = P_{n+1}(a)$ is also impossible.

COROLLARY. *Given any set of a_r^*'s positive if and only if $|r| \leqslant R$, and if k is a positive integer, and*

$$P_n = \sum_{n_1+n_2+\ldots+n_{2k}=n} a_{n_1}^* a_{n_2}^* \ldots a_{n_{2k}}^*,$$

then $P_{|n|} > P_{|n|+1}$ if $|n| \leqslant 2kR$.

For write $$Q_r = \sum_{n_1+n_2+\ldots+n_k=r} a_{n_1}^* a_{n_2}^* \ldots a_{n_k},$$

then the Q's are in "star" order, as in the proof of Theorem 2, and $Q_r > 0$ if and only if $|r| \leqslant kR$. Now

$$P_n = \sum_{r+s=-n} Q_r^* Q_s^*$$

and our corollary follows from the lemma.

LEMMA 3. *Given any four sets of positive numbers p_r, q_r, P_r, Q_r, $|r| \leqslant M$, in "star" order and such that*

$$\sum_{|r|\leqslant N} p_r \leqslant \sum_{|r|\leqslant N} P_r, \quad \sum_{|r|\leqslant N} q_r \leqslant \sum_{|r|\leqslant N} Q_r \quad (0 \leqslant N \leqslant M),$$

then $$\sum_{|r|\leqslant M} p_r q_r \leqslant \sum_{|r|\leqslant M} P_r Q_r.$$

Further, if equality occurs in this last, then

$$\sum_{r\leqslant N} q_r = \sum_{r\leqslant N} Q_r$$

for all N, such that $P_{|N|} > P_{|N|+1}$.

If $r \geqslant 0$, write

$$p_r = a_{2r}, \quad q_r = b_{2r}, \quad P_r = A_{2r}, \quad Q_r = B_{2r},$$

and if $r<0$,

$$p_{-r}=a_{2r-1},\quad q_{-r}=b_{2r-1},\quad P_{-r}=A_{2r-1},\quad Q_{-r}=B_{2r-1}.$$

It is clear that the a's are in descending order, and that $a_{2r}=a_{2r-1}$ if $r>0$. Similarly with the b's, A's, and B's.

$$\sum_{r=0}^{2R} a_r=\sum_{|r|\leqslant R} p_r\leqslant \sum_{|r|\leqslant R} P_r=\sum_{r=0}^{2R} A_r \quad \text{if} \quad R\leqslant M,$$

$$\sum_{r=0}^{2R+1} a_r=\tfrac{1}{2}\left\{\sum_{r=0}^{2R} a_r+\sum_{r=0}^{2R+2} a_r\right\}\leqslant \tfrac{1}{2}\left\{\sum_{r=0}^{2R} A_r+\sum_{r=0}^{2R+2} A_r\right\}$$

$$=\sum_{r=0}^{2R+1} A_r \quad \text{if} \quad R<M,$$

since $$a_{2r+1}=a_{2r+2} \quad \text{and} \quad A_{2r+1}=A_{2r+2}.$$

Hence $$\sum_{r=0}^{N} a_r\leqslant \sum_{r=0}^{N} A_r \quad (0\leqslant N\leqslant 2M),$$

and similarly $$\sum_{r=0}^{N} b_r\leqslant \sum_{r=0}^{N} B_r \quad (0\leqslant N\leqslant 2M).$$

Write $$\xi_r=A_r-a_r,\quad \eta_r=B_r-b_r.$$

Then

$$X_N=\sum_{r=0}^{N}\xi_r\geqslant 0,\quad Y_N=\sum_{r=0}^{N}\eta_r\geqslant 0 \quad \text{if} \quad 0\leqslant N\leqslant 2M,$$

$$\sum_{|r|\leqslant M}(P_rQ_r-p_rq_r)=\sum_{r=0}^{2M}(A_rB_r-a_rb_r)$$

$$=\sum_{r=0}^{2M}[A_r(b_r+\eta_r)-b_r(A_r-\xi_r)]$$

$$=\sum_{r=0}^{2M}(A_r\eta_r+b_r\xi_r)$$

$$=A_0\eta_0+b_0\xi_0+\sum_{r=1}^{2M}[A_r(Y_r-Y_{r-1})+b_r(X_r-X_{r-1})]$$

$$=\sum_{r=0}^{2M-1}[Y_r(A_r-A_{r+1})+X_r(b_r-b_{r+1})]+A_{2M}Y_{2M}+b_{2M}X_{2M}$$

$$\geqslant 0,$$

since the A's and b's are in descending order.

Thus $$\sum_{|r|\leqslant M} p_rq_r=\sum_{|r|\leqslant M} P_rQ_r$$

is clearly possible only if

$$Y_r = 0 \quad \text{whenever} \quad A_r > A_{r+1},$$

and similarly if $X_r = 0$ whenever $B_r > B_{r+1}$,

in particular only if

$$\sum_{|r| \leqslant N} (Q_r - q_r) = \sum_{r=0}^{2N} (B_r - b_r) = 0 \quad \text{whenever} \quad P_{N+1} < P_N,$$

and similarly if

$$\sum_{|r| \leqslant N} (P_r - p_r) \sum_{r=0}^{2N} (A_r - a_r) = 0 \quad \text{whenever} \quad Q_{N+1} < Q_N.$$

This completes the lemma.

8. THEOREM 5. *Suppose that we have two finite strictly symmetrical sets of non-negative a's and c's respectively, such that*

$$a_r^* > 0 \quad \textit{if and only if} \quad |r| \leqslant R,$$
$$c_t^* > 0 \quad \textit{if and only if} \quad |t| \leqslant 2R,$$
$$c_{|t|}^* > c_{|t|+1}^* \quad \textit{if} \quad |t| \leqslant 2R,$$

then, if

$$\Sigma = \Sigma a_r a_s c_t = \Sigma a_r^* a_s^* c_t^* = \Sigma^*,$$

the a's and c's must be in "star" order save for a trivial modification.

By Theorem 1 with its conditions the maximum value of

$$\Sigma / \Sigma^* = \Sigma x_r y_s z_t / \Sigma x_r^* y_s^* z_t^*$$

is attained if we take $x_r = a_r$, $y_s = a_s$, $z_t = c_t$. Hence, by Lemma 1, we may replace all the positive a's considered as x's by unity, then all positive a's considered as y's by unity, and finally all the positive c's by unity, without altering the relation $\Sigma = \Sigma^*$. Thus, if we write

$$X_r = Y_r = 1 \quad \text{if} \quad a_r > 0,$$
$$= 0 \quad \text{otherwise},$$
$$Z_t = 1 \quad \text{if} \quad c_t > 0,$$
$$= 0 \quad \text{otherwise},$$

we have

$$X_r^* = Y_r^* = 1 \quad \text{if} \quad r| \leqslant R,$$
$$= 0 \quad \text{otherwise},$$
$$Z_t^* = 1 \quad \text{if} \quad |t| \leqslant 2R,$$
$$= 0 \quad \text{otherwise},$$

and

$$\Sigma(X, Y, Z) = \Sigma X_r Y_s Z_t = \Sigma X_r^* Y_s^* Z_t^* = \Sigma^*(X, Y, Z).$$

We now find it convenient to work again in terms of the plane $x+y+z=0$ in three dimensions. If we consider the figure appropriate to $\Sigma^*(X, Y, Z)$, namely that consisting of the lines $x=r$ $(|r| \leqslant R)$, $y=s$ $(|s| \leqslant R)$, $z=t$ $(|t| \leqslant 2R)$, we see that every one of the $(2R+1)^2$ intersections of the x and y lines is a point of triple intersection. Hence $\Sigma(X, Y, Z) = \Sigma^*(X, Y, Z)$ is possible only if the same is true of the figure appropriate to $\Sigma(X, Y, Z)$, which, in turn, is possible only if the x, y, and z lines in this figure are evenly spaced and are such that the distance λ between adjacent x lines is equal to each of the corresponding distances between y and z lines. Since the a's and c's are strictly symmetrical, so are the X's, Y's, and Z's, and we have

$$X_r = Y_r > 0 \quad \text{if and only if} \quad r = \lambda\rho \ \ (0 \leqslant |\rho| \leqslant R),$$

$$Z_t > 0 \quad \text{if and only if} \quad t = \lambda\tau \ \ (0 \leqslant |\tau| \leqslant 2R),$$

i.e.

$$a_r > 0 \quad \text{if and only if} \quad r = \lambda\rho \ \ (0 \leqslant |\rho| \leqslant R),$$

$$c_t > 0 \quad \text{if and only if} \quad t = \lambda\tau \ \ (0 \leqslant |\tau| \leqslant 2R),$$

where ρ and τ are integers.

Thus, if we perform the trivial rearrangement of writing $a_{r/\lambda}$ in place of a_r wherever $a_r > 0$, and $c_{t/\lambda}$ in place of c_t wherever $c_t > 0$, we may assume that

$$a_r > 0 \quad \text{if and only if} \quad |r| \leqslant R,$$

$$c_t > 0 \quad \text{if and only if} \quad |t| \leqslant 2R.$$

Throughout the remainder of the proof we shall suppose this trivial rearrangement to have been effected; the a^*'s and c^*'s are, of course, unaffected by any rearrangement of the a's and c's.

By Lemma 1 we may replace every positive a_r by α_r equal to unity without upsetting the equation $\Sigma = \Sigma^*$.

$$\alpha_r = \alpha_r^* = 1 \quad \text{if} \quad |r| \leqslant R,$$

$$= 0 \quad \text{otherwise,}$$

since $\alpha_r > 0$ if and only if $|r| \leqslant R$. If

$$\mu_n = \sum_{r+s=n} \alpha_r \alpha_s = \sum_{r+s=n} \alpha_r^* \alpha_s^*,$$

by Lemma 2, $\mu_{|n|} > \mu_{|n|+1}$ if $|n| \leqslant 2R$. Now

$$\sum_{n=-2R}^{2R} \mu_n c_n = \Sigma a_r a_s c_t = \Sigma = \Sigma^*$$
$$= \Sigma a_r^* a_s^* c_t^* = \sum_{n=-2R}^{2R} \mu_n c_n^*.$$

Since $\mu_{|n|} > \mu_{|n|+1}$ if $|n| \leqslant 2R$ and the c's form a symmetrical set, this is possible only if the c's are in "star" order, and $c_t = c_t^*$, $|t| \leqslant 2R$.

Write

$$p_n = \sum_{r+s=n} a_r a_s, \quad P_n = \sum_{r+s=n} a_r^* a_s^*.$$

Now

$$p_{-n} = p_n = \sum_{r+s=n} a_r a_s \leqslant \Sigma a_r^2 = p_0.$$

Hence the p's form a strictly symmetrical set, and the P's, as a special case of the Q's of the proof of Theorem 2, are in "star" order. Suppose that $q_r = q_{\rho(r)}^*$ and take $x_r = 1$ if $|\rho(r)| \leqslant m$, $x_r = 0$ otherwise; the x's form a symmetrical set.

$$\sum_{|\rho| \leqslant m} p_\rho^* = \Sigma x_r p_r = \Sigma x_r a_s a_t$$
$$\leqslant \Sigma x_r^* a_s^* a_t^* = \Sigma x_r^* P_r = \sum_{|r| \leqslant m} P_r.$$

This is true for all m. Hence, by Lemma 3,

$$\sum_{|\rho| \leqslant 2R} c_\rho^* p_\rho^* \leqslant \sum_{|\rho| \leqslant 2R} c_\rho^* P_\rho^*,$$

and equality is possible only if

$$\sum_{\rho| \leqslant m} p_\rho^* = \sum_{|\rho| \leqslant m} P_\rho^*$$

for every m such that $c_m^* > c_{m+1}^*$, *i.e.* for $0 \leqslant m \leqslant 2R$. Thus equality is possible only if

$$p_\rho^* = P_\rho^* \quad (|\rho| \leqslant 2R).$$

Now, in any case,

$$\Sigma = \sum_{|\rho| \leqslant 2R} c_\rho p_\rho \leqslant \sum_{|\rho| \leqslant 2R} c_\rho^* p_\rho^* \leqslant \sum_{|\rho| \leqslant 2R} c_\rho^* P_\rho = \Sigma^*,$$

and, since $\Sigma = \Sigma^*$, we must have

$$\sum_{|\rho| \leqslant 2R} c_\rho p_\rho = \sum_{|\rho| \leqslant 2R} c_\rho^* p_\rho^* = \sum_{|\rho| \leqslant 2R} c_\rho^* P_\rho.$$

Hence $p_\rho^* = P_\rho$, and, since $c_\rho = c_\rho^*$, and $c_{|\rho|+1}^* < c_{|\rho|}^*$ if $|\rho| \leqslant 2R$, we must also have $p_\varrho = p_\rho^*$, *i.e.* $p_\rho = P_\rho$ for $|\rho| \leqslant 2R$.

Consider first $n = 2R$.

$$p_n = a_R^2 = P_n = a_R^{*2}.$$

Hence $a_R = a_R^*$. Next, by considering $n = 2R-1$, we can obtain $a_{R-1} = a_{R-1}^*$. In fact, suppose that we have shown that

$$a_r = a_r^* \quad \text{for} \quad R \geqslant r \geqslant R-\rho > 0,$$

and consider $p_n = P_n$ with $n = R-\rho-1$. We have

$$\sum_{r=R-\rho-1}^{R} a_r a_{2R-\rho-1-r} = \sum_{r=R-\rho-1}^{R} a_r^* a_{2R-\rho-1-r}^*,$$

i.e.

$$a_{R-\rho-1}a_R + \sum_{r=R-\rho}^{R} a_r a_{2R-\rho-1-r} = a_{R-\rho-1}^* a_R^* + \sum_{r=R-\rho}^{R} a_r^* a_{2R-\rho-1-r}^*.$$

Hence, by our last supposition,

$$a_{R-\rho-1}a_R = a_{R-\rho-1}^* a_R^*,$$

whence

$$a_{R-\rho-1} = a_{R-\rho-1}^*.$$

Thus, by induction, we have $a_r = a_r^*$ for all $r \geqslant 0$, and, since $a_r = a_{-r}$ and $a_r^* = a_{-r}^*$, the same is true for all r. This proves the theorem.

THEOREM 6. *Given any finite strictly symmetrical set of non-negative a_r's such that*

$$\Sigma = \Sigma a_{n_1} a_{n_2} \dots a_{n_{2k}} = \Sigma a_{n_1}^* a_{n_2}^* \dots a_{n_{2k}}^*,$$

where k is any positive integer greater than unity, then the a_r's are in "star" order save for a trivial modification.

Write

$$q_r = \sum_{n_1+n_2+\dots+n_{2k-2}=r} a_{n_1} a_{n_2} \dots a_{n_{2k-2}},$$

$$Q_r = \sum_{n_1+n_2+\dots+n_{2k-2}=r} a_{n_1}^* a_{n_2}^* \dots a_{n_{2k-2}}^*.$$

Clearly $q_r = q_{-r}$, and if

$$X_n = \sum_{n_1+n_2+\dots+n_{k-1}=n} a_{n_1} a_{n_2} \dots a_{n_{k-1}},$$

then

$$q_r = \sum_{n_1+n_2=r} X_{n_1} X_{n_2} \leqslant \Sigma X_n^2 = q_0.$$

Hence the q's form a strictly symmetrical set. As we have shown in the proof of Theorem 2, the Q_r's are in "star" order. Further, if $a_r^* > 0$ if and only if $|r| \leqslant R$, then, by Lemma 2, corollary, $Q_{|r|} > Q_{|r+1|}$ if $|r| \leqslant R(2k-2)$.

Now, given any symmetrical set of x's, by Theorem 2 we have

$$\Sigma x_r q_r \leqslant \Sigma x_r^* Q_r.$$

Suppose $q_r = q^*_{\rho(r)}$ and take $x_r = 1$ if $|\rho(r)| \leqslant m$, $x_r = 0$ otherwise. Hence

$$\sum_{|\rho| \leqslant m} q_\rho^* \leqslant \sum_{|r| \leqslant m} Q_r \quad \text{(all } m\text{)}.$$

We now write

$$p_r = \sum_{n_1+n_2=r} a_{n_1} a_{n_2}, \quad P_r = \sum_{n_1+n_2=r} a_{n_1}^* a_{n_2}^*,$$

and, as a special case of the above, $P_{|r|} > P_{|r|+1}$ if $|r| \leqslant 2R$ and

$$\sum_{|\rho| \leqslant m} p_\rho^* \leqslant \sum_{|r| \leqslant m} P_r \quad \text{(all } m\text{)}.$$

Hence, by Lemma 3,

$$\Sigma p_\rho^* q_\rho^* \leqslant \Sigma P_\rho Q_\rho \dagger,$$

and equality is possible only if

$$\sum_{\rho| \leqslant m} q_\rho^* = \sum_{\rho \leqslant m} Q_\rho$$

for all m such that $P_m > P_{m+1}$, *i.e.* for $0 \leqslant m \leqslant 2R$, and if

$$\sum_{|\rho| \leqslant m} p_\rho^* = \sum_{|\rho| \leqslant m} P_\rho$$

for all m such that $Q_m > Q_{m+1}$, *i.e.* for $0 \leqslant m \leqslant R(2k-2)$.

Now, in any case,

$$\Sigma = \Sigma p_r q_r \leqslant \Sigma p_\rho^* q_\rho^* \leqslant \Sigma P_\rho Q_\rho = \Sigma^*,$$

and, since $\Sigma = \Sigma^*$, we must have

$$\Sigma p_r q_r = \Sigma p_\rho^* q_\rho^* = \Sigma P_\rho Q_\rho.$$

Hence

$$\sum_{|\rho| \leqslant m} p_\rho^* = \sum_{|\rho| \leqslant m} P_\rho \quad [0 \leqslant m \leqslant (2k-2)R],$$

i.e.

$$p_\rho^* = P_\rho \qquad [0 \leqslant m \leqslant (2k-2)R].$$

Since $P_\rho = 0$ if $\rho > 2R$, we must have $p_\rho^* = 0$ if $\rho > 2R$, for otherwise we should have

$$\sum_{|\rho| \leqslant m} p_\rho^* > \sum_{|\rho| \leqslant 2R} p_\rho^* = \sum_{|\rho| \leqslant 2R} P_\rho = \sum_{|\rho| \leqslant m} P_\rho,$$

† There is only a finite number of terms in each of these summations, but on the left-hand side we have yet to show that the suffix ρ is confined to $|\rho| \leqslant 2R$.

which is impossible, since we have shown that

$$\sum_{|\rho|\leqslant m} p_\rho^* \leqslant \sum_{|\rho|\leqslant m} P_\rho \quad (\text{all } m).$$

Thus $p_\rho^* = P_\rho$ for all ρ, and the p's are therefore a rearrangement of the P's. Similarly, $q_\rho^* = Q$ if $0 \leqslant \rho \leqslant 2R$, and, since $\Sigma p_r q_r = \Sigma p_\rho^* q_\rho^*$, the q_r's appearing in the summation on the left of this last equation must be a rearrangement of the q_ρ^*'s appearing on the right, *i.e.* of the Q_ρ's where $|\rho| \leqslant 2R$. Remembering that

$$p_n = \sum_{n_1+n_2=n} a_{n_1} a_{n_2}, \qquad P_n = \sum_{n_1+n_2=n} a_{n_1}^* a_{n_2}^*,$$

we have

$$\Sigma a_r a_s \gamma_t = \Sigma a_r^* a_s^* \gamma_t^*,$$

where

$$\gamma_t^* = q_t^* \quad \text{if} \quad |t| \leqslant 2R,$$

$$= 0 \quad \text{otherwise},$$

and the γ's are a rearrangement of the γ^*'s. It will be seen that the a's and γ's conform to the conditions of Theorem 5, and hence the a's are in "star" order save for a trivial rearrangement.

From Theorem 6 it is possible to deduce the following almost immediately.

THEOREM 7. *Suppose that we are given any finite set of numbers a_n such that $|a_n| = a_n$, where $a_n \leqslant a_0$ hold for all n, and let us write*

$$f(\theta) = \tfrac{1}{2} a_0 + \Sigma a_n \cos n\theta,$$

$$f^*(\theta) = \tfrac{1}{2} a_0^* + \Sigma a_n^* \cos n\theta.$$

If now

$$\int_0^{2\pi} |f(\theta)|^{2k} d\theta = \int_0^{2\pi} |f^*(\theta)|^{2k} d\theta,$$

where k is a positive integer greater than unity, then

$$a_n = e^{Ai} a_n^*$$

for all n, and

$$f(\theta) = e^{Ai} f^*(\lambda\theta),$$

where A is a real constant and λ is a positive integer.

ON THE QUARTIC DEVELOPABLE

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Introduction.

1. It has been well known, since the researches of Veronese and Segre*, that a rational ruled surface of order n in $[3]$ is the projection of a normal rational ruled surface of order n in $[n+1]$. The tangents of a twisted cubic form a rational ruled surface of the fourth order, so that this surface—the quartic developable—must be obtainable by projection from a normal rational quartic ruled surface R_2^4 in $[5]$. The projection must be from some line l which does not meet R_2^4.

It is the main purpose of this paper precisely to specify this projection. The surface R_2^4 has ∞^1 conics upon it whose planes form a three-dimensional cubic locus V_3^3; and the tangents of R_2^4 form a four-dimensional quartic locus V_4^4. V_3^3 is a double locus and R_2^4 a cuspidal surface on V_4^4. V_4^4 is the projection of a well known four-dimensional quartic locus formed by ∞^1 solids in $[7]$; the solids on V_4^4 are the *tangent solids* of R_2^4, and V_4^4 has ∞^3 directrices (lines which meet all its solids). Of these ∞^3 directrices of V_4^4 ∞^2 are double directrices and lie on V_3^3, and of these ∞^2 double directrices ∞^1 are the generators of R_2^4. *The projection of R_2^4 from any one of the simple directrices of V_4^4 is a quartic developable.*

This may also be stated in another way. There are ∞^3 cubic curves on R_2^4; *the tangent planes of R_2^4 at the points of any one of these cubic curves are such that there is one (and only one) line meeting them all, and the projection of R_2^4 from this line is a quartic developable.*

2. The paper is divided into four sections; the second section is concerned with the geometry of R_2^4 and the details of the projection, the third with the polarity between the generators and tangent solids of R_2^4; the

* Veronese: "Behandlung der projektivischen Verhältnisse der Räume von verschiedenen Dimensionen durch das Prinzip des Projizierens und Schneidens", *Math. Annalen*, 19 (1882), 161. Segre: "Sulle rigate razionali in uno spazio lineare qualunque", *Atti Torino*, 19 (1884) 355.

fourth section gives the algebra corresponding to some of the geometrical reasoning of the second and third sections.

The first section is of a different nature; it is concerned with the locus derived from two loci V and V', each formed by ∞^1 linear spaces, by a (1, 1) correspondence between their generating spaces. Particular cases of the results of this first section are used in developing the argument in the second section, and it has seemed worth while to state them in some detail.

I.

3. If two curves C and C', of respective orders m and m', are in (1, 1) correspondence, the lines which join pairs of corresponding points form a ruled surface whose order is, in general, $m+m'$. If there is a point P common to the two curves, and if the correspondence is such that to P, regarded as a point of C, corresponds the point P of C', and to P, regarded as a point of C', corresponds the point P of C, then P is said to be a *united point* of the correspondence. In general, there are no united points; there certainly cannot be any except at intersections of the two curves. But if the correspondence has i united points, the order of the ruled surface is reduced to $m+m'-i$.

This result can be extended to loci of higher dimensions. Suppose, for example, that there is a (1, 1) correspondence between the generators of a ruled surface of order m and the points of a curve of order m'. Then, if there are i generators of the ruled surface which contain the points of the curve corresponding to them, the planes which contain the corresponding pairs of generators and points form a three-dimensional locus of order $m+m'-i$. If there is a (1, 1) correspondence between the generators of two ruled surfaces, the solids which join corresponding pairs of generators form a four-dimensional locus whose order is, in general, the sum of the orders of the two surfaces. Each time that two corresponding generators intersect, the order must be reduced by one; and each time that two corresponding generators coincide, the order must be reduced by two.

The general result may be stated. Suppose that there is a (1, 1) correspondence between the generating spaces of a locus V^m_{n+1} generated by ∞^1 $[n]$'s and the generating spaces of a locus $V^{m'}_{n'+1}$ generated by ∞^1 $[n']$'s. Suppose, further, that there are i pairs of corresponding spaces which intersect, the intersections being spaces $[h_1]$, $[h_2]$, ..., $[h_i]$ respectively. Then the ∞^1 spaces $[n+n'+1]$ which join pairs of corresponding spaces form a locus $V_{n+n'+2}$ whose order is

$$m+m'-\sum_{\rho=1}^{\rho=i}(h_\rho+1).$$

This result can be deduced forthwith from that in which all the numbers h_ρ are zero. For suppose that there is just one pair of corresponding spaces which intersect, an $[n]$ of V^m_{n+1} meeting the corresponding $[n']$ of $V^{m'}_{n'+1}$ in an $[h]$. The two loci V^m_{n+1} and $V^{m'}_{n'+1}$ belong to some space $[N]$; we can regard them as the projections of two loci U^m_{n+1} and $U^{m'}_{n'+1}$, in a space $[N+h+1]$, there being a (1, 1) correspondence between the spaces $[n]$ of U^m_{n+1} and the spaces $[n']$ of $U^{m'}_{n'+1}$ for which no two corresponding spaces intersect. The spaces $[n+n'+1]$ which join corresponding spaces of U^m_{n+1} and $U^{m'}_{n'+1}$ form a locus $U_{n+n'+2}$; the loci V are the projections of the loci U from an $[h]$ lying in an $[n+n'+1]$ of $U_{n+n'+2}$. The order of the locus $V_{n+n'+2}$ must be less by $h+1$ than the order of $U_{n+n'+2}$; and the order of $U_{n+n'+2}$ is $m+m'$, because all numbers h are zero for the correspondence between the spaces of U^m_{n+1} and $U^{m'}_{n'+1}$. Hence, if corresponding spaces of V^m_{n+1} and $V^{m'}_{n'+1}$ have an $[h]$ in common, the order of $V_{n+n'+2}$ is reduced by $h+1$.

4. A curve on a ruled surface which is met by each generator in only one point is called a *directrix*; similarly, a curve on a locus generated by ∞^1 spaces of dimension n is called a directrix curve of the locus if it meets each generating $[n]$ in one and only one point. We can also call a ruled surface a directrix surface of such a locus if there is one generator of the surface in each $[n]$, and so on. A locus V_{n+1} generated by ∞^1 spaces of dimension n is called a directrix of a locus $V_{\nu+1}$, generated by ∞^1 spaces of dimension ν (where $\nu \geq n$) if each $[\nu]$ of $V_{\nu+1}$ contains one $[n]$ of V_{n+1}.

Suppose now that on $V_{\nu+1}$ we have two directrix loci V^m_{n+1} and $V^{m'}_{n'+1}$ of orders m and m', and suppose that $n+n'=\nu-1$. Then in each space $[\nu]$ of $V_{\nu+1}$ there is a space $[n]$ of V^m_{n+1} and a space $[n']$ of $V^{m'}_{n'+1}$; we suppose that, in general, these two spaces $[n]$ and $[n']$ do not intersect. There will be a finite number, say i, of spaces $[\nu]$ which contain pairs of spaces $[n]$ and $[n']$ which do intersect. A space $[\nu]$ of $V_{\nu+1}$ is determined as the space which contains the spaces $[n]$ and $[n']$; we thus have a (1, 1) correspondence between the generating spaces of V^m_{n+1} and the generating spaces of $V^{m'}_{n'+1}$. Hence the order of $V_{\nu+1}$ must be

$$m+m'-\sum_{\rho=1}^{\rho=i}(h_\rho+1).$$

If either of the directrix loci is a curve, that is, if either n or n' is zero, each of the numbers h_ρ must be zero. Hence the order of $V_{\nu+1}$ in this case is $m+m'-i$.

We have then the following result: on a locus $V^M_{\nu+1}$ of order M, generated by ∞^1 spaces $[\nu]$, a directrix curve of order m and a directrix

locus $V_\nu^{m'}$ of order m' intersect in $m+m'-M$ points. It follows, as a corollary, that we cannot have on $V_{\nu+1}^{M}$ a directrix curve and a directrix locus V_ν the sum of whose orders is less than M without the curve lying entirely on V_ν.

The special cases $\nu = 1$ and $\nu = 2$ are of most interest for us.

On a ruled surface of order M two directrix curves of orders m and m' intersect in $m+m'-M$ points.*

On a locus of order M formed by ∞^1 planes a directrix curve of order m and a directrix ruled surface of order m' intersect in $m+m'-M$ points unless the curve lies entirely on the surface.

II.

5. Let us take in [5] two non-intersecting planes, and in each plane a conic. A (1, 1) correspondence between these conics gives a ruled surface R_2^4 of the fourth order. The solid which contains any two of the generators of R_2^4 is the base of a pencil of primes; these cut out ∞^1 directrix conics on R_2^4, there being one conic through each point of R_2^4. If we take three pairs of generators of R_2^4 we see that the planes of the directrix conics are determined as the intersections of corresponding primes of three projectively related pencils; they thus form a three-dimensional locus V_3^3 of the third order†.

A solid through a generator of R_2^4 meets it again, in general, in two points; for a prime through the solid meets R_2^4 in the generator and a directrix cubic curve, this meeting the solid in three points of which one is on the generator. We can establish a projectivity between the two line-stars whose bases are any two generators of R_2^4, primes of the two stars corresponding when they contain the same directrix cubic of R_2^4. The solids which contain the directrix cubics of R_2^4 are intersections of corresponding primes of the stars, and the chords of R_2^4 are the intersections of corresponding solids of the stars. Of the ∞^4 pairs of corresponding solids there are ∞^1 pairs which intersect in planes; these are the planes of the directrix conics, which are thus again seen to form a locus of the third order‡. Corresponding planes of the two stars do

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* For a more general result concerning curves which are not directrices see Segre, *Rend. dell'Accad. dei Lincei* (4), 3^2 (1887), 3. Concerning the intersection of two loci on a locus formed by a single infinity of linear spaces see B. Levi, *Atti Torino*, 34 (1899), 745.

† The locus is of the third order, because the section by a [3] is a curve generated by the intersections of corresponding planes of three projectively related pencils; this is a cubic curve.

‡ The section by a [3] is the curve generated by intersections of corresponding lines of two projectively related point-stars.

not meet in general; but there are ∞^2 pairs of planes which do so, these meeting in the points of R_2^4 itself*.

There is *one* chord of R_2^4 passing through a point O of general position in [5]. For let g and g' be the two generators of R_2^4 which are selected as the bases of the line-stars. To the plane α which joins g to O corresponds a plane α' through g'; if α and α' intersect their intersection must be on R_2^4; they do not intersect if O is of general position. To the plane β' which joins g' to O corresponds a plane β through g. The solids $\alpha\beta$ and $\alpha'\beta'$ are corresponding solids in the projectivity, and therefore intersect in a chord of R_2^4; this passes through O, and, if O is of general position, is the only chord of R_2^4 which does. If the two solids $\alpha\beta$ and $\alpha'\beta'$ have a plane of intersection O lies on V_3^3; there are then ∞^1 chords of R_2^4 passing through O, all of them lying in the plane of V_3^3 which passes through O.

6. At each point of R_2^4 there is a tangent plane† ; it may be defined as the plane containing the generator and the tangent of the directrix conic which passes through the point. The tangent planes at different points of the same generator lie in a solid† which we call a *tangent solid* of R_2^4. When R_2^4 is generated by a (1, 1) correspondence between two of its directrix conics the tangent solids are determined by the pairs of tangents of the two conics at corresponding points. The tangent solids thus join pairs of corresponding lines in a (1, 1) correspondence between two plane envelopes, both of the second class; they therefore form a four-dimensional locus V_4^4 of order four.

7. Let us now consider a line l of [5]. There is a chord of R_2^4 passing through each point of l, and four of these chords are tangents, viz., those passing through the four intersections of l and V_4^4. The chords meet R_2^4 in points of a curve C; they give on C a rational involution with four double points, so that C must be elliptic. C meets each generator in two points, since the solid containing l and a generator meets R_2^4 in two further points. Also, since a prime through l meets R_2^4 in a rational normal quartic curve, three of whose chords meet l, C is of the sixth order. We have thus on R_2^4 an elliptic sextic curve C meeting each generator in two points.

The projection of R_2^4 from l on to a solid is a quartic ruled surface with a twisted cubic for its double curve; the twisted cubic is the projection of

* For the projective generation of a ruled surface of order $n-1$ in $[n]$ see Veronese, *Math. Annalen*, 19 (1882), 226–227.

† See Del Pezzo, *Rend. di Palermo*, 1 (1887), 243–245.

C, and four of its points are pinch points*. If we wish to obtain a developable surface in [3] we must choose l, so that it is met not only by four but by an infinity of tangents of R_2^4; in other words, l must lie on V_4^4. Then, instead of obtaining an elliptic sextic curve which meets each generator of R_2^4 in two points, we should obtain a rational cubic curve which meets each generator of R_2^4 in one point.

8. Conversely, let us consider any directrix cubic curve Γ_3 on R_2^4; there are ∞^3 such curves, given by the primes through any fixed generator. The tangent planes of R_2^4 at the points of Γ_3 are determined as the planes which join the tangents of Γ_3 to corresponding points of any directrix conic of R_2^4, a point of the conic corresponding to a tangent of Γ_3 when it is on the same generator of R_2^4 as is the point of contact of the tangent with Γ_3. Hence we have a (1, 1) correspondence between the generators of a ruled surface of the fourth order (formed by the tangents of Γ_3) and the points of a conic; since the conic meets Γ_3 in one point† there is one generator of the ruled surface passing through the corresponding point of the conic. Hence the tangent planes of R_2^4 at the points of Γ_3 form a locus of three dimensions and of order $4+2-1=5$. Denote it by V_3^5.

There are two types of loci V_3^5, of the fifth order, formed by rational singly-infinite families of planes‡. A locus of the first type is generated by means of two lines and a cubic curve; it has ∞^1 directrix lines, but no directrix conics. A locus of the second type is generated by means of a line and two conics; it has ∞^3 directrix conics, but only one directrix line. The locus V_3^5 formed by the tangent planes of R_2^4 at the points of Γ_3 contains R_2^4, and therefore directrix conics; hence it has one and only one directrix line. Hence, corresponding to the ∞^3 cubic curves of R_2^4, we have ∞^3 loci V_3^5, each with a directrix line; we have thus ∞^3 lines from which R_2^4 may be projected into a developable surface. The ∞^3 loci V_3^5 all lie on V_4^4; their directrix lines are directrices§ of V_4^4.

Of the ∞^8 lines of [5] there are ∞^3 from which R_2^4 can be projected into a developable surface. Five conditions are necessary for a quartic

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* For a full account of the different projections of the surface R_2^4 see *The theory of ruled surfaces* (Cambridge, 1931), 55, *et seq.*

† *Cf.* the end of § 4.

‡ Loci formed by rational singly-infinite families of planes are studied by Segre, *Atti Torino*, 21 (1885), 95. Such a locus can be generated in one and only one way by means of three curves, the sum of whose orders is equal to the order of the locus. See the foot-note to a paper by Segre, *Atti Torino*, 19 (1884), 892. The loci V_3^5 are normal in [7].

§ V_4^4 is the projection of a normal locus M_4^4 formed by ∞^1 solids in [7]; this normal locus has ∞^3 directrix lines.

ruled surface in [3] to be a developable surface. Given any twisted cubic curve in [3], those of its chords which belong to a linear complex form a quartic ruled surface with the curve as double curve*. If, of the ∞^5 linear complexes in [3], we take that one which contains the tangents of the cubic, the quartic ruled surface is a developable.

III.

9. We can establish a polarity between the generators and the tangent solids of R_2^4, a tangent solid being dual to the generator along which it touches R_2^4. A plane containing a directrix conic meets every generator and lies in a prime with every tangent solid; its polar must be a plane lying in a prime with every tangent solid and meeting every generator. Hence the planes of V_3^3 are polars of each other; there must be two planes ϖ_1 and ϖ_2 of V_3^3 which are self-polar, meeting R_2^4 in two of its directrix conics γ_1 and γ_2.

The polar of a tangent plane of R_2^4 is another tangent plane passing through the same generator of R_2^4; the points of contact are separated harmonically by the intersections of the generator with γ_1 and γ_2. The tangent planes of R_2^4 at the points of γ_1 and γ_2 are all self-polar.

It may be stated at once that the polarity which we are considering is an ordinary polarity in regard to a quadric. There are ∞^5 quadrics Ω which contain R_2^4; any one of these meets V_3^3 in a sextic surface consisting of two of its planes together with R_2^4. Consider a quadric Ω which contains R_2^4; let γ_1 and γ_2 be the two directrix conics of R_2^4 whose planes ϖ_1 and ϖ_2 lie on Ω, and let g be any generator of R_2^4 meeting γ_1 and γ_2 in P_1 and P_2 respectively. In order that the polar solid of g, in regard to Ω, should be the tangent solid of R_2^4 along g, it is necessary and sufficient that Ω should contain the tangent planes of R_2^4 at P_1 and P_2. If it contains either of these tangent planes it must contain the other. In order that the polar solid of each generator of R_2^4 in regard to Ω should be the tangent solid of R_2^4 along that generator, it is necessary and sufficient that Ω should contain the tangent planes of R_2^4 at all the points of γ_1 (and therefore also the tangent planes of R_2^4 at all the points of γ_2).

The tangent planes of R_2^4 at the points of any one of its directrix conics γ form a quartic locus W_3^4; they join the tangents of γ to the points of any other directrix conic, and are thus determined by a (1, 1) correspondence between the lines of a plane envelope of class two and the points of a curve of order two. Any quadric meets W_3^4 in a surface of order eight; if the quadric contains both R_2^4 and the plane ϖ which contains γ, these, since ϖ

* Cayley, *Papers*, 6, 316.

is a double plane on W_3^4, make up a surface of order six common to the quadric and W_3^4. Hence, if the quadric is made to contain the tangent planes of R_2^4 at three points of γ, it must contain the tangent planes of R_2^4 at *all* the points of γ.

Of the ∞^5 quadrics which contain R_2^4 there are ∞^4 which also contain the plane of a directrix conic γ; of these there are ∞^1 which contain the tangent planes of R_2^4 at all the points of γ, forming a pencil whose quartic base locus is W_3^4. Each of these ∞^1 quadrics contains the tangent planes of R_2^4 at the points of some other directrix conic; given any two directrix conics γ_1 and γ_2 there is a unique quadric which contains all the tangent planes of R_2^4 at the points of these two conics. Thus, corresponding to the pairs of directrix conics of R_2^4, we have ∞^2 quadrics in regard to which the generators of R_2^4 are the polars of the tangent solids.

10. We consider, then, the polarity determined by the quadric Ω which contains the tangent planes of R_2^4 at all the points of two of its directrix conics γ_1 and γ_2. To the tangent planes of R_2^4 at the points of a directrix cubic Γ correspond, in the polarity, the tangent planes at the points of a second directrix cubic Γ'; Γ and Γ' have two intersections, one of these being on γ_1 and the other on γ_2. Thus the loci V_3^5 formed by tangent planes of R_2^4 at points of its directrix cubics are dual to each other in pairs.

The polar of the directrix of a locus V_3^5 is a solid meeting each plane of the dual locus in a line. We have seen that V_3^5 has one and only one directrix line; hence there is one and only one solid which meets each plane of V_3^5 in a line. This is evidently the solid containing the directrix cubic of R_2^4. Hence the directrices of the ∞^3 loci V_3^5 are polars of the solids containing the ∞^3 directrix cubics of R_2^4. Since the directrices of the loci V_3^5 generate V_4^4 there are four of them which meet an arbitrary line; it follows, by duality, that there are four of the ∞^3 solids containing the cubic curves of R_2^4 which meet an arbitrary solid in planes. This can be seen directly, for the ∞^3 solids which contain the cubic curves of R_2^4 are the intersections of corresponding primes of two projectively related line-stars. These primes meet the arbitrary solid in planes, so that there is thus established a collineation, in the solid, which is known to have four self-corresponding planes.

A plane which passes through a directrix of V_4^4 necessarily meets V_4^4 further in three lines, each line lying in a solid of V_4^4. Hence the polar of a plane through a directrix of V_4^4 is a trisecant plane of R_2^4. The polars of planes through the same directrix of V_4^4 are trisecant planes all of whose intersections with R_2^4 lie on the same directrix cubic.

11. The tangent solids of R_2^4 meet an arbitrary plane in the points of a rational quartic curve. This curve is the intersection of the plane with V_4^4, its three double points being the three intersections of the plane with V_3^3; it is of class six, and has six points of inflection and four double tangents. Dually, the primes which join an arbitrary plane to the generators of R_2^4 meet another arbitrary plane in the tangents of a rational curve of class four with three double tangents; the curve is of the sixth order, having six cusps and four double points.

This last statement is equivalent to the following: those points of R_2^4 which are points of contact of tangents which meet an arbitrary plane π lie on a rational sextic curve C, six of whose tangents meet π. The fact that the points of contact lie on a *sextic* curve follows at once if we consider a section of R_2^4 by a prime which contains π; for such a section is a rational normal quartic curve, *six** of whose tangents meet π. C meets each generator of R_2^4 in one point; it is a directrix of R_2^4.

We can prove that six tangents of C meet π by using the results of § 4. The tangents of C form a ruled surface of order ten*. The tangent plane of R_2^4 at a point of C is the plane which contains the tangent of C and the generator of R_2^4; it may therefore be determined by joining the tangent of C to that point of any directrix conic γ of R_2^4 which lies on the generator through the point of contact of the tangent with C. We have thus a (1, 1) correspondence between the generators of a ruled surface of order ten and the points of a curve of order two; since C meets γ in four points†, there are four generators of the ruled surface which contain the points of the conic which correspond to them. Hence‡ the tangent planes of R_2^4 at the points of C form a locus of three dimensions and order $10+2-4=8$.

Now the quartic curve in π is a directrix curve of this locus, and the ruled surface formed by the tangents of C is a directrix ruled surface of the locus. Hence† the ruled surface and the quartic curve meet in $10+4-8=6$ points, so that there are six tangents of C which meet π.

IV.

12. If we take two planes in [5] to be $x_3=x_4=x_5=0$ and $x_0=x_1=x_2=0$, then, if two conics are taken, one in each of the planes, and placed in (1, 1) correspondence, we may take corresponding points of

* The tangents of a rational curve of order n, without cusps, in space of any number of dimensions, form a ruled surface of order $2n-2$.

† § 4.

‡ § 3.

the two conics to be $(\theta^2, \theta, 1, 0, 0, 0)$ and $(0, 0, 0, \theta^2, \theta, 1)$. The ruled surface R_2^4 determined by the correspondence is the locus of points whose coordinates are

$$(\lambda\theta^2, \lambda\theta, \lambda, \mu\theta^2, \mu\theta, \mu).$$

Points for which θ is constant are on a line, and points for which $\lambda : \mu$ is constant are on a conic. The equations of the plane of the conic are

$$\frac{x_0}{x_3} = \frac{x_1}{x_4} = \frac{x_2}{x_5} = \frac{\lambda}{\mu},$$

so that the planes of the ∞^1 conics form the V_3^3 whose equations are

$$\frac{x_0}{x_3} = \frac{x_1}{x_4} = \frac{x_2}{x_5}.$$

The tangent plane of R_2^4 is the plane of the three points

$$(\theta^2, \theta, 1, 0, 0, 0), \quad (0, 0, 0, \theta^2, \theta, 1), \quad \text{and} \quad (2\lambda\theta, \lambda, 0, 2\mu\theta, \mu, 0).$$

Hence the tangent planes of R_2^4 at the different points of the generator $\theta = \alpha$ all lie in the solid

$$x_0 - 2\alpha x_1 + \alpha^2 x_2 = x_3 - 2\alpha x_4 + \alpha^2 x_5 = 0.$$

Eliminating α from these equations we obtain

$$(x_0x_5 - x_2x_3)^2 = 4(x_2x_4 - x_1x_5)(x_1x_3 - x_0x_4),$$

the equation of V_4^4. Clearly V_3^3 is a double locus on V_4^4.

13. The equation of any prime which contains the generator $\theta = \alpha$ is of the form

$$A(x_0 - \alpha x_1) + B(x_1 - \alpha x_2) + C(x_3 - \alpha x_4) + D(x_4 - \alpha x_5) = 0,$$

and it meets R_2^4 again in points for which

$$(A\theta + B)\lambda + (C\theta + D)\mu = 0.$$

These points lie on a directrix cubic curve, having coordinates

$$\{\theta^2(C\theta + D), \ \theta(C\theta + D), \ C\theta + D, \ -\theta^2(A\theta + B), \ -\theta(A\theta + B), \ -(A\theta + B)\}.$$

We have thus, for different values of $A : B : C : D$, ∞^3 directrix cubic curves on R_2^4.

The equations of the solid which contains the cubic curve are

$$Ax_0 + Bx_1 + Cx_3 + Dx_4 = 0,$$

$$Ax_1 + Bx_2 + Cx_4 + Dx_5 = 0.$$

It meets an arbitrary solid, whose equations are

$$px_0+qx_1+rx_2+sx_3+tx_4+ux_5=0,$$
$$p'x_0+q'x_1+r'x_2+s'x_3+t'x_4+u'x_5=0,$$

in a plane if the four equations are linearly dependent, and this requires

$$\begin{Vmatrix} A & B & 0 & C & D & 0 \\ 0 & A & B & 0 & C & D \\ p & q & r & s & t & u \\ p' & q' & r' & s' & t' & u' \end{Vmatrix} = 0.$$

The system of equations given by the vanishing of the determinants of this matrix has four solutions in $A : B : C : D$; hence there are four of the ∞^3 solids containing cubic curves of R_2^4 which meet an arbitrary solid in planes.

The coordinates of the points of the plane which touches R_2^4 at a point of the cubic curve are

$$\{\rho\theta^2+2\theta(C\theta+D),\ \rho\theta+(C\theta+D),\ \rho,\ \sigma\theta^2-2\theta(A\theta+B),\ \sigma\theta-(A\theta+B),\ \sigma\};$$

different points of the plane being given by different values of ρ and σ. If we allow ρ, σ, and θ *all* to vary, we obtain the ∞^3 points of the locus V_3^5 generated by the tangent planes of R_2^4 at different points of the cubic curve. The coordinates of a point of V_3^5 satisfy the equation of V_4^4.

If we keep ρ and σ fixed and allow θ to vary we obtain the points of a directrix curve of V_3^5. This curve is, in general, a conic; but if we choose the special values $\rho=-2C$ and $\sigma=2A$ we obtain

$$(2D\theta,\ D-C\theta,\ -2C,\ -2B\theta,\ A\theta-B,\ -2A),$$

the coordinates of the points of a line. This is the directrix line of V_3^5.

If, for a point on this directrix line of V_3^5, we allow the ratios $A : B : C : D$, as well as θ, to vary, we obtain all the points of V_4^4. Thus a directrix of V_4^4 joins the point

$$(2D,\ -C,\ 0,\ -2B,\ A,\ 0)$$

of the tangent solid $x_2=x_5=0$ to the point

$$(0,\ D,\ -2C,\ 0,\ -B,\ -2A)$$

of the tangent solid $x_0=x_3=0$. V_4^4 is thus generated by the joins of pairs of corresponding points in a projectivity between the two tangent solids.

14. The equations of R_2^4 are

$$\frac{x_0}{x_1} = \frac{x_1}{x_2} = \frac{x_3}{x_4} = \frac{x_4}{x_5};$$

and the equation of a quadric which contains it is of the form

$$a(x_0x_2-x_1^2)+b(x_0x_4-x_1x_3)+c(x_0x_5-x_1x_4)$$
$$+d(x_1x_4-x_2x_3)+e(x_1x_5-x_2x_4)+f(x_3x_5-x_4^2) = 0.$$

The polar solid of each generator of R_2^4 in regard to this quadric is the tangent solid of R_2^4 along the generator provided that

$$b = c+d = e = 0.$$

We have then a quadric whose equation is of the form

$$a(x_0x_2-x_1^2)+\beta(x_0x_5-2x_1x_4+x_2x_3)+\gamma(x_3x_5-x_4^2) = 0;$$

if $\alpha:\beta:\gamma$ vary we obtain the quadrics of a net whose base surface is R_2^4 counted twice. The quadric contains the plane

$$\frac{x_0}{x_3} = \frac{x_1}{x_4} = \frac{x_2}{x_5} = \frac{\lambda}{\mu}$$

of V_3^3 if

$$a\lambda^2+2\beta\lambda\mu+\gamma\mu^2 = 0$$

(so that there are two planes of V_3^3 on the quadric), and, when this relation is satisfied, the quadric also contains all the tangent planes of R_2^4 at points of the corresponding directrix conic. The polar plane of the plane

$$\frac{x_0}{x_3} = \frac{x_1}{x_4} = \frac{x_2}{x_5} = \frac{\lambda}{\mu}$$

is the plane

$$\frac{x_0}{x_3} = \frac{x_1}{x_4} = \frac{x_2}{x_5} = \frac{\lambda'}{\mu'},$$

where

$$a\lambda\lambda'+\beta(\lambda\mu'+\lambda'\mu)+\gamma\mu\mu' = 0.$$

The polar solid of the line whose points have coordinates

$$(2D\theta,\ D-C\theta,\ -2C,\ -2B\theta,\ A\theta-B,\ -2A)$$

is the solid which meets R_2^4 in points for which

$$\lambda\{aD-\beta B+\theta(aC-\beta A)\}+\mu\{\beta D-\gamma B+\theta(\beta C-\gamma A)\} = 0.$$

Thus the directrices of the loci V_3^5 are the polars of the solids which contain the cubic curves of R_2^4. The planes of pairs of loci V_3^5 are dual to each

other; the duality may be expressed by

$$a:b:c:d = aC-\beta A : aD-\beta B : \beta C-\gamma A : \beta D-\gamma B,$$

which gives also

$$A:B:C:D = ac-\beta a : ad-\beta b : \beta c-\gamma a : \beta d-\gamma b,$$

unless $a\gamma = \beta^2$.

There is singularity if $a\gamma = \beta^2$. The quadric, whose discriminant is a multiple of $(a\gamma-\beta^2)^3$, is then a cone whose vertex is the plane of the directrix conic of R_2^4 for which $\lambda/\mu = \beta/a = \gamma/\beta$. We omit this special case.

The correspondence between the pairs of loci V_3^5 may be regarded as an involutory collineation between the points (a, b, c, d) and (A, B, C, D) of a three-dimensional space Σ. The self-corresponding points are given by

$$\rho x = az-\beta x,$$
$$\rho y = at-\beta y,$$
$$\rho z = \beta z-\gamma x,$$
$$\rho t = \beta t-\gamma y,$$

and ρ must satisfy $(\rho^2+a\gamma-\beta^2)^2 = 0$. The self-corresponding points of the collineation lie on two lines; the line joining a pair of corresponding points P and P' meets both these lines, and P and P' are separated harmonically by them.

The planes of the directrix conics are dual in pairs. The two self-dual planes are given by

$$a\lambda^2+2\beta\lambda\mu+\gamma\mu^2 = 0,$$

and the algebra is simplified by taking these two planes (which do not coincide) to be $x_3 = x_4 = x_5 = 0$ and $x_0 = x_1 = x_2 = 0$. Then $a = \gamma = 0$, and we have polarity in regard to the quadric

$$x_0x_5-2x_1x_4+x_2x_3 = 0.$$

The polar plane of the plane

$$\lambda x_0-\mu x_3 = \lambda x_1-\mu x_4 = \lambda x_2-\mu x_5 = 0$$

is

$$\lambda x_0+\mu x_3 = \lambda x_1+\mu x_4 = \lambda x_2+\mu x_5 = 0.$$

The duality between the pairs of loci V_3^5 is now expressed by

$$a:b:c:d = -A:-B:C:D.$$

We have in Σ a harmonic inversion in regard to the two lines $x = y = 0$ and $z = t = 0$.

The points of Σ are in (1, 1) correspondence with the directrix cubics of R_2^4. A directrix cubic degenerates into a directrix conic and a generator if $A/C = B/D$, hence those points of Σ which correspond to degenerate directrix cubics lie on the quadric

$$xt = yz.$$

To the points of the generator $x/y = z/t = k$ of this quadric correspond the cubic curves on R_2^4 which consist of the generator for which $\theta = -1/k$ and the different directrix conics.

To the points of the generator $x/z = y/t = k'$ of the quadric correspond the cubic curves on R_2^4 which consist of the directrix conic for which $\lambda/\mu = -1/k'$ and the different generators of R_2^4. The generators of this system on the quadric include the two lines which are fixed for the collineation; the directrix conics of R_2^4 which arise from these two generators are those whose planes are self-polar in regard to $x_0x_5 - 2x_1x_4 + x_2x_3 = 0$.

NORMAL COVARIANT DERIVATIVES

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1. *Introduction.*

The geometry associated with the analysis appearing in the present paper is Riemannian. The problem attacked is that of finding a natural tensor generalization of the partial derivative of the elementary differential calculus. The generalization arrived at occupies a position midway between the ordinary covariant derivative (upon which Levi-Civita based his theory of parallelism) and that which arises from the more recent theories of teleparallelism. For it actually reduces in certain special cases to the ordinary covariant derivative, and yet can be used to define a teleparallelism.

The next paragraph contains an outline of the theory of normal coordinates (upon which the ideas of this paper are based). In §§ 3, 4, 5 the new covariant derivative is defined and its properties investigated. In §§ 6, 7 appears the consequent theory of parallelism, while the last paragraph contains an extension to general Riemannian space of the idea of homogeneous tensors introduced for the case of a Euclidean space in a previous paper†.

Finally, it should be stated that the chief disadvantage of the present theory is that it gives a special importance to a particular point of space, namely the origin of normal coordinates.

2. *Preliminary definitions.*

Consider a Riemannian space of n dimensions specified by the metric‡

$$ds^2 = g_{\mu\nu}\,dx^\mu\,dx^\nu. \tag{1}$$

† *Proc. London Math. Soc.* (2), 31 (1930), 225.

‡ The summation convention is adopted throughout.

Let $(\bar{x}^i) \equiv (\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n)$ be a fixed point of the space, which we shall call the *base point*. Then, as is well known, the differential equations of the geodesics are

$$\frac{d^2x^\mu}{ds^2} + \{\alpha\beta, \mu\} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0. \tag{2}$$

The solution of these equations which is such that $(x^i) = (\bar{x}^i)$ and $dx^i/ds = a^i$ when $s = 0$ is†

$$x^i = \bar{x}^i + a^i s - \frac{1}{2!} c^i_{jk} a^j a^k s^2 - \frac{1}{3!} c^i_{jkl} a^j a^k a^l s^3 - \ldots, \tag{3}$$

where c^i_{jk}, c^i_{jkl}, ... are certain determinate constants.

In other words, equations (3) are those of the geodesics passing through the point $(\bar{x}^i)$, any particular geodesic being specified by the values chosen for the arbitrary constants a^i. s is the length of the arc of the geodesic measured from $(\bar{x}^i)$ to (x^i).

If then we introduce new variables (y^i) defined by the relations

$$x^i = \bar{x}^i + y^i - \frac{1}{2!} c^i_{jk} y^j y^k - \ldots, \tag{4}$$

the equations of the geodesics through $(\bar{x}^i)$, which is the origin $(y^i) = 0$ of the new variables, become simply

$$y^i = a^i s. \tag{5}$$

The coordinates (y^i) are said to be *normal*. The components of a tensor in these normal coordinates will be denoted by the affixing of an asterisk. Thus, for example,

$$^*g_{\mu\nu} = \frac{\partial x^\alpha}{\partial y^\mu} \frac{\partial x^\beta}{\partial y^\nu} g_{\alpha\beta}.$$

From the definition of a^i we get, by (1),

$$\bar{g}_{\mu\nu} a^\mu a^\nu = 1,$$

the superposed bar indicating (as always hereafter) evaluation at the point $(\bar{x}^i)$, that is, at $(y^i) = 0$.

Hence, from (5),

$$s^2 = \bar{g}_{\mu\nu} y^\mu y^\nu. \tag{6}$$

† Veblen, "Invariants of quadratic differential forms", *Camb. Mathematical Tract*, No. 24 (1927), 84. Our notation differs slightly from that of Veblen.

Thus, defining the scalar Ω to be one-half of the square of the geodesic distance between the points $(\bar{x}^i)$ and (x^i) in the x-coordinate system, and between the origin and the point (y^i) in the y system, we have

$$\Omega = \tfrac{1}{2}\bar{g}_{\mu\nu} y^\mu y^\nu. \tag{7}$$

It should be noticed that the $\bar{g}_{\mu\nu}$ are functions of the $\bar{x}^i$ only.

From the definition it is evident that Ω is a function of both the x^i and $\bar{x}^i$; now it has been proved elsewhere† that

$$y^i = -\bar{g}^{i\alpha}\frac{\partial\Omega}{\partial\bar{x}^\alpha}.$$

Following the notation of the paper referred to, $\partial\Omega/\partial\bar{x}^\alpha$ will be denoted by $\Omega_{(\alpha)}$, and $\bar{g}^{i\alpha}\Omega_{(\alpha)}$ by $\Omega^{(i)}$. Unbracketed suffixes will be used to denote covariant differentiations with respect to the (x^i). Thus

$$y^i = -\Omega^{(i)}. \tag{8}$$

Hence
$$\frac{\partial y^i}{\partial x^j} = -\frac{\partial\Omega^{(i)}}{\partial x^j} = -\Omega^{(i)}_j, \quad \text{say.} \tag{9}$$

Again, from (7), we get

$$\frac{\partial\Omega}{\partial y^\mu} = \bar{g}_{\mu\nu} y^\nu,$$

that is
$$*\Omega_\mu = \bar{g}_{\mu\nu} y^\nu$$
$$= *g_{\mu\nu} y^\nu, \tag{10}$$

since†
$$\bar{g}_{\mu\nu} y^\nu = *g_{\mu\nu} y^\nu.$$

Raising the suffix μ in (10), we get at once

$$*\Omega^\mu = y^\mu. \tag{11}$$

This equation differs from (8) in that it gives a relation between the y^i and the components in normal coordinates of the contravariant vector Ω^i, whereas (8) defines the y^i as functions of the x's.

Finally, if the coordinates x^i are transformed by the arbitrary relations

$$x^i = x^i(x'^\mu), \tag{12}$$

then the normal coordinates (y'^μ) corresponding to the new variables

† *Proc. London Math. Soc.* (2), 32 (1931), 87.

‡ Veblen, *loc. cit.*, 96 (14.8).

x'^{μ} are connected with the (y^{μ}) by the linear relations†

$$y'^{i} = \left(\frac{\partial x'^{i}}{\partial x^{j}}\right)_0 y^{j}, \tag{13}$$

the suffix 0 indicating evaluation at the common y origin, that is at $(\bar{x}^{i})$.

3. *Normal covariant derivatives.*

In any given coordinate system (x^{i}), let the functions $\Gamma^{\lambda}_{\mu\nu}$ be defined by the relations

$$\Gamma^{\lambda}_{\mu\nu} = \frac{\partial x^{\lambda}}{\partial y^{\alpha}} \frac{\partial^2 y^{\alpha}}{\partial x^{\mu} \partial x^{\nu}}, \tag{14}$$

where the (y^{μ}) are the corresponding normal coordinates defined by either of the equations (4), (8). To find the law of transformation of these functions, change the coordinates to another system (x'^{i}), letting (y'^{i}) be the corresponding normal coordinates having the same origin as the (y^{i}). Then

$$\begin{aligned}
\Gamma'^{\lambda}_{\mu\nu} &= \frac{\partial x'^{\lambda}}{\partial y'^{\alpha}} \frac{\partial^2 y'^{\alpha}}{\partial x'^{\mu} \partial x'^{\nu}} \\
&= \frac{\partial x'^{\lambda}}{\partial x^{\beta}} \frac{\partial x^{\beta}}{\partial y^{\sigma}} \frac{\partial y^{\sigma}}{\partial y'^{\alpha}} \frac{\partial}{\partial x'^{\mu}} \left(\frac{\partial y'^{\alpha}}{\partial y^{\tau}} \frac{\partial y^{\tau}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial x'^{\nu}}\right) \\
&= \frac{\partial x'^{\lambda}}{\partial x^{\beta}} \frac{\partial x^{\beta}}{\partial y^{\sigma}} \left(\frac{\partial x^{\sigma}}{\partial x'^{\alpha}}\right)_0 \frac{\partial}{\partial x'^{\mu}} \left[\left(\frac{\partial x'^{\alpha}}{\partial x^{\tau}}\right)_0 \frac{\partial y^{\tau}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial x'^{\nu}}\right],
\end{aligned}$$

by (13). Remembering that $(\partial x'^{\alpha}/\partial x^{\tau})_0$ is a constant so far as differentiation with respect to x'^{μ} is concerned, we quickly get

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\beta}} \frac{\partial x^{\rho}}{\partial x'^{\nu}} \frac{\partial x^{\kappa}}{\partial x'^{\mu}} \Gamma^{\beta}_{\kappa\rho} + \frac{\partial x'^{\lambda}}{\partial x^{\beta}} \frac{\partial^2 x^{\beta}}{\partial x'^{\mu} \partial x'^{\nu}}. \tag{15}$$

From this it at once follows that, if T^{λ} be an arbitrary contravariant vector, then

$$T^{\lambda}_{:\mu} \equiv \frac{\partial T^{\lambda}}{\partial x^{\mu}} + \Gamma^{\lambda}_{\alpha\mu} T^{\alpha} \tag{16}$$

is a tensor‡.

This we shall call the *normal covariant derivative of the vector* T^{λ} *with* $(\bar{x}^{\lambda})$ *as base point.* And in general the normal covariant derivative

† Veblen, *loc. cit.*, 86 (3.5).

‡ Veblen, *loc. cit.*, ch. 3, §§ 10, 11.

of any tensor $T^{\lambda_1\lambda_2\cdots}_{\mu_1\mu_2\cdots}$ will be defined to be

$$T^{\lambda_1\lambda_2\cdots}_{\mu_1\mu_2\cdots;\nu} = \frac{\partial T^{\lambda_1\lambda_2\cdots}_{\mu_1\mu_2\cdots}}{\partial x^\nu} + \Gamma^{\lambda_1}_{\alpha\nu} T^{\alpha\lambda_2\cdots}_{\mu_1\mu_2\cdots} + \ldots - \Gamma^{\alpha}_{\mu_1\nu} T^{\lambda_1\lambda_2\cdots}_{\alpha\mu_2\cdots} - \ldots. \tag{17}$$

The normal covariant derivative of a scalar is defined to be the ordinary partial derivative of the scalar.

It should be noticed that, by (14), it follows at once that $\Gamma^\lambda_{\mu\nu}$ is in general symmetrical in μ and ν; *i.e.*

$$\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}. \tag{18}$$

To find an explicit formula for the functions Γ in terms of the x's, we notice that, since

$$\frac{\partial x^\lambda}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial x^\mu} = \delta^\lambda_\mu,$$

δ^λ_μ being the Kronecker symbol, it follows that

$$\frac{\partial x^\lambda}{\partial y^\alpha} = \left(\text{cofactor of } \frac{\partial y^\alpha}{\partial x^\lambda} \text{ in the Jacobian } \frac{\partial(y^i)}{\partial(x^j)}\right) \div \frac{\partial(y^i)}{\partial(x^j)}.$$

By (9),
$$\frac{\partial(y^i)}{\partial(x^j)} = (-)^n \| \Omega^{(i)}_j \|$$
$$= (-)^n J, \quad \text{say,}$$

where $\| \Omega^{(i)}_j \|$ denotes the determinant of the $\Omega^{(i)}_j$.

Hence
$$\frac{\partial x^\lambda}{\partial y^\alpha} = -J^{-1} \times (\text{cofactor of } \Omega^{(\alpha)}_\lambda \text{ in } J),$$

so that, by (9) and (14),

$$\Gamma^\lambda_{\mu\nu} = J^{-1} \times (\text{cofactor of } \Omega^{(\alpha)}_\lambda \text{ in } J) \times \frac{\partial^2 \Omega^{(\alpha)}}{\partial x^\mu \partial x^\nu}. \tag{19}$$

And since Ω can be calculated for a given metric directly from the differential equations (2), this gives an explicit formula for the Γ's in terms of the x's.

4. THEOREM I. *In the y coordinate system the normal covariant derivative of a tensor is the ordinary partial derivative.*

For, by the law of transformation (15),

$$\Gamma^\lambda_{\mu\nu} = \frac{\partial x^\lambda}{\partial y^\alpha}\frac{\partial y^\beta}{\partial x^\mu}\frac{\partial y^\gamma}{\partial x^\nu} {}^*\Gamma^\alpha_{\beta\gamma} + \frac{\partial x^\lambda}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu}$$
$$= \frac{\partial x^\lambda}{\partial y^\alpha}\frac{\partial y^\beta}{\partial x^\mu}\frac{\partial y^\gamma}{\partial x^\nu} {}^*\Gamma^\alpha_{\beta\gamma} + \Gamma^\lambda_{\mu\nu},$$

whence
$${}^*\Gamma^\alpha_{\beta\gamma} = 0. \tag{20}$$

Hence, by (16),
$$*T^{\lambda}{}_{:\mu} = \frac{\partial^* T^{\lambda}}{\partial y^{\mu}}, \tag{21}$$
and so for any tensor†.

THEOREM II. *The "normal Riemann tensor", defined to be*
$$B^i_{jkl} = \frac{\partial \Gamma^i_{jk}}{\partial x^l} - \frac{\partial \Gamma^i_{jl}}{\partial x^k} + \Gamma^b_{jk}\Gamma^i_{bl} - \Gamma^b_{jl}\Gamma^i_{bk},$$
is everywhere identically zero‡.

For, by (20), it at once follows that
$$*B^i_{jkl} = 0,$$
and this, being a tensor equation, must be true in all systems of coordinates. Hence
$$B^i_{jkl} = 0. \tag{22}$$
From this it follows that $T^{\mu}{}_{:\lambda:\nu} = T^{\mu}{}_{:\nu:\lambda}$, and so for any tensor.

THEOREM III. *In a flat space* $\Gamma^{\lambda}_{\mu\nu} = \{\mu\nu, \lambda\}$.

By a "flat" space is meant one for which the ordinary Riemann tensor, namely
$$K^i_{jkl} = \frac{\partial}{\partial x^l}\{jk, i\} - \frac{\partial}{\partial x^k}\{jl, i\} + \{jk, b\}\{bl, i\} - \{jl, b\}\{bk, i\},$$
is everywhere identically zero.

For in a flat space the normal coordinates are ordinary Cartesian coordinates (not necessarily rectangular). Hence $*\{\mu\nu, \lambda\} = 0$. It follows that if T^{μ} is an arbitrary vector and $T^{\mu}_{\nu} = \frac{\partial T^{\mu}}{\partial x^{\nu}} + \{\alpha\nu, \mu\} T^{\alpha}$, then
$$*T^{\mu}{}_{:\nu} = \frac{\partial^* T^{\mu}}{\partial y^{\nu}}$$
$$= *T^{\mu}_{\nu}.$$
This is a tensor formula, so
$$T^{\mu}{}_{:\nu} = T^{\mu}_{\nu}. \tag{23}$$
The theorem follows on account of the arbitrariness of T^{μ}.

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† It is worth remarking that the *ordinary* covariant derivative $T^{\lambda}_{\mu} = \partial T^{\lambda}/\partial x^{\mu} + \{\alpha\mu, \lambda\} T^{\alpha}$ possesses the property that $*T^{\lambda}_{\mu} = \partial^* T^{\lambda}/\partial y^{\mu}$ at the origin $y^i = 0$, but this equation, unlike (20), is not in general true at all points of space.

‡ That B^i_{jkl} is indeed a tensor follows from Veblen, *loc. cit.*, 41, where a proof appears for the case of *any* set of functions Γ^i_{jk}, symmetrical in the lower suffixes, which satisfy the law of transformation (15).

In a flat space, therefore, the normal covariant derivative is the ordinary covariant derivative, and the functions $\Gamma^{\lambda}_{\mu\nu}$ are independent of the base point $(\bar{x}^{\lambda})$. In a general Riemannian space, however, the Γ's are functions of the $\bar{x}^{\lambda}$ as well as of the x^{λ}. Hence the base point is of special importance in the present theory, a different set of Γ's being obtained by a different choice of base point.

5. *The normal covariant derivatives of the fundamental tensor and of the scalar* Ω.

Since, in general, $\partial^* g_{\mu\nu}/\partial y^{\lambda} \neq 0$ unless the $^*g_{\mu\nu}$ are constants (which is the case only when the space is flat), it follows that

$$g_{\mu\nu:\lambda} \neq 0 \tag{24}$$

in general. Similarly

$$g^{\mu\nu}{}_{:\lambda} \neq 0. \tag{25}$$

But since $^*g^{\mu}_{\nu} = \delta^{\mu}_{\nu}$, we have $\partial^* g^{\mu}_{\nu}/\partial y^{\lambda} = 0$, whence

$$g^{\mu}_{\nu:\lambda} = 0. \tag{26}$$

In forming normal covariant derivatives we may therefore treat the mixed tensor g^{μ}_{ν} as a constant, but not the covariant and contravariant tensors $g_{\mu\nu}$, $g^{\mu\nu}$.

Now consider the scalar Ω. By (11),

$$^*\Omega^{\mu} = y^{\mu},$$

whence

$$\frac{\partial^*\Omega^{\mu}}{\partial y^{\nu}} = \delta^{\mu}_{\nu} = {}^*g^{\mu}_{\nu}.$$

From this we deduce at once that

$$\Omega^{\mu}{}_{:\nu} = g^{\mu}_{\nu}, \tag{27}$$

where

$$\Omega^{\mu} = g^{\mu\alpha}\frac{\partial\Omega}{\partial x^{\alpha}}. \tag{28}$$

Again, by (10),

$$\frac{\partial^*\Omega_{\mu}}{\partial y^{\nu}} = {}^*g_{\mu\nu} + y^{\lambda}\frac{\partial^* g_{\mu\lambda}}{\partial y^{\nu}}$$

$$= {}^*g_{\mu\nu} + {}^*\Omega^{\lambda}\frac{\partial^* g_{\mu\lambda}}{\partial y^{\nu}}$$

by (11). Hence

$$\Omega_{\mu:\nu} = g_{\mu\nu} + \Omega^{\lambda} g_{\mu\lambda:\nu}. \tag{29}$$

If the space is flat, this becomes

$$\Omega_{\mu\nu} = g_{\mu\nu},$$

where $\Omega_{\mu\nu}$ is the ordinary second covariant derivative of Ω. This result was found in a previous paper†.

6. *Parallelism.*

Let T^μ be any contravariant vector. Then we shall say that T^μ has been carried by *normal parallel transport over the displacement* (dx^μ) if the increment in T^μ is given by

$$dT^\mu = -\Gamma^\mu_{\alpha\beta} T^\alpha dx^\beta. \tag{30}$$

The conditions of integrability of these equations are $B^i_{jkl} = 0$, which are satisfied (§ 4, Theorem II). Equations (30) therefore define a tele-parallelism. In fact, by (14) and (30),

$$\frac{\partial T^\mu}{\partial x^\beta} + \frac{\partial x^\mu}{\partial y^\kappa} \frac{\partial^2 y^\kappa}{\partial x^\alpha \partial x^\beta} T^\alpha = 0.$$

Multiplying by $\partial y^\sigma / \partial x^\mu$ (and summing for μ) we get

$$\frac{\partial y^\sigma}{\partial x^\mu} \frac{\partial T^\mu}{\partial x^\beta} + \frac{\partial^2 y^\sigma}{\partial x^\alpha \partial x^\beta} T^\alpha = 0,$$

or

$$\frac{\partial}{\partial x^\beta} \left[\frac{\partial y^\sigma}{\partial x^\alpha} T^\alpha \right] = 0;$$

whence

$$\frac{\partial y^\sigma}{\partial x^\alpha} T^\alpha = \text{constant},$$

or, by (9),

$$\Omega^{(\sigma)}_{\alpha} T^\alpha = \text{constant}. \tag{31}$$

These are therefore the integrated equations of parallel transport.

7. *The autoparallels.*

Putting $T^\mu = dx^\mu / ds$ in (30), we get the differential equations of the autoparallels, namely

$$\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0. \tag{32}$$

† *Proc. London Math. Soc.* (2), 31 (1930), 225.

In normal coordinates (y^i) these are

$$\frac{d^2 y^\mu}{ds^2} = 0,$$

which on integration give

$$y^\mu = b^\mu + a^\mu s, \tag{33}$$

where the b^μ and a^μ are arbitrary constants. In particular, the autoparallels through the base point are given by

$$y^\mu = a^\mu s,$$

and thus, by (5), coincide with the geodesics through that point. But in general the autoparallels *not* passing through the base point will not coincide with the geodesics.

Substituting from (8) in (33), we get

$$\Omega^{(\mu)} = -(b^\mu + a^\mu s),$$

which are thus the equations of the autoparallels in the x coordinates, that is, the solutions of equations (32).

8. *Homogeneous tensors.*

We now extend to general Riemannian space the idea of homogeneous tensors outlined in the case of a flat space in the paper quoted above (§ 5).

Consider an arbitrary tensor T^α_β, which, without loss of generality, we may take to be of the second rank only. We shall call T^α_β a *homogeneous tensor of degree m, with respect to the base point* $(\bar{x}^i)$, if it satisfies the differential equations

$$\Omega^\mu T^\alpha_{\beta:\mu} = m T^\alpha_\beta, \tag{34}$$

where Ω^μ is defined by equation (28).

A homogeneous scalar V of degree m is similarly defined to be one satisfying the equation

$$\Omega^\mu V_\mu = mV. \tag{35}$$

In the y coordinates (34) becomes, by (11) and (21),

$$y^\mu \frac{\partial^* T^\alpha_\beta}{\partial y^\mu} = m^* T^\alpha_\beta, \tag{36}$$

so that each component of the tensor is homogeneous in the ordinary sense.

THEOREM A. *If T^α_β is homogeneous of degree m, its first normal covariant derivative is homogeneous of degree $m-1$.*

For, by (36),

$$y^\mu \frac{\partial^2 {}^*T^\alpha_\beta}{\partial y^\nu \partial y^\mu} = (m-1) \frac{\partial {}^*T^\alpha_\beta}{\partial y^\nu} \tag{37}$$

from which we deduce at once that

$$\Omega^\mu T^\alpha_{\beta:\nu:\mu} = (m-1) T^\alpha_{\beta:\nu}, \tag{38}$$

which proves the theorem.

THEOREM B. *Euler's general theorem on homogeneous functions.*

Multiplying (37) by y^ν and summing for ν, we get, using (36),

$$y^\mu y^\nu \frac{\partial^2 {}^*T^\alpha_\beta}{\partial y^\mu \partial y^\nu} = m(m-1) {}^*T^\alpha_\beta,$$

which gives $$\Omega^\mu \Omega^\nu T^\alpha_{\beta:\mu:\nu} = m(m-1) T^\alpha_\beta.$$

Proceeding in this way we could obtain the tensor form of Euler's general theorem on homogeneous functions, namely

$$\Omega^{\mu_1} \Omega^{\mu_2} \ldots \Omega^{\mu_r} T^\alpha_{\beta:\mu_1:\mu_2:\ldots:\mu} = m(m-1)\ldots(m-r+1) T^\alpha_\beta.$$

THEOREM C. *The scalar Ω is itself homogeneous of degree 2.*

For, as is well known, Ω satisfies the differential equation†

$$\Omega^\mu \Omega_\mu = 2\Omega,$$

and the theorem follows from the definition (35).

THEOREM D. *If T^α_β, U^α_β are two homogeneous tensors of degree m of the same kind, so also is $T^\alpha_\beta + U^\alpha_\beta$.*

This results at once from the definition (34).

THEOREM E. *If T^α_β, H^ε_κ are homogeneous tensors of degree m and m' respectively, then the product $T^\alpha_\beta H^\varepsilon_\kappa$ is homogeneous of degree $m+m'$*

For it is easily proved that

$$\begin{aligned}(T^\alpha_\beta H^\varepsilon_\kappa)_{:\mu} &= T^\alpha_{\beta:\mu} H^\varepsilon_\kappa + T^\alpha_\beta H^\varepsilon_{\kappa:\mu} \\ &= (m+m') T^\alpha_\beta H^\varepsilon_\kappa.\end{aligned}$$

† This is indeed an immediate consequence of equations (7), (10), and (11).

THEOREM F. *The fundamental tensors* $g_{\mu\nu}$, $g^{\mu\nu}$ *are in general not homogeneous.*

This is obvious, since it is not in general true that there exists a constant m such that

$$y^\lambda \frac{\partial {}^*g_{\mu\nu}}{\partial y^\lambda} = m\,{}^*g_{\mu\nu} \quad \text{or} \quad y^\lambda \frac{\partial {}^*g^{\mu\nu}}{\partial y^\lambda} = m\,{}^*g^{\mu\nu}.$$

In the case when the space is flat we have, however,

$$\frac{\partial {}^*g_{\mu\nu}}{\partial y^\lambda} = 0 = \frac{\partial {}^*g^{\mu\nu}}{\partial y^\lambda},$$

since all the ${}^*g$'s are constants. Hence *in a flat space* the $g_{\mu\nu}$ and $g^{\mu\nu}$ are homogeneous of degree zero, as was found in the earlier paper quoted above.

THEOREM G. *The operation of contracting a covariant with a contravariant tensor-suffix may be performed without affecting the degree or homogeneity of a homogeneous tensor.*

This is an immediate consequence of the fact that a "contraction" is a summation.

THEOREM H. *The operations of raising and lowering the suffixes of a tensor may be performed without altering the degree or homogeneity of the tensor only in the case when the space is flat.*

For these operations consist of multiplying the tensor by $g_{\mu\nu}$ or $g^{\mu\nu}$ and contracting. And the lack of homogeneity of the latter tensors except in the case of flat space is proved above.

ON SOME INEQUALITIES IN THE THEORY OF FUNCTIONS

By H. FRAZER.

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1. Hardy and Littlewood* have shown that, *if* $f(z)=a_1z+a_2z^2+\ldots$ *is regular in* $|z|<1$ *and takes no value more than* p *times, then*

$$|a_n|<A(p)n^{\beta(p)}\mu$$

where A *and* β *are functions of* p *only, and*

$$\mu=\text{Max}\,(|a_1|,\ |a_2|,\ \ldots,\ |a_p|).$$

In this paper we consider the function $f(z)$ which is regular in $|z|<1$, takes no value more than p times, and has q zeros $(0\leqslant q\leqslant p)$; and are concerned mainly with the inequality for $|f(z)|$ from which we obtain a refinement of the above inequality for $|a_n|$.

More precisely, we show that, *if* $f(z)=a_0+a_1z+\ldots$ *is regular in* $|z|<1$, *takes no value more than* p *times, and has* q *zeros* $(0\leqslant q\leqslant p)$, *then*

(*a*) $$|f(z)|<\frac{\mu}{(1-r)^{Ap(q+1)}}\quad(\tfrac{1}{2}\leqslant|z|<1),$$

(*b*) $$|a_n|\leqslant(n+1)^{Ap(q+1)}\mu,$$

(*c*) $$|f(z)|<\frac{C}{(1-r)^{Ap}}\quad(|z|<1),$$

where A *is an absolute constant,*

$$\mu=\text{Max}\,(|a_0|,\ |a_1|,\ \ldots,\ |a_q|),$$

and C *is a constant depending only on the function.*

2. We denote by A an absolute constant, and by μ

$$\text{Max}\,(|a_0|,\ |a_1|,\ \ldots,\ |a_q|).$$

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* "A theorem concerning series of positive terms, with applications to the theory of functions", *Det Kgl. Danske Videnskabernes Selskab, Mathematisk-fysiske Meddelelser*, 7 (1925), 4.

An A, unless distinguished by a suffix, does not necessarily denote the same constant at different occurrences.

We deduce our results from the following theorem, which deals with the special case $q = 0$.

THEOREM 1. *Suppose that, in* $|z|<1$, $f(z) = a_0+a_1z+\ldots$ *is regular, takes no value more than* p *times, and has no zeros. Then*

$$|a_1|<Ap|a_0|.$$

Let

$$\psi(z) = \frac{f(z)}{a_0} = 1+\frac{a_1}{a_0}z+\ldots = e^{\phi(z)},$$

where $\phi(z) = b_1z+b_2z^2+\ldots$ is regular in $|z|<1$. $\phi(z) = 0$ when $z = 0$. Therefore $\phi(z)$ takes at most $p-1$ of the values $2\nu\pi i$ $(1\leqslant\nu\leqslant p)$. For if $\phi(z)$ took all these values, $f(z)$ would take the value a_0 $p+1$ times. That is, there is at least one of these values, $2m\pi i$ say, which $\phi(z)$ does not take. Similarly, there is at least one of the values $2(2p+\nu)\pi i$ $(1\leqslant\nu\leqslant p)$, $2(2p+r)\pi i$ say, which $\phi(z)$ does not take.

Consider the function

$$\vartheta(z) = \frac{\phi(z)-2m\pi i}{2(2p+r-m)\pi i} = -\frac{m}{2p+r-m}+\frac{b_1z}{2(2p+r-m)\pi i}+\ldots.$$

This is regular in $|z|<1$, and takes neither of the values 0 nor 1. Also

$$\left|\frac{m}{2p+r-m}\right|\leqslant\left|\frac{p}{2p+1-p}\right|<1.$$

Therefore, applying Schottky's theorem*,

$$\left|\frac{b_1}{2(2p+r-m)}\right|<A.$$

Since $b_1 = (a_1/a_0)$ the result follows.

This result is best possible except for the absolute constant A, as is shown by

$$f(z) = \left(\frac{1+z}{1-z}\right)^{2p} = 1+4pz+\ldots.$$

* Verschärfter Schottkyscher Satz; E. Landau, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie* (2 ed. 1929), 103.

3. We shall require the following two lemmas.

LEMMA 1. *Suppose that, in* $|z| \leqslant \rho \leqslant 1$, $f(z) = a_0 + a_1 z + \ldots$ *is regular and has* q *zeros, then*

$$P = \operatorname*{Min}_{|z|=\rho} |f(z)| \leqslant (q+2)2^{q-1}\mu.$$

This is a slight extension of a result obtained by Landau* [in which $\rho = 1$ and $(q+2)2^{q-1}$ is replaced by an undetermined constant depending on q]. The extension presents no difficulty.

LEMMA 2. *Suppose that, in* $|z| < 1$, $f(z) = a_0 + a_1 z + \ldots$ *is regular, takes no value more than* p *times, and has* q *zeros* $(0 \leqslant q \leqslant p)$. *Then*

$$|f(z)| < e^{Ap(q+1)}\mu \quad (|z| = \tfrac{1}{2}).$$

In the unit circle subdivide the annulus bounded by $|z| = \frac{1}{2}$ and $|z| = \frac{3}{4}$ into $q+1$ smaller annuli by circles $|z| = \frac{1}{2} + \frac{1}{4}\nu/(q+1)$ $(1 \leqslant \nu \leqslant q)$. Since $f(z)$ has q zeros only, there is at least one of these annuli in which $f(z)$ has no zeros. Choose one such annulus, which can be denoted by $\{\frac{1}{2} + \frac{1}{4}r/(q+1), \frac{1}{2} + \frac{1}{4}(r+1)/(q+1)\}$ $(0 \leqslant r \leqslant q)$. Describe the circle $|z| = \frac{1}{2} + \frac{1}{8}(2r+1)/(q+1)$. Let z_1 be any point on this circle. Then with centre z_1 and radius $A_1/(q+1)$ we can describe a circle lying entirely in $\{\frac{1}{2} + \frac{1}{4}r/(q+1), \frac{1}{2} + \frac{1}{4}(r+1)/(q+1)\}$, that is in the annulus in which $f(z)$ has no zeros.

Consider the function

$$F(z) = f\left(z_1 + \frac{A_1}{q+1}z\right) = f(z_1) + \frac{A_1}{q+1}f'(z_1)z + \ldots.$$

This is regular in $|z| < 1$, takes no value more than p times, and has no zeros. Therefore

$$\left|\frac{A_1}{q+1}f'(z_1)\right| < Ap|f(z_1)|.$$

That is

$$\frac{|f'(z_1)|}{|f(z_1)|} < Ap(q+1).$$

Let z_0 be the point on $|z| = \frac{1}{2} + \frac{1}{8}(2r+1)/(q+1)$ at which $f(z)$ is a minimum. Now

$$\log\frac{f(z_1)}{f(z_0)} = \int_C \frac{f'(u)}{f(u)}\,du = \int_\alpha^\beta \frac{f'(u)}{f(u)}\frac{du}{d\theta}\,d\theta,$$

* E. Landau, Bemerkungen zu der Arbeit des Herrn Bieberbach, "Über die Verteilung der Null- und Einstellen analytischer Funktionen", *Math. Annalen*, 86 (1922), 159.

where C is the arc of $|z|=\frac{1}{2}+\frac{1}{8}(2r+1)/(q+1)$ joining z_0 and z_1, and $z_0=|z_0|e^{i\alpha}$, $z_1=|z_1|e^{i\beta}$ $(\alpha\leqslant\beta\leqslant\alpha+2\pi)$. That is

$$\begin{aligned}\log\frac{|f(z_1)|}{|f(z_0)|}&=\int_\alpha^\beta \mathfrak{R}\frac{f'(u)}{f(u)}\frac{du}{d\theta}\,d\theta\\&\leqslant\int_\alpha^\beta\frac{|f'(u)|}{|f(u)|}\left|\frac{du}{d\theta}\right|d\theta\\&<\int_\alpha^\beta Ap(q+1)\{\tfrac{1}{2}+\tfrac{1}{8}(2r+1)/(q+1)\}\,d\theta\\&<Ap(q+1).\end{aligned}$$

Therefore

$$|f(z_1)|<|f(z_0)|e^{Ap(q+1)}\leqslant\mu e^{Ap'(q+1)},$$

by Lemma 1. This inequality is true at all points on

$$|z|=\tfrac{1}{2}+\tfrac{1}{8}(2r+1)/(q+1),$$

and therefore by the maximum modulus principle at all points in $0\leqslant|z|\leqslant\frac{1}{2}+\frac{1}{8}(2r+1)/(q+1)$, and therefore, in particular, for $|z|=\frac{1}{2}$.

4. THEOREM 2. *Suppose that, in* $|z|<1$, $f(z)=a_0+a_1z+\ldots$ *is regular, takes no value more than* p *times, and has* q *zeros* $(0\leqslant q\leqslant p)$. *Then*

$$|f(z)|<\frac{\mu}{(1-r)^{Ap(q+1)}}\quad(\tfrac{1}{2}\leqslant|z|<1).$$

Let $z_0=r_0e^{i\theta_0}$ be any point such that $\frac{3}{4}\leqslant r_0<1$. Denote the points

$$\left(r_0+\frac{1-r_0}{2}\right)e^{i\theta_0},\ \left(r_0+\frac{1-r_0}{4}\right)e^{i\theta_0},\ \left(r_0-\frac{1-r_0}{4}\right)e^{i\theta_0},\ \left(r_0-\frac{1-r_0}{2}\right)e^{i\theta_0}\text{ and }r_0e^{i\theta_0}$$

by B, D, C, A, and Z. Let P and P' be the two points whose distances from A and B are each equal to $AB=1-r_0$. With centre P describe two concentric circles with radii PB and PD. These two circles define an annulus. With the same centre P describe q equidistant circles within this annulus. In this way we obtain $q+1$ annuli. Consider only the minor arcs of the concentric circles cut off by AB. Similarly consider the $q+1$ minor arcs of the concentric circles with centre P'. In this way we obtain $q+1$ independent regions each bounded by four minor arcs. $f(z)$ has q zeros only. Therefore there exists at least one region in which $f(z)$ has no zeros. Choose one such region and, using an obvious notation, denote it by $(r_2,\ r_3)$ $\{\frac{1}{4}(1-r_0)\sqrt{13}\leqslant r_2<r_3\leqslant 1-r_0\}$. With centre P and radius $\frac{1}{2}(r_2+r_3)$ describe the minor arc of the circle cut off by AB. Similarly, with centre P' and the same radius, describe the minor arc of

the circle cut off by AB. Denote these arcs by R and R'. We note that, if z_R is any point on R or R', a circle centre z_R and radius $(1-|z_R|)/A_2(q+1)$ can be described, lying entirely in (r_2, r_3), that is in the region for which $f(z)$ has no zeros.

Draw the two tangents ZT and ZT' to the circle $|z|=\frac{1}{2}$. Bisect the angle TZO by ZU, where U is on $|z|=\frac{1}{2}$. Subdivide the angle UZO into $q+1$ equal parts by ZT_ν $(1\leqslant\nu\leqslant q)$, where T_ν is on $|z|=\frac{1}{2}$. Denote U by T_0 and V by T_{q+1}, where V is the point where OZ intersects $|z|=\frac{1}{2}$. Similarly bisect the angle $T'ZO$ by ZU', and define points T'_ν $(0\leqslant\nu\leqslant q+1)$. Let ZT_ν intersect R at a_ν and let ZT'_ν intersect R' at a'_ν. As before, there exists a region $(T_\nu a_\nu a_{\nu+1} T_{\nu+1})$ and also a region $(T'_\nu a'_\nu a'_{\nu+1} T'_{\nu+1})$ in which $f(z)$ has no zeros $(0\leqslant\nu\leqslant q)$. Bisect the angle $T_\nu ZT_{\nu+1}$ by ZQ, where Q is a point on $|z|=\frac{1}{2}$. Let the bisector intersect R at W. Bisect the angle $T'_\nu ZT'_{\nu+1}$ by ZQ', where Q' is on $|z|=\frac{1}{2}$. Let the bisector intersect R' at W'. Denote QW by Λ and $Q'W'$ by Λ'. Let C denote the path QWR_2 and C' the path $Q'W'R_2$, where R_2 is the intersection of ZB and R (or ZB and R').

Let z_1 be any point on C or C'. Then, with centre z_1 and radius $(1-|z_1|)/A_2(q+1)$, a circle can be described in which $f(z)$ has no zeros. For z_1 on WR_2, or $W'R_2$, we have seen that this is satisfied. For z_1 on Λ, let $H=z_1$. Draw HI, the perpendicular to ZT_ν or $ZT_{\nu+1}$. Then, if $c=ZH$ and ϕ is the angle TZO,

$$\begin{aligned}\frac{HI}{1-|z_1|} &= \frac{c\sin\{\phi/2(q+1)\}}{1-|z_1|}\\ &\geqslant \frac{c\sin\{\phi/2(q+1)\}}{c+(1-r_0)} = \frac{\sin\{\phi/2(q+1)\}}{1+(1-r_0)/c}\\ &> A\sin\frac{\phi}{2(q+1)} > \frac{A}{(4r_0)2(q+1)}\\ &> \frac{1}{A_2(q+1)}.\end{aligned}$$

Similarly for z_1 on Λ'.

Also the angle which OU makes with ZU is less than $\frac{1}{4}\pi$. For, denoting it by θ, we have

$$\frac{OU}{\sin\frac{1}{2}\phi} = \frac{r_0}{\sin\theta}.$$

Therefore

$$\sin\theta < \frac{r_0\sin\frac{1}{2}\phi}{\frac{1}{4}} < \frac{1}{\sqrt{2}}.$$

Similarly for OU' and ZU'.

Let $|z_1| = r_1$. Consider the function

$$F(z) = f\left(z_1 + \frac{1-r_1}{A_2(q+1)}z\right) = f(z_1) + \frac{1-r_1}{A_2(q+1)}f'(z_1)z + \ldots.$$

This is regular in $|z| < 1$, takes no value more than p times, and has no zeros. Therefore

$$\frac{|f'(z_1)|}{|f(z_1)|} < \frac{Ap(q+1)}{1-r_1}.$$

Now

$$\log\frac{f(z_1)}{f(Q)} = \int_C \frac{f'(u)}{f(u)}\,du = \int_{\frac{1}{2}}^{r_1} \frac{f'(u)}{f(u)}\frac{du}{dr}\,dr,$$

where the contour is the part of C joining Q to z_1. Therefore

$$\log\frac{|f(z_1)|}{|f(Q)|} = \int_{\frac{1}{2}}^{r_1} \mathfrak{R}\,\frac{f'(u)}{f(u)}\frac{du}{dr}\,dr \quad (u = re^{i\theta})$$

$$\leqslant \int_{\frac{1}{2}}^{r_1} \left|\frac{f'(u)}{f(u)}\right| \left|\frac{du}{dr}\right| dr.$$

We note that $|du/dr| = \sec\phi$, where ϕ is the angle which the radius vector through O makes with the tangent to the path. By construction $|du/dr| < 2$. Therefore

$$\log\frac{|f(z_1)|}{|f(Q|} < \int_{\frac{1}{2}}^{r_1} \frac{Ap(q+1)}{1-|u|}\,2dr$$

$$< \int_{\frac{1}{2}}^{r_1} \frac{Ap(q+1)}{1-r}\,dr$$

$$< Ap(q+1)\log\left(\frac{\frac{1}{2}}{1-r_1}\right).$$

Therefore

$$|f(z_1)| < \frac{|f(Q)|}{(1-r_1)^{Ap(q+1)}} < \frac{\mu e^{A_3 p(q+1)}}{(1-r_1)^{Ap(q+1)}},$$

by Lemma 2. Similarly for z_1 on C'. Also

$$|f(z)| < \mu e^{A_3 p(q+1)},$$

for all z on the path QQ', where by the path QQ' we mean the path along $|z| = \frac{1}{2}$. Therefore the maximum value of $|f(z)|$ on the boundary (QQ', C, C') is less than

$$\frac{\mu e^{A_3 p(q+1)}}{(1-r_{R_2})^{Ap(q+1)}} < \frac{\mu}{(1-r_{R_2})^{Ap(q+1)}},$$

where r_{R_2} is the value of $|z|$ at R_2. Since z_0 lies inside the domain bounded by (QQ', C, C'),

$$|f(z_0)| < \frac{\mu}{(1-r_{R_2})^{Ap(q+1)}} < \frac{\mu}{(1-r_0)^{Ap(q+1)}} \quad (\tfrac{3}{4} \leqslant r_0 < 1)$$

by the maximum modulus principle.

The inequality can easily be extended to $\frac{1}{2} \leqslant r < 1$. For

$$|f(z)| < \frac{\mu}{(1-\frac{3}{4})^{Ap(q+1)}} \quad (\tfrac{1}{2} \leqslant |z| < \tfrac{3}{4})$$

$$< \frac{\mu}{(1-\frac{1}{2})^{Ap(q+1)}}.$$

This proves the theorem.

THEOREM 3. *Suppose that, in* $|z|<1$, $f(z) = a_0+a_1z+\ldots$ *is regular, takes no value more than* p *times, and has* q *zeros* $(0 \leqslant q \leqslant p)$. *Then*

$$|a_n| \leqslant (n+1)^{Ap(q+1)}\mu.$$

$$|a_n| \leqslant \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|\, r^{-n}\, d\theta$$

$$< \frac{\mu}{r^n(1-r)^{Ap(q+1)}} \quad (r \geqslant \tfrac{1}{2}).$$

Let $r = n/(n+1)$. Then $r \geqslant \frac{1}{2}$ $(n \geqslant 1)$,

$$|a_n| < \frac{\mu}{\{n/(n+1)\}^n \{1/(n+1)\}^{Ap(q+1)}}$$

$$= \mu\left(1+\frac{1}{n}\right)^n (n+1)^{Ap(q+1)}$$

$$< \mu e(n+1)^{Ap(q+1)}$$

$$< \mu(n+1)^{Ap(q+1)} \quad (n \geqslant 1).$$

The inequality is also true for $n = 0$ if we replace $<$ by $\leqslant$. For clearly

$$|a_0| \leqslant \mu.$$

In our final theorem we are concerned with replacing the index $Ap(q+1)$ of $1-r$ in Theorem 2 by Ap. We cannot do this, however, without replacing μ by a different constant C. The index Ap cannot be reduced further, except as regards the absolute constant A, as is shown by

$$f(z) = \frac{z^p}{(1-z)^{2p}}.$$

THEOREM 4. *Suppose that, in* $|z|<1$, $f(z)=a_0+a_1z+\ldots$ *is regular, takes no value more than* p *times, and has* q *zeros* $(0\leqslant q\leqslant p)$. *Then*

$$|f(z)|<\frac{C}{(1-r)^{Ap}} \quad (|z|<1),$$

where C *is a constant depending only on the function.*

Let the zero of the function with the largest modulus in $|z|<1$ be on the circle $|z|=a$. Then $f(z)$ has no zeros in $0<a<1$. Let $z_1=r_1e^{i\theta_1}$ be any point in $\frac{1}{2}(1+a)\leqslant r_1<1$.

Consider the function

$$F(z)=f\{z_1+(1-r_1)z\}=f(z_1)+(1-r_1)f'(z_1)z+\ldots.$$

This is regular in $|z|<1$, takes no value more than p times, and has no zeros. Therefore

$$\frac{|f'(z_1)|}{|f(z_1)|}<\frac{Ap}{1-r_1}.$$

Now

$$\log\frac{f(z_1)}{f[\frac{1}{2}(1+a)e^{i\theta_1}]}=\int_C\frac{f'(u)}{f(u)}\,du=\int_{\frac{1}{2}(1+a)}^{r_1}\frac{f'(u)}{f(u)}\frac{du}{dr}\,dr,$$

where C is the straight line joining $\frac{1}{2}(1+a)e^{i\theta_1}$ and $r_1e^{i\theta_1}$. That is

$$\begin{aligned}\log\frac{|f(z_1)|}{|f[\frac{1}{2}(1+a)e^{i\theta_1}]|}&=\int_{\frac{1}{2}(1+a)}^{r_1}\mathfrak{R}\frac{f'(u)}{f(u)}\frac{du}{dr}\,dr\\&\leqslant\int_{\frac{1}{2}(1+a)}^{r_1}\left|\frac{f'(u)}{f(u)}\right|\left|\frac{du}{dr}\right|dr\\&<Ap\log\left(\frac{\frac{1}{2}(1+a)}{1-r_1}\right).\end{aligned}$$

Therefore

$$\begin{aligned}|f(z_1)|&<\frac{|f[\frac{1}{2}(1+a)e^{i\theta}]|}{(1-r_1)^{Ap}} \quad [\tfrac{1}{2}(1+a)\leqslant|z_1|<1]\\&\leqslant\frac{C}{(1-r_1)^{Ap}},\end{aligned}$$

where C is the maximum of $|f(z)|$ on $|z|=\frac{1}{2}(1+a)$.

Also

$$|f(z)|<C \quad [0\leqslant|z|<\tfrac{1}{2}(1+a)].$$

Therefore

$$|f(z)|<\frac{C}{(1-r)^{Ap}} \quad (|z|<1).$$

I should like to express my thanks to Mr. A. E. Ingham for suggesting the problem to me and assisting me considerably throughout the investigations.

ON POLAR SINGULARITIES OF POWER SERIES AND OF DIRICHLET SERIES

By G. PÓLYA.

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1. This paper consists of three sections, each of which may be read independently of the others. But actually they illustrate from various sides the same fact, expressed by the following theorem†.

THEOREM A. *Let $\omega(u)$ be a real-valued function, defined for $u \geqslant 1$, that undergoes a finite number $W(x)$ of changes of sign as u varies from 1 to x.*

Suppose that the integral

$$\int_1^\infty \omega(u)\, u^{-s}\, du = \Phi(s)$$

converges for some values of s, and that the function $\Phi(s)$ defined by it is meromorphic in a certain half-plane $\mathfrak{R}s > b$.

If $W(x) = O(\log x)$ we have the following alternatives: either $\Phi(s)$ is regular for $\mathfrak{R}s > b$, or it has a pole $\beta + i\gamma$ of maximum abscissa β [i.e. $\Phi(s)$ is regular for $\mathfrak{R}s > \beta$], and such that

$$0 \leqslant \gamma \leqslant \pi \varlimsup_{x\to\infty} \frac{W(x)}{\log x}.$$

In the first of the following three sections I shall obtain, by an easy transformation of Theorem A, a relation between the polar singularities of a Dirichlet series and the zeros of the partial sums of the corresponding power series. The second section deals with a theorem on the zeros of trigonometrical sums that, in certain particular cases, leads to more precise conclusions than Theorem A. In the last section a theorem on power series with polar singularities is considered that is analogous to

† G. Pólya, *Göttinger Nachrichten*, Math.-Phys. Klasse, 1930, 19–27.

Theorem A. The proofs in the second and third section exhibit a certain parallelism.

I. *On the power series corresponding to a Dirichlet series.*

2. An ordinary Dirichlet series

$$a_1 1^{-s}+a_2 2^{-s}+\ldots+a_n n^{-s}+\ldots = D(s) \tag{1}$$

is linked to the corresponding power series

$$a_1 z+a_2 z^2+\ldots+a_n z^n+\ldots = P(z) \tag{2}$$

by the well known formula

$$\Gamma(s)\, D(s) = \int_0^\infty P(e^{-x})\, x^{s-1}\, dx, \tag{3}$$

to which Theorem A may be applied (put $x^{-1} = u$). It gives a sufficient condition for the existence of a pole of maximum abscissa of the function $\Gamma(s)\, D(s)$ supposed meromorphic, the condition involving the positive roots of $P(z)$, or rather its positive roots of odd order, counted without multiplicity. This application is so immediate that it is unnecessary to give details. I shall prove a further transformation of this condition that involves, instead of the roots of $P(z)$, the roots of its partial sums or "sections". It may be stated thus:

Theorem I. *Suppose that*

(*a*) $a_1, a_2, a_3, \ldots$ *are real numbers such that*

$$|a_n| \leqslant a \quad (n = 1, 2, 3, \ldots). \tag{4}$$

(*b*) *The Dirichlet series* (1) [which is, in virtue of (4), convergent for $\Re s > 1$] *represents a function* $D(s)$ *meromorphic for* $\Re s > b$, *where* $b < 1$.

(*c*) *Let* p_n *be the number of the roots of the polynomial*

$$f_n(x) = a_1 x+a_2 x^2+\ldots+a_n x^n \tag{5}$$

in the interval $0 < x < 1$, *and suppose that there is a sequence of integers* $n_1, n_2, n_3, \ldots$ *such that*

$$0 < n_1 < n_2 < n_3 < \ldots$$

$$\lim_{m\to\infty} \frac{\log n_{m+1}}{\log n_m} = 1, \tag{6}$$

$$p_{n_m} = O(\log n_m) \tag{7}$$

as m *tends to infinity.*

Under these conditions (*a*), (*b*), (*c*) *we have the following alternatives: either* $\Gamma(s)\,D(s)$ *is regular for* $\Re s > b$, *or it has a pole* $\beta+i\gamma$ *of maximum abscissa* β *whose ordinate* γ *satisfies*

$$0 \leqslant \gamma \leqslant \pi \varlimsup_{m\to\infty} \frac{p_{n_m}}{\log n_m}. \tag{8}$$

3. Putting

$$P(e^{-1/u}) = a_1 e^{-1/u} + a_2 e^{-2/u} + \ldots + a_n e^{-n/u} + \ldots = F(u),$$

we obtain from (3)

$$\Gamma(s)\,D(s) = \int_0 F(u)\,u^{-1-s}\,du. \tag{9}$$

Take a positive number B, such that

$$-B < b, \tag{10}$$

and put

$$F^*(u) = \begin{cases} F(u) & \text{if } |F(u)| > u^{-B}, \\ 0 & \text{if } |F(u)| \leqslant u^{-B}; \end{cases}$$

then

$$\int_1^\infty |F(u) - F^*(u)|\,u^{-1-\sigma}\,du \leqslant \int_1^\infty u^{-(B+\sigma)-1}\,du$$

converges for $\sigma > -B$. Thus we obtain from (9)

$$\int_1^\infty F^*(u)\,u^{-1-s}\,du = \Gamma(s)\,D(s) + \phi(s), \tag{11}$$

where $\phi(s)$ is regular for $\Re s \geqslant b$, so that, by hypothesis (*b*), the function (11) is meromorphic for $\Re s > b$.

Let $W(x)$ be the number of changes of sign of $F^*(u)$ in the interval $1 < u < x$. In order to apply Theorem A, stated in the introduction, to (11) we have to examine the order of magnitude of $W(x)$.

Let ϵ be a given number, $0 < \epsilon < 1$, and put

$$a = 1-\epsilon. \tag{12}$$

If $n_{m-1}^a \leqslant x \leqslant n_m^a$, we have

$$\frac{W(x)}{\log x} \leqslant \frac{W(n_m^a)}{\log n_m^a} \frac{\log n_m}{\log n_{m-1}},$$

and therefore, using (6), we obtain

$$\varlimsup_{x\to\infty} \frac{W(x)}{\log x} \leqslant \varlimsup_{m\to\infty} \frac{W(n_m^a)}{a \log n_m}. \tag{13}$$

(We could replace $\leqslant$ by $=$, but this is not necessary for our present purpose.)

In order to examine the right-hand side of (13) we restrict u to the interval

$$1 \leqslant u \leqslant n^a. \tag{14}$$

We put

$$x = e^{-1/u}; \tag{15}$$

x is restricted to the interval

$$e^{-1} \leqslant x \leqslant e^{-n^{-a}}. \tag{16}$$

The variables u and x being connected by (15) and restricted by (14) and (16), we find, by using (4), that

$$\begin{aligned} |F(u)-f_n(x)| &= |a_{n+1}x^{n+1}+a_{n+2}x^{n+2}+\dots| \\ &< \frac{ax^n}{1-x} \leqslant \frac{ae^{-nn^{-a}}}{1-e^{-n^{-a}}} \\ &< 2n^a a e^{-n^{1-a}} < n^{-aB} \leqslant u^{-B}, \end{aligned}$$

if n is sufficiently large. We conclude

If $F^*(u) \neq 0$ for a u in the interval (14) *then $F^*(u)$ and $f_n(x)$ are of the same sign.* Thus each change of sign of $F^*(u)$ in the interval (14) involves a corresponding root of $f_n(x)$ in the interval (16), so that

$$W(n^a) \leqslant p_n$$

and

$$\frac{W(n^a)}{a \log n} \leqslant \frac{p_n}{a \log n}. \tag{17}$$

We infer from (13), (17), and (12) that

$$\overline{\lim_{x\to\infty}} \frac{W(x)}{\log x} \leqslant \frac{1}{1-\epsilon} \overline{\lim_{m\to\infty}} \frac{p_{n_m}}{\log n_m}.$$

From this and Theorem A we reach the conclusion (8), observing that ϵ may be as small as we please.

4. I shall show by an example that equality in (8) is possible. The function

$$\Gamma(s)\,D(s) = \Gamma(s)\,\frac{1-2^{-s}}{1+2^{-s}} = \Gamma(s)(1-2\,.\,2^{-s}+2\,.\,4^{-s}-2\,.\,8^{-s}+\dots)$$

is meromorphic in the whole plane. The abscissae of its poles have a maximum, namely 0, and among all its poles of this maximum abscissa 0, contained in the closed upper half-plane, the minimum of the ordinates is reached by

$$\beta+i\gamma=\frac{\pi i}{\log 2}. \tag{18}$$

If we put

$$n_m=2^m, \tag{19}$$

condition (6) is fulfilled. The polynomial

$$f_{2^m}(x)=x-2x^2+2x^4-2x^8+\ldots+(-1)^m 2x^{2^m}$$

has, by Descartes' rule, no more than m positive zeros, so that certainly

$$p_{2^m}\leqslant m=\frac{\log 2^m}{\log 2} \tag{20}$$

and the condition (7) is satisfied. Comparing the values (18), (19), (20), we see that, in the present case, equality is attained in (8).

The series

$$\mu(1)1^{-s}+\mu(2)2^{-s}+\ldots+\mu(n)n^{-s}+\ldots=1/\zeta(s)$$

may be used as a further illustration of Theorem I. It fulfils the conditions (a), (b); we do not yet know exactly the situation of its poles, but if there is a pole of maximum abscissa its ordinate is certainly not zero, and consequently, by Theorem I, the number of the real roots of the polynomial

$$\mu(1)x+\mu(2)x^2+\mu(3)x^3+\ldots+\mu(n)x^n \tag{21}$$

between 0 and 1 is certainly not $o(\log n)$. We may even affirm that the number of these roots is not $O(\log n)$ if there is no pole of maximum abscissa. We should have this latter case if the Riemann hypothesis were so compietely false, that 1 was the upper bound of the real parts of the zeros of $\zeta(s)$.

II. *On the zeros of certain trigonometrical sums.*

5. The starting point of this section was the following well known fact:

The trigonometrical polynomial

$$c_m\cos(mx+\delta_m)+c_{m+1}\cos\{(m+1)x+\delta_{m+1}\}+\ldots+c_n\cos(nx+\delta_n)$$

(*where* c_μ, δ_μ *are real,* $c_m^2>0$, $c_n^2>0$, $0\leqslant m\leqslant n$) *has at least* $2m$ *and*

at most $2n$ *zeros in the interval* $0 \leqslant x < 2\pi$. This fact, mentioned by Hurwitz†, may be generalized in various ways. It is contained in a general theorem, on Sturm-Liouville functions, which is indeed historically prior: Hurwitz obtained his theorem by specialization from the work of Sturm‡ (and gave an independent proof afterwards). The present section is concerned with various extensions, of which I shall prove first the following theorem:

THEOREM II. *Let* $c_1, c_2, c_3, \ldots, \gamma_1, \gamma_2, \gamma_3, \ldots, \delta_1, \delta_2, \delta_3, \ldots$ *be real numbers,*

$$|c_1|+|c_2|+|c_3|+\ldots \text{ convergent}, \quad c_1^2 > 0, \quad 0 < \gamma_1 < \gamma_2 < \gamma_3 < \ldots;$$

and assume that the real-valued function $g(x)$ *is continuous for* $x > 0$, *and that*

$$g(x) = O(e^{-ax}) \tag{1}$$

as x *tends to* ∞, a *being a positive constant. Then the function*

$$f(x) = g(x)+c_1 \cos(\gamma_1 x+\delta_1)+c_2 \cos(\gamma_2 x+\delta_2)+\ldots$$

changes sign at least

$$\frac{\gamma_1 t}{\pi} - A$$

times, as x *increases from* 0 *to* t, A *being independent of* t.

After having illustrated Theorem II by an example taken from the theory of the ξ-function (Art. 7), I shall state some analogous results with a brief indication of the proof (Arts. 8, 9).

6. In order to prove Theorem II, we make use of Rolle's theorem in the following form:

LEMMA 1. *If* $f(x)$ *undergoes* V *changes of sign as* x *varies from* 0 *to* t, *then*

$$f^{(-1)}(x) = \int_0^x f(x)\,dx + C,$$

where C *is any constant, undergoes at most* $V+1$ *changes of sign in the same interval.*

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† A. Hurwitz, *Math. Annalen*, 57 (1903), 425–446 (444).

‡ Ch. Sturm, *Journal de Math.*, 1 (1836), 373–444 (431–444).

Put

$$g^{(0)}(x) = g(x), \tag{2}$$

$$g^{(-l-1)}(x) = -\int_x^\infty g^{(-l)}(x)\,dx \quad (l = 0, 1, 2, 3, \ldots). \tag{3}$$

The definition (3) is legitimate; starting from the hypothesis (1), we may prove by complete induction that

$$g^{(-l)}(x) = O(e^{-\alpha x}) \quad (l = 1, 2, 3, \ldots), \tag{4}$$

and so the integral (3) converges. Put

$$f^{(-l)}(x) = g^{(-l)}(x) + \sum_{\mu=1}^{\infty} \frac{c_\mu}{\gamma_\mu^l} \cos\left(\gamma_\mu x + \delta_\mu - l\frac{\pi}{2}\right) \quad (l = 1, 2, 3, \ldots). \tag{5}$$

By this definition
$$\frac{df^{(-l-1)}(x)}{dx} = f^{(-l)}(x),$$

that is, $f^{(-l)}(x)$ arises from $f(x)$ by repeated integration. Applying Lemma 1 l times, we see that, in any interval, the number of changes of sign of $f^{(-l)}(x)$ exceeds that of $f(x)$ at most by l units.

Now take l so large that

$$|c_1| > \left(\frac{\gamma_1}{\gamma_2}\right)^l |c_2| + \left(\frac{\gamma_1}{\gamma_3}\right)^l |c_3| + \left(\frac{\gamma_1}{\gamma_4}\right)^l |c_4| + \ldots;$$

in virtue of (4), there is an x_0, such that

$$|g(x)| < \frac{|c_1|}{\gamma_1^l} - \frac{|c_2|}{\gamma_2^l} - \frac{|c_3|}{\gamma_3^l} - \ldots \quad \text{for} \quad x \geqslant x_0$$

and
$$\cos(\gamma_1 x_0 + \delta_1 - \tfrac{1}{2}\pi l) = 1.$$

This being so, $f^{(-l)}(x)$, defined by (5), becomes alternately positive and negative as x takes the values

$$x_0, \quad x_0 + \frac{\pi}{\gamma_1}, \quad x_0 + \frac{2\pi}{\gamma_1}, \quad x_0 + \frac{3\pi}{\gamma_1}, \quad \ldots.$$

Therefore $f^{(-l)}(x)$ changes sign at least once between two successive points $x_0 + (m\pi/\gamma_1)$ and $x_0 + \{(m+1)\pi/\gamma_1\}$, the number of its variations (= changes of sign) in the interval $0 < x < t$ is not less than $(\gamma_1 t/\pi) - O(1)$, and the number of variations of $f(x)$ in the same interval is, by what we said before, not less than $(\gamma_1 t/\pi) - O(1) - l$. Thus we have proved Theorem II.

The method of proof which we have used is due to Liouville†; we shall have other applications of it in Arts. 8 and 10–13.

7. Theorem II gives information about the behaviour of the series

$$\sum_{n=1}^{\infty} \{\Lambda(n)-1\}\, e^{-ny} \tag{6}$$

as the positive variable y tends to 0. The $\Lambda(n)$ are defined by

$$\sum_{1}^{\infty} \Lambda(n)\, n^{-s} = -\zeta'(s)/\zeta(s),$$

and the behaviour of (6) is closely connected with the remainder term in the prime number theorem.

Assuming the Riemann hypothesis, we have‡

$$y^{\frac{1}{2}} \sum_{n=1}^{\infty} \{\Lambda(n)-1\}\, e^{-ny} = -\sum_{m=1}^{\infty} \{\Gamma(\tfrac{1}{2}+i\gamma_m)\, y^{-i\gamma_m} + \Gamma(\tfrac{1}{2}-i\gamma_m)\, y^{i\gamma_m}\} + O(y^{\frac{1}{4}}) \tag{7}$$

$$= 2\Sigma' c_\mu \cos\left(\gamma_\mu \log\frac{1}{y} + \delta_\mu\right) + O(e^{-\frac{1}{4}\log(1/y)}),$$

where $\frac{1}{2}+i\gamma_1$, $\frac{1}{2}+i\gamma_2$, ... are the roots of $\zeta(s)$ in the upper half-plane, $0 < \gamma_1 \leqslant \gamma_2 \leqslant \gamma_3 \leqslant \ldots$. The sum Σ' is extended over all *different* roots of $\zeta(s)$ in the upper half-plane; δ_μ is the amplitude of $-\Gamma(\frac{1}{2}+i\gamma_\mu)$ and c_μ is its modulus, multiplied by the multiplicity of the root $\frac{1}{2}+i\gamma_\mu$. Thus $c_\mu > 0$ and $\Sigma' c_\mu$ is convergent.

Let $N(x)$ be the number of different roots of odd order of (6) in the infinite interval $y > x$. Applying Theorem II to (7), we find that

$$N(x) \geqslant \frac{\gamma_1}{\pi} \log\frac{1}{x} - O(1) \tag{8}$$

as x tends to 0.

The application of Theorem A to

$$-\Gamma(s)\left(\frac{\zeta'(s)}{\zeta(s)}+\zeta(s)\right) = \int_0^\infty \sum_1^\infty \{\Lambda(n)-1\}\, e^{-n/u}\, u^{-1-s}\, du$$

gives only

$$\overline{\lim_{x\to 0}} \frac{N(x)}{\log(1/x)} \geqslant \frac{\gamma_1}{\pi}, \tag{9}$$

but *without* assuming the Riemann hypothesis. The meaning of γ_1 is in

† J. Liouville, *Journal de Math.*, 1 (1836), 269–277.

‡ G. H. Hardy and J. E. Littlewood, *Acta Math.*, 41 (1918), 119–196 (134–137).

(9) not necessarily the same as in (8). In (9) γ_1 means either ∞ if there is no root of $\xi(s)$ having a maximum abscissa; or, if such roots exist, γ_1 means the ordinate of that root which, among all roots having this maximum abscissa and situated in the upper half-plane, has the minimum ordinate†. Thus γ_1 means the same thing in (8) and (9) *if* the Riemann hypothesis is assumed.

It may be observed that, if the Riemann hypothesis is true, (7) is, in a certain strip, the sum of an almost periodic function of $\log(1/y)$ and of a function that tends uniformly to 0 with y. From this, it follows by well known methods‡ that the left-hand side of (9) is finite.

We could add analogous remarks on other series connected with the ξ-function. *E.g.* if we assume more than the Riemann hypothesis, namely the so-called Mertens hypothesis§, results analogous to (7) and (8) can be proved for the series

$$\mu(1)\,e^{-y}+\mu(2)\,e^{-2y}+\mu(3)\,e^{-3y}+\ldots.$$

8. The infinite trigonometrical sum arising in Theorem II may be considered as a limiting case of the trigonometrical polynomials mentioned at the beginning of Art. 5. In other limiting cases terms of the form

$$P(x)\cos\gamma x+Q(x)\sin\gamma x \tag{10}$$

may arise, where $P(x)$ and $Q(x)$ designate rational integral functions of x. Let us call the higher of the degrees of $P(x)$ and $Q(x)$ (or their common degree) *the degree of* (10).

The following theorem completes Theorem II in some respects.

THEOREM II*. *Let* $\gamma_1, \gamma_2, \ldots, \gamma_M$ *be real numbers*

$$0<\gamma_1<\gamma_2<\ldots<\gamma_M, \tag{11}$$

$P_m(x)$, $Q_m(x)$ *polynomials with real coefficients, and* $N(t)$ *the number of the zeros of the function*

$$f(x)=\sum_{m=1}^{M}\{P_m(x)\cos\gamma_m x+Q_m(x)\sin\gamma_m x\}+g(x) \tag{12}$$

in the interval $0\leqslant x\leqslant t$. *We assume that* $g(z)$ *is an analytic function of the complex variable* $z=x+iy$, *real valued for* $y=0$, *regular in the*

† Compare Pólya, *Göttinger Nachrichten*, Math. Phys. Klasse, 1930, 19–21.

‡ By using a suitable generalization of Jensen's theorem on the zeros of analytic functions, see, *e.g.*, H. Bohr und B. Jessen, *Acta Math.*, 54 (1930), 1–35, Hilfssatz 2 on 18.

§ See H. Cramér and E. Landau, *Arkiv för Mat.*, 15 (1921), No. 28.

strip $-1 \leqslant y \leqslant 1$, $x \geqslant 0$, *and that*

$$g(x+iy) = O(e^{-ax}) \tag{13}$$

uniformly in this strip as x *tends to* ∞; a *is a positive constant.*

Consider now the terms of highest degree among those arising in the sum on the right-hand side of (12), *and,* m *being restricted to these highest terms, let* γ' *be the minimum and* γ'' *the maximum of* γ_m. *Then there is a constant* A *such that*

$$\frac{\gamma' t}{\pi} - A < N(t) < \frac{\gamma'' t}{\pi} + A. \tag{14}$$

We have tacitly assumed that $P_m^2(x)+Q_m^2(x)$ $(m = 1, 2, \ldots, M)$ does not vanish identically.

Theorem II* could be completed in various ways. The first inequality (14) is true even if we count in $N(t)$ only the changes of sign, that is the zeros of odd order, considered without multiplicity, and the second inequality (14) remains true if we count in $N(t)$ all zeros of the interval with multiplicity; the hypotheses on $g(x)$ could be lessened if we had in view only one half of the double inequality (14); and so on. But I have stated Theorem II* in this rather clumsy form, in order to emphasize its analogy with the more interesting Theorem III in Art. 10, where, in fact, the same function (12) is considered, but only for integral values, not for continuously increasing values of the variable.

The proof of Theorem II* proceeds on the same lines as that of Theorem II. Since we have now to estimate $N(t)$ from *both* sides, we must consider not only the repeated integrals $f^{(-1)}$, $f^{(-2)}$, ... as for Theorem II, but also the successive derivatives. By Lemma 1, if we pass along the series of functions

$$f^{(-l)},\ f^{(-l+1)},\ \ldots,\ f^{(-1)},\ f,\ f',\ \ldots,\ f^{(l-1)},\ f^{(l)}$$

from the left to the right and *disregard quantities which remain bounded for* $x \to \infty$, the number of roots *increases.* Theorem II* can be proved by showing that for a sufficiently large positive l

(I) in $f^{(-l)}(x)$ the term corresponding to γ' is predominant, and the number of zeros in the interval $0 \leqslant x \leqslant t$ is $(\gamma' t/\pi)+O(1)$;

(II) in $f^{(l)}(x)$ the term corresponding to γ'' is predominant, and the number of zeros in the interval $0 \leqslant x \leqslant t$ is $(\gamma'' t/\pi)+O(1)$;

(III) both $f^{(-l)}(x)$ and $f^{(l)}(x)$ have only a finite number of multiple positive zeros.

I do not give details because the whole argument must be repeated in Arts. 11–13 for a variable which takes only the values 0, 1, 2, 3, ... instead of increasing continuously. I mention only the following lemmas which are useful in establishing (I), (II), (III).

LEMMA 2. *If* $\gamma > 0$, δ *is real, and*

$$f(z) = \sin(\gamma z + \delta) + g(z),$$

where the function $g(z)$ *is analytic in the strip*

$$x \geqslant 0, \quad -h \leqslant y \leqslant h \tag{15}$$

($z = x+iy$, $h > 0$) *and satisfies the inequality*

$$|g(z)| < \text{Min}\{1, \tfrac{1}{2}(e^{\gamma h} - e^{-\gamma h})\},$$

then the positive zeros of $f(z)$ *are, except possibly a finite number, simple, and their number in the interval* $0 \leqslant x \leqslant t$ *is*

$$\frac{\gamma t}{\pi} + O(1).$$

LEMMA 3. *If* C, C_1, ..., C_M, δ, δ_1, ..., δ_M *are real,* γ, γ_1, ..., γ_M *are positive,*

$$f(z) = C\cos(\gamma z + \delta) + \sum_{m=1}^{M} C_m \cos(\gamma_m z + \delta_m) + g(z),$$

$g(z)$ *is analytic in the strip* (15) *and tends there uniformly to* 0 *as* $x \to \infty$, *and finally*

$$\sum_{m=1}^{M} |C_m|(e^{\gamma_m h} + e^{-\gamma_m h}) < \text{Min}\{2|C|, \ |C|(e^{\gamma h} - e^{-\gamma h})\},$$

then the number of zeros of $f(z)$ *in the interval* $0 \leqslant x \leqslant t$ *is*

$$\frac{\gamma t}{\pi} + O(1).$$

Lemma 2 is easily obtained by applying Rouché's theorem to the rectangle whose sides are

$$x = \{(m-\tfrac{1}{2})\pi - \delta\}/\gamma, \quad x = \{(m+\tfrac{1}{2})\pi - \delta\}/\gamma, \quad y = -h, \quad y = h,$$

m being a positive integer. Lemma 3 is an immediate consequence of Lemma 2.

9. I observe that the constant A arising in (14) may be made numerically precise in certain particular cases, as is shown by the following theorem:

THEOREM II**. *Let* γ_m, $P_m(z)$, $Q_m(z)$, $f(z)$, $N(t)$ *have the same meaning as in Theorem II*, but suppose now that* $g(z)$ *vanishes identically. Let*

p_m be the degree of the general term of the sum on the right-hand side of (12), *and put*

$$p_1+p_2+\ldots+p_M+M=n\ ;$$

then

$$\frac{\gamma_1 t}{\pi}-(2n-1)\leqslant N(t)\leqslant\frac{\gamma_M t}{\pi}+(2n-1). \tag{16}$$

We may prove the first inequality by considering the expression

$$w=\sum_{m=1}^{M} e^{i\gamma_m z}\{P_m(z)-iQ_m(z)\},$$

whose real part for a real $z=x$ is just $f(x)$, and the rectangle whose corners are 0, t, $t+iH$, and iH, H being a sufficiently large positive number. If z follows the boundary of this rectangle in the positive direction avoiding the zeros of w, then w describes a curve in the w-plane that may cross the imaginary axis. We prove the first inequality (16) by counting these crossings, in the same manner as the second inequality, which is known, was proved†.

I mention finally the following result.

THEOREM II***. *Suppose that $0<a<b$, that the real-valued functions $\phi(x)$ and $\psi(x)$ possess integrable derivatives for $a\leqslant x\leqslant b$, and that*

$$\phi(a)^2+\psi(a)^2>0,\quad \phi(b)^2+\psi(b)^2>0.$$

Let $N(t)$ be the number of zeros of the function

$$f(x)=\int_a^b\{\phi(u)\cos xu+\psi(u)\sin xu\}\,du$$

in the interval $0\leqslant x\leqslant t$. Then there is a constant A such that

$$\frac{at}{\pi}-A<N(t)<\frac{bt}{\pi}+A.$$

To prove Theorem II***, integrate by parts and follow the lines of Arts. 6 and 8‡.

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† See Problem 24 proposed by G. Pólya, *Jahresbericht d. deutschen Mathem. Verein*, 34 (1925), *97*, and its solution by N. Obreschkoff, *ibid.*, 37 (1928), *82–84*.

‡ The upper limitation of $N(t)$ is not new; see Theorem VIII of a paper by Mary L. Cartwright, *Quart. J. of Math.* (Oxford Series), 1 (1930), 38–59. See also G. Pólya, *Math. Zeitschrift*, 2 (1918), 352–383, and E. C. Titchmarsh, *Proc. London Math. Soc.* (2), 25 (1926), 283–302.

III. *On the changes of sign of the coefficients in certain power series.*

10. The starting point of this section was a theorem due to J. König† which may be generalized as follows:

THEOREM III. *Consider the power series*

$$a_0+a_1z+a_2z^2+\ldots+a_nz^n+\ldots=f(z), \tag{1}$$

whose coefficients $a_0, a_1, a_2, \ldots$ *are real and whose radius of convergence is* 1. *Suppose that* $f(z)$ *has only poles and no singularities of any other kind on the circle* $|z|=1$; *it has then (since the* a_n *are real) some poles on the closed upper half of this circle. Consider those among these poles whose order is a maximum, and let* $e^{i\gamma'}$ *be the nearest one to the point* 1 *and* $e^{i\gamma''}$ *the farthest one from it* $(0\leqslant\gamma'\leqslant\gamma''\leqslant\pi)$. *Let the first* $n+1$ *coefficients*

$$a_0,\ a_1,\ a_2,\ \ldots,\ a_n$$

have V_n *variations (i.e. changes of sign;* $0\leqslant V_n\leqslant n$.) *Then*

$$\frac{\gamma'}{\pi}\leqslant\varliminf_{n\to\infty}\frac{V_n}{n}\leqslant\varlimsup_{n\to\infty}\frac{V_n}{n}\leqslant\frac{\gamma''}{\pi}. \tag{2}$$

I shall prove this theorem on the supplementary hypothesis that *the point* $z=1$ *is not a pole of* $f(z)$ *if* $\gamma'>0$‡.

The case considered by J. König was the very special one in which $f(z)$ is the reciprocal of a polynomial having only simple roots and just two roots of maximum modulus conjugate to each other, so that $0<\gamma'=\gamma''<\pi$ and V_n/n has a limit. I shall prove the theorem by an argument parallel to that used in the preceding section. In fact, I shall prove more than (2), namely the existence of a constant A, independent of n, such that

$$\frac{\gamma' n}{\pi}-A<V_n<\frac{\gamma'' n}{\pi}+A \tag{3}$$

for $n=1, 2, 3, \ldots$.

11. The following Lemmas 1′, 2′, 3′ are analogous to the lemmas used in the preceding section.

LEMMA 1′. *If*

$$b_n=b_0+a_1+a_2+\ldots+a_n\quad(n=1, 2, 3, \ldots),$$

then the sequence $b_0, b_1, b_2, \ldots, b_n$ *has at most one change of sign more than the sequence* $a_0, a_1, a_2, \ldots, a_n$.

† J. König, *Math. Annalen* 9 (1876), 530–540. The analogous statement I gave formerly in the *Göttinger Nachrichten* (1930), 27, was less precise than the present one.

‡ The theorem can be proved *without* the supplementary hypothesis, even for algebraic-logarithmic singularities instead of poles.

Assume that b_j is positive and that b_j and b_k *terminate* two *successive* changes of sign; that is (if we assume $k-j>1$)

$$b_{j-1}\leqslant 0,\quad b_j>0,\quad b_{j+1}\geqslant 0,\quad \ldots,\quad b_{k-1}\geqslant 0,\quad b_k<0.$$

Then $$a_j=b_j-b_{j-1}>0,\qquad a_k=b_k-b_{k-1}<0.$$

and therefore there is at least one change of sign in

$$a_j,\quad a_{j+1},\quad \ldots,\quad a_{k-1},\quad a_k.$$

This conclusion is, of course, independent of the assumption $b_j>0$, and can be stated in a form analogous to Rolle's theorem: between two consecutive changes of sign of the b there is at least one change of sign of the a. Hence the lemma.

LEMMA 2′. *If $\gamma>0$, δ is real,*

$$a_n=\sin(\gamma n+\delta)+b_n\quad (n=0,\ 1,\ 2,\ \ldots),$$

and there is an η such that

$$0<2\eta<\gamma<\pi-2\eta, \tag{4}$$

$$|b_n|<\sin\eta, \tag{5}$$

then the number of changes of sign in the sequence $a_0, a_1, \ldots, a_n$ is

$$\frac{\gamma n}{\pi}+O(1).$$

If the difference between $\gamma n+\delta$ and the nearest multiple of π is not less than η, then

$$\operatorname{sg} a_n=\operatorname{sg}\sin(\gamma n+\delta)$$

in virtue of (5)†. If for an integer m

$$-\eta<\gamma n+\delta-m\pi<\eta,$$

then a_n and $\sin(\gamma n+\delta)$ may have different signs; but it follows from (4) that

$$-\pi+\eta<\gamma(n-1)+\delta-m\pi<-\eta,\quad \eta<\gamma(n+1)+\delta-m\pi<\pi-\eta,$$

and therefore, by (5),

$$\operatorname{sg} a_{n-1}=\operatorname{sg}\sin\{\gamma(n-1)+\delta\}=-\operatorname{sg}\sin\{\gamma(n+1)+\delta\}=-\operatorname{sg} a_{n+1};$$

† As usual, $\operatorname{sg} x$ means $+1$, 0, or -1 according as $x>0$, $x=0$, or $x<0$.

thus a_{n-1}, a_n, a_{n+1} show one change of sign just as $\sin\{\gamma(n-1)+\delta\}$, $\sin(\gamma n+\delta)$, $\sin\{\gamma(n+1)+\delta\}$ do.

To sum up, a_n and $\sin(\gamma n+\delta)$ are either of the same sign or, *if* they are included between other terms, they cannot occasion any change of sign. Therefore, the two sequences

$$a_0, \quad a_1, \quad a_2, \quad \ldots, \quad a_n,$$

$$\sin\delta, \quad \sin(\gamma+\delta), \quad \sin(2\gamma+\delta), \quad \ldots, \quad \sin(n\gamma+\delta)$$

cannot differ by more than two changes of sign (at both ends). Hence the lemma.

LEMMA 3′. *If* C, C_1, ..., C_M, δ, δ_1, ..., δ_M *are real,* γ, γ_1, ..., γ_M *are positive,*

$$a_n = C\cos(\gamma n+\delta)+\sum_{m=1}^{M} C_m\cos(\gamma_m n+\delta)+b_n,$$

$$\lim_{n\to\infty} b_n = 0,$$

$$|C_1|+|C_2|+\ldots+|C_M| < |C|\sin\eta,$$

and (4) *is valid, then the number of changes of sign in the sequence* a_0, a_1, ..., a_n *is*

$$\frac{\gamma n}{\pi}+O(1).$$

Disregard a suitably chosen finite number of terms and apply Lemma 2′ to a_n/C.

12. In order to prove Theorem III, we consider the series

$$f_k(z) = a_0^{(k)}+a_1^{(k)}z+a_2^{(k)}z^2+\ldots+a_n^{(k)}z^n+\ldots, \tag{6}$$

defined by

$$f_k(z) = (1-z)^k f(z) \quad \text{if} \quad k = 0, 1, 2, 3, \ldots, \tag{7}$$

and by

$$f_k(z) = (1-z)^{-l}\left\{f(z)-f(1)-\frac{f'(1)}{1!}(z-1)-\ldots-\frac{f^{(l-1)}(1)}{(l-1)!}(z-1)^{l-1}\right\} \tag{8}$$

$$\text{if} \quad k = -l = -1, \ -2, \ -3, \ \ldots;$$

the definition of $f_k(z)$ for a negative k presupposes that $f(z)$ is regular for $z = 1$.

By definition $f_0(z) = f(z)$, $a_n^{(0)} = a_n$. We have

$$f_{k-1}(z) = \frac{f_k(z)+b}{1-z},$$

where
$$b = \begin{cases} 0 & \text{for } k \geqslant 1, \\ \dfrac{(-1)^{k+1} f^{(-k)}(1)}{(-k)!} & \text{for } k \leqslant 0, \end{cases}$$

and therefore

$$a_n^{(k-1)} = (a_0^{(k)}+b)+a_1^{(k)}+a_2^{(k)}+\ldots+a_n^{(k)}. \tag{9}$$

We pass from $a_n^{(k)}$ to $a_n^{(k-1)}$ by summation, that is by a process analogous to integration. The passage from $a_n^{(k-1)}$ to $a_n^{(k)}$ is analogous to differentiation, and in this passage, by Lemma 1′, no more than one change of sign can be lost, a result analogous to Rolle's theorem. If we *disregard a quantity that remains bounded for* $n \to \infty$, we may express the result of the application of Lemma 1′ thus: *The number of variations of* $a_0^{(k)}$, $a_1^{(k)}$, $a_2^{(k)}$, ... *increases with* k. We shall use the fact that the number of variations of a_0, a_1, a_2, ... is included between the number of variations of $a_0^{(-l)}$, $a_1^{(-l)}$, $a_2^{(-l)}$, ..., and that of $a_0^{(l)}$, $a_1^{(l)}$, $a_2^{(l)}$, ..., where l is a large positive integer.

13. The general coefficient a_n of the series (1) may be put in the form

$$a_n = b_n+c_n,$$

where $b_n = O(e^{-\alpha n})$ with $\alpha > 0$, and c_n is the contribution of the poles on the circle of convergence; that is, c_n is the coefficient of z^n in the expansion of the rational function that has the same singularities as $f(z)$ on $|z| = 1$, and no other singularities. If $e^{i\gamma}$ is a pole of $f(z)$ of order p, then

$$f(z) = C(1-ze^{-i\gamma})^{-p}\{1+(z-e^{i\gamma})\mathfrak{P}(z-e^{i\gamma})\}, \tag{10}$$

where $\mathfrak{P}(z-e^{i\gamma})$ is a power series in $z-e^{i\gamma}$ and

$$C = |C|e^{-i\delta} \neq 0.$$

If $0 < \gamma < \pi$ the pair of poles $e^{i\gamma}$ and $e^{-i\gamma}$ contribute to c_n a term

$$\frac{2|C|}{(p-1)!}n^{p-1}\cos(\gamma n+\delta)+O(n^{p-2}), \tag{11}$$

the contribution being one half of the expression (11) if it comes from one pole situated in $z = 1$ ($\gamma = 0$) or $z = -1$ ($\gamma = \pi$).

If we except the point $z = 1$, $f(z)$ and $f_k(z)$, defined by (7) and (8), have the same poles. If k is positive and sufficiently large, $f_k(z)$ is regular for

$z = 1$, and so is $f_k(z)$ for a negative k if it can be defined at all. We easily find from (7), (8), (10), that the contribution of the couple of poles $e^{i\gamma}$ and $e^{-i\gamma}$ to $a_n^{(-l)}$ is

$$(12) \qquad \frac{1}{(2\sin\frac{1}{2}\gamma)^l}\frac{2|C|}{(p-1)!}n^{p-1}\cos[\gamma n+\delta-\tfrac{1}{2}l(\pi-\gamma)]+O(n^{p-2}),$$

and their contribution to $a_n^{(l)}$ is

$$(13) \qquad (2\sin\tfrac{1}{2}\gamma)^l\frac{2|C|}{(p-1)!}n^{p-1}\cos[\gamma n+\delta+\tfrac{1}{2}l(\pi-\gamma)]+O(n^{p-2}).$$

We see from (12) and (13) that, as l increases, the contributions of the poles on $|z| = 1$ to $a_n^{(-l)}$ and to $a_n^{(l)}$ are altered in a systematic way; in $a_n^{(-l)}$, the poles near to the point $z = 1$ are favoured, in $a_n^{(l)}$ those far from the point $z = 1$. Let P be the order of the poles $e^{i\gamma'}$ and $e^{i\gamma''}$; by hypothesis, no pole of $f(z)$ on $|z| = 1$ is of a higher order. Suppose l sufficiently large; among the poles of order P, $e^{i\gamma'}$ and $e^{-i\gamma'}$ will give the preponderant contribution to $a_n^{(-l)}$ and $e^{i\gamma''}$ to $e^{-i\gamma''}$ to $a_n^{(l)}$. The meaning of "preponderant" may be made precise by applying Lemma 3′ to $a_n^{(-l)}n^{-P+1}$ and to $a_n^{(l)}n^{-P+1}$; we find that $a_0^{(-l)}$, $a_1^{(-l)}$, ..., $a_n^{(-l)}$ show

$$\frac{\gamma' n}{\pi}+O(1)$$

changes of sign, and $a_0^{(l)}$, $a_1^{(l)}$, ..., $a_n^{(l)}$

$$\frac{\gamma'' n}{\pi}+O(1)$$

changes of sign. By what we have said in Art. 12, the number of variations of $a_0, a_1, \ldots, a_n$ is contained between these limits.

To prove Theorem III completely, we have only to add that, in virtue of the supplementary hypothesis concerning the existence of a pole at the point $z = 1$, the construction of $f_{-l}(z)$ cannot break down except if $\gamma' = 0$, in which case it is unnecessary to prove the first inequality (3).

[*Added 26 October*, 1931.—Theorem II on p. 90 remains true if we replace (1) by the less restrictive hypothesis that $g(x) = o(1)$.]

ON THE ROOTS OF CERTAIN ALGEBRAIC EQUATIONS

By A. Bloch *and* G. Pólya.

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1. This paper is concerned with polynomials of the form

$$f(x) = 1+\epsilon_1 x+\epsilon_2 x^2+\epsilon_3 x^3+\ldots+\epsilon_n x^n, \tag{1}$$

where

$$\epsilon_\nu^3 = \epsilon_\nu \quad (\nu = 1, 2, 3, \ldots, n), \tag{2}$$

that is ϵ_ν takes one of the values -1, 0, 1. The total number of polynomials of this form (1) is 3^n. Among these 3^n polynomials there is one (or more) possessing a *maximum* number of real zeros in the open interval $0<x<1$; let this maximum number be P_n. It is immediately evident that $0 \leqq P_n \leqq n$, and that P_n cannot decrease as n increases. This paper aims at determining the order of magnitude of P_n as n tends to infinity. Since this question seems very particular and rather out of the way, we begin by enumerating some of the examples and observations which led to it.

(i) Let p be an odd prime, and consider the polynomial, involving Legendre symbols,

$$\left(\frac{1}{p}\right)+\left(\frac{2}{p}\right)x+\left(\frac{3}{p}\right)x^2+\ldots+\left(\frac{p-1}{p}\right)x^{p-2}.$$

If it has no roots in the interval $0<x<1$, we know that the Dirichlet series

$$L(s) = \sum_1^\infty \left(\frac{n}{p}\right)\frac{1}{n^s}$$

possesses no positive real root. The polynomial has been examined for all odd primes under 300, that is in 61 cases. In 54 cases there are no

roots, in four cases there are just two roots in the interval $0<x<1$, and in the remaining three cases further computations must be carried out in order to decide whether the number of roots in question is 0 or 2†.

(ii) Denote as usual by $\lambda(1)$, $\lambda(2)$, $\lambda(3)$, ... the Liouville numbers defined by the series

$$\lambda(1)1^{-s}+\lambda(2)2^{-s}+\ldots+\lambda(n)n^{-s}+\ldots = \xi(2s)/\xi(s).$$

Numerical computations show that the polynomial

$$\lambda(1)+\lambda(2)x+\lambda(3)x^2+\ldots+\lambda(n)x^{n-1}$$

has just one real root in the interval $0<x<1$ if $2<n\leqslant 1500$‡.

(iii) The polynomial

$$\mu(1)+\mu(2)x+\mu(3)x^2+\ldots+\mu(n)x^{n-1},$$

involving the Möbius numbers, has more than $o(\log n)$ roots, as follows from the consideration of the series

$$\mu(1)1^{-s}+\mu(2)2^{-s}+\ldots+\mu(n)n^{-s}+\ldots = 1/\xi(s),$$

by a theorem of the preceding paper§ and the simplest properties of the ζ-function.

(iv) The polynomial $1-x+x^2-\ldots+(-1)^n x^n$, the n-th partial sum, or briefly the n-th section of the geometric series whose sum is $1/(1+x)$, has no root between 0 and 1.

(v) Consider the power series

$$1-x-x^2-x^3+x^4+\ldots+x^8-x^9-\ldots = \frac{1-2x+2x^4-\ldots+(-1)^n 2x^{n^2}+\ldots}{1-x}.$$

The first coefficient is $+1$, and is followed by three coefficients -1, next five coefficients $+1$, then seven coefficients -1, and so on. It may be shown, by a not quite obvious application of a theorem due to Laguerre, that no section of this power series has more than two positive roots or more than one negative root.

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† G. Grimm, "Ueber die reellen Nullstellen Dirichletscher L-Reihen", *Thesis* (Zürich, 1931).

‡ G. Pólya, *Jahresbericht der deutschen Math. Ver.*, 28 (1919), 31–40; see 38.

§ See the end of Art. 4, 89, *supra*.

(vi) In the power series

$$x-x^2-x^3+x^4+\ldots+x^7-x^8-\ldots = \frac{x-2x^2+2x^4-\ldots+(-1)^n 2x^{2^n}+\ldots}{1-x}$$

the first non-vanishing coefficient is $+1$, then follow two coefficients -1, next four coefficients $+1$, then eight coefficients -1, and so on. The number of real roots of the n-th section in the interval $0<x<1$ is† $(\log n/\log 2)+O(1)$.

(vii) If we change x into $-x$ we obtain from (1) a polynomial of the same kind, as also we do if we change x into $1/x$ and multiply by a suitable factor. Thus for a *general* polynomial (1) it is nearly the same thing to find the number of the real roots between 0 and 1 as to find the total number of real roots, the roots -1 and $+1$ being excepted. Now it is easy to see that *if* $\epsilon_n \neq 0$ *and* $n>3$, *all roots of* (1) *cannot be real.* For suppose that $\alpha_1, \alpha_2, \ldots, \alpha_n$, the roots of (1), are all real. Applying the inequality connecting arithmetic and geometric means to the positive numbers $\alpha_1^2, \alpha_2^2, \ldots, \alpha_n^2$ we find that

$$1 = (\alpha_1^2\alpha_2^2\ldots\alpha_n^2)^{1/n} \leqslant \frac{\alpha_1^2+\alpha_2^2+\ldots+\alpha_n^2}{n} = \frac{\epsilon_{n-1}^2-2\epsilon_n\epsilon_{n-2}}{\epsilon_n^2 n} \leqslant \frac{3}{n},$$

that is $n \leqslant 3$. Observe that all roots of

$$(1-x)^2(1+x) = 1-x-x^2+x^3$$

are real.

(viii) *The average number of real roots of the polynomial* (1) *does not exceed* $O(\sqrt{n})$. We give a proof for the analogous result in the case of equations with coefficients ± 1, which is, in this respect, a little simpler. Consider the equation

$$\eta_0+\eta_1 x+\eta_2 x^2+\ldots+\eta_n x^n = 0, \tag{3}$$

where $\eta_\nu^2 = 1$, $\nu = 0, 1, 2, \ldots, n$. The number of roots of the equation (3) in the *closed* interval $0 \leqslant x \leqslant 1$ exceeds at most by a unit the number of changes of sign in the sequence‡

$$\eta_0, \quad \eta_0+\eta_1, \quad \eta_0+\eta_1+\eta_2, \quad \ldots, \quad \eta_0+\eta_1+\ldots+\eta_n. \tag{4}$$

Therefore it will be sufficient to prove that the average number of these

† Compare the preceding paper, the beginning of Art. 4, 88–89.

‡ This follows from a theorem of E. Laguerre, *Œuvres* 1 (Paris, 1898), 6, if $x=1$ is not a root, and from the ordinary Descartes' rule if $x=1$ is a root.

changes of sign is $O(\sqrt{n})$. But this problem is identical with a classical problem of the calculus of probability. We may consider a player who plays heads and tails, throwing a coin $n+1$ times, and betting each time a shilling. We may interpret $\eta_0, \eta_1, \ldots, \eta_n$ as the (positive or negative) gains of the player at the successive events, and $\eta_0+\eta_1+\ldots+\eta_\nu$ as his total gain in the first $\nu+1$ events. The probability of having

$$\eta_0+\eta_1+\eta_2+\ldots+\eta_{2k-1}=0$$

is, as is well known,

$$\frac{1}{2^{2k}}\binom{2k}{k}=\frac{1\cdot 3\cdot 5\ldots(2k-1)}{2\cdot 4\cdot 6\ldots 2k},$$

and the probability that $\eta_0+\eta_1+\ldots+\eta_{2k-2}$ and $\eta_0+\eta_1+\ldots+\eta_{2k}$ have opposite signs is therefore

$$\frac{1}{2}\cdot\frac{1\cdot 3\cdot 5\ldots(2k-1)}{2\cdot 4\cdot 6\ldots 2k}.$$

Determine the integer m by the inequality $2m-1<n\leqq 2m+1$; then the average number of changes of sign in the sequence (4) is

$$\sum_{k=1}^{m}\frac{1}{2}\cdot\frac{1\cdot 3\cdot 5\ldots(2k-1)}{2\cdot 4\cdot 6\ldots 2k}\sim\sum_{k=1}^{m}\frac{1}{2\sqrt{(k\pi)}}\sim\sqrt{\left(\frac{n}{2\pi}\right)}.$$

The result in (viii) "explains" up to a certain point the foregoing observations (i)–(vii). We shall prove that not only the average number, but also the maximum number of roots of (1) between 0 and 1 is only an infinitely small fraction of n as n becomes infinitely large, that is $P_n=o(n)$. The crucial question is whether P_n is of an order as low as $\log n$. [See examples (ii), (iii), and (vi).] We found that the answer is in the negative. Though we have not been able to determine the exact order of magnitude of P_n, we can limit it on both sides.

There is a positive constant A such that, for $n>3$,

$$\frac{1}{A}\frac{n^{\frac{1}{4}}}{(\log n)^{\frac{1}{2}}}<P_n<A\frac{n\log\log n}{\log n}. \tag{5}$$

We shall prove this double inequality in the two subsequent sections.

2. The second inequality (5) is contained in the following more general proposition.

Assume that the coefficients $a_0, a_1, \ldots, a_n$ of the polynomial

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_n=f(x) \tag{6}$$

are real and satisfy the following conditions:

$$a_\nu^2 \leqslant 1 \quad (\nu = 0, 1, 2, \ldots, n), \tag{7}$$

$$a_0^2 \geqslant a, \quad a_n^2 \geqslant a. \tag{8}$$

Denote by R_n the number of real roots of the polynomial (6). *To any given a, such that $0 < a < 1$, corresponds a positive constant A such that*

$$R_n < A \frac{n \log \log n}{\log n} \tag{9}$$

for $n \geqslant 3$†.

We observe that $f(-x)$ and $x^n f(1/x)$ satisfy the same conditions as $f(x)$. This observation permits us to change x into $-x$ or into $1/x$ in certain parts of the proof. We observe further that, if the inequality (9) holds with a certain A for sufficiently great n, it evidently holds with some other A for $n \geqslant 3$.

The following proof could be presented in a real and purely algebraical form, but we shall make free use of the theory of analytic functions of a complex variable z. We need the following lemmas.

LEMMA 1. *All roots of* (6) *are situated in the circular ring*

$$\frac{a}{1+a} < |z| < \frac{1+a}{a}.$$

Put

$$\frac{1+a}{a} = \beta. \tag{10}$$

If $|z| \geqslant \beta$, then we obtain, using (7) and (8),

$$\begin{aligned} |a_0 z^n + a_1 z^{n-1} + \ldots + a_n| &\geqslant |a_0 z^n| \left(1 - \frac{|a_1|}{|a_0 z|} - \frac{|a_2|}{|a_0 z^2|} - \ldots - \frac{|a_n|}{|a_0 z^n|}\right) \\ &> |a_0| \beta^n \left(1 - \frac{1}{a\beta} - \frac{1}{a\beta^2} - \ldots\right) \\ &= |a_0| \beta^n \left(1 - \frac{1}{a(\beta-1)}\right) = 0. \end{aligned}$$

Thus we have proved that $|f(z)| > 0$ for $|z| \geqslant \beta$. This establishes the complete Lemma 1, since we may change z in $1/z$, as we observed above.

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† That $R_n = o(n)$ could be deduced from a result of G. Szegö, *Berliner Sitzungsberichte*, 21 (1922), 59–64, by observing that the polynomials (6) form a "normal family".

LEMMA 2. *If* $1 < q < \beta$, *then the number of roots of* (6) OUTSIDE *the circular ring*

$$\frac{1}{q} < |z| < q \tag{11}$$

does not exceed

$$\frac{B}{q-1} \log \frac{1}{q-1}, \tag{12}$$

where B *depends only on* α.

Denote by N the number of the roots of (6) whose modulus does not exceed $1/q$. Put

$$\frac{1}{q} = \rho, \quad \frac{1+\rho}{2} = r, \tag{13}$$

and apply Jensen's formula to (6). We obtain

$$\begin{aligned} N \log \frac{r}{\rho} &\leqslant -\log |f(0)| + \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\phi})| \, d\phi \\ &\leqslant -\tfrac{1}{2} \log \alpha + \tfrac{1}{2} \log \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\phi})|^2 \, d\phi \right) \\ &= -\tfrac{1}{2} \log \alpha + \tfrac{1}{2} \log (|a_0|^2 r^{2n} + \ldots + |a_{n-1}|^2 r^2 + |a_n|^2) \\ &< \tfrac{1}{2} \log \frac{1}{\alpha(1-r^2)}. \end{aligned} \tag{14}$$

We have here used (8), the inequality between the geometric and arithmetic means of a function, and (7). It follows from (13) and (14) that

$$N \leqslant \tfrac{1}{2} \left(\log \frac{4q^2}{\alpha(3q+1)} + \log \frac{1}{q-1} \right) \Big/ \log \left(1 + \frac{q-1}{2} \right).$$

Thus the number of the zeros of (6) inside the circle $|z| \leqslant 1/q$ does not exceed an expression of the form (12). We complete the proof of Lemma 2 by changing z into $1/z$.

LEMMA 3. *Denote by* $z_1, z_2, \ldots, z_n$ *the roots of the polynomial* (6) *and put*

$$z_1^m + z_2^m + \ldots + z_n^m = s_m,$$

m *being any positive integer. Then*

$$|s_m| < \left(1 + \frac{1}{\alpha} \right)^m. \tag{15}$$

In fact,

$$\sum_1^\infty \frac{s_m}{m}\frac{1}{z^m} = -\sum_{\nu=1}^{n}\log\left(1-\frac{z_\nu}{z}\right) = -\log\frac{f(z)}{a_0 z^n}$$

$$= -\log\left(1+\frac{a_1}{a_0}\frac{1}{z}+\frac{a_2}{a_0}\frac{1}{z^2}+\ldots+\frac{a_n}{a_0}\frac{1}{z^n}\right)$$

$$\ll -\log\left(1-\frac{1}{az}-\frac{1}{az^2}-\ldots-\frac{1}{az^n}-\ldots\right)$$

$$= -\log\left\{1-\left(1+\frac{1}{a}\right)\frac{1}{z}\right\}+\log\left(1-\frac{1}{z}\right)$$

$$\ll -\log\left\{1-\left(1+\frac{1}{a}\right)\frac{1}{z}\right\}.$$

We have here used (7), (8), and the current notation for majorant series. Comparing the coefficient of z^{-m} in the first and the last series we get (15).

In the following proof we assume that the polynomial (6) has not less positive zeros than negative zeros; this is no limitation, since we may change x into $-x$ We put

$$l = [a\log n] \tag{16}$$

($[x]$ means here the integral part of x), and

$$q = 1+\frac{\log l}{l}. \tag{17}$$

We denote the total number of zeros of (6) *outside* the circular ring (11) by Q_n and the number of *positive* zeros of (6) *inside* the circular ring (11) by P_n^*. Then we have at most $P_n^*+Q_n$ positive zeros in all, and thus

$$R_n \leqslant 2(P_n^*+Q_n). \tag{18}$$

It follows from Lemma 2 and from (16) and (17) that

$$Q_n < \frac{Bl}{\log l}\log\frac{l}{\log l} < Bl < B\log n, \tag{19}$$

for sufficiently great n.

We put further

$$k = q^2, \tag{20}$$

and consider the real-valued function

$$T(z) = \frac{2}{k} r\cos\phi + \frac{2}{k^2} r^2\cos 2\phi + \ldots + \frac{2}{k^l} r^l \cos l\phi \tag{21}$$

of the complex variable

$$z = |z|\, e^{i\phi} = re^{i\phi}.$$

If $|z| \leqslant \beta$ [see (10)], then we get, from (16) and (21),

$$\begin{aligned} |T(z)| \leqslant |T(\beta)| &< 2l\beta^l \leqslant 2a\log n\, \beta^{a\log n} \\ &= 2an^{a\log\beta}\log n = 2an^{\gamma}\log n. \end{aligned} \tag{22}$$

We put

$$\gamma = a\log\beta = a\log\left(1+\frac{1}{a}\right) < a\,.\,\frac{1}{a} = 1. \tag{23}$$

If

$$r = |z| \leqslant q,$$

$$\begin{aligned} &T(z)+1-\frac{k^2-r^2}{k^2-2kr\cos\phi+r^2} \\ &= -2\left\{\left(\frac{r}{k}\right)^{l+1}\cos(l+1)\phi + \left(\frac{r}{k}\right)^{l+2}\cos(l+2)\phi+\ldots\right\} \\ &\geqslant -2\left\{\left(\frac{q}{k}\right)^{l+1} + \left(\frac{q}{k}\right)^{l+2}+\ldots\right\} \\ &= -2(q^{-l-1}+q^{-l-2}+\ldots) = -\frac{2}{q^l(q-1)}; \end{aligned} \tag{24}$$

we have here used (20). Now, if $x > 0$,

$$\log(1+x) > x - \frac{x^2}{2} > x - x^2, \quad 1+x > e^{x-x^2}.$$

Therefore

$$\begin{aligned} q^l = \left(1+\frac{\log l}{l}\right)^l &> e^{\log l - (\log l)^2/l} \\ &= le^{-(\log l)^2/l} > \tfrac{1}{2}l, \end{aligned} \tag{25}$$

if n is sufficiently large. It follows from (24) and (25) that

$$T(z) > -1 + \frac{k^2-r^2}{k^2-2kr\cos\phi+r^2} - \frac{2}{\frac{1}{2}l\log l/l} > -1 - \frac{4}{\log l}. \tag{26}$$

Finally, if z is positive and inside the circular ring (11), then, by (26), (20), and (17),

$$T(z) \geqslant T\left(\frac{1}{q}\right) > -1 + \frac{k^2 - q^{-2}}{(k - q^{-1})^2} - \frac{4}{\log l} \tag{27}$$

$$= \frac{2}{kq - 1} - \frac{4}{\log l} = \frac{2}{(1 + \log l/l)^3 - 1} - \frac{4}{\log l}.$$

We collect the inequalities (22), (23), (26), (27) in the following lemma.

Lemma 4. *If we define k and q by* (20), (17), *and* (16), *the function* (21) *satisfies the following inequalities for sufficiently large n:*

$$T\left(\frac{1+\alpha}{\alpha}\right) < Cn^{\gamma} \log n, \tag{28}$$

$$T(z) > -Cn^{\gamma} \log n \quad \textit{if} \quad \frac{\alpha}{1+\alpha} < |z| < \frac{1+\alpha}{\alpha}, \tag{29}$$

$$T(z) > -2 \quad \textit{if} \quad q^{-1} < |z| < q, \tag{30}$$

$$T(z) > \frac{1}{C} \frac{\log n}{\log \log n} \quad \textit{if} \quad z \textit{ is real, } q^{-1} < z < q, \tag{31}$$

where the positive numbers C and γ depend only on α (in particular, they are independent of n) and

$$\gamma < 1. \tag{32}$$

The situation is this : the circular ring (11) becomes infinitely narrow as n increases. Outside it the number of zeros is very small [see (19)], inside it the positive zeros z give rise to a comparatively large positive $T(z)$ [see (31)], and all the other zeros give rise to a $T(z)$ which is bounded from below [see (30)]. We shall be able to use all these remarks in considering the formula

$$T(z_1) + T(z_2) + \ldots + T(z_n) = \frac{2}{k} s_1 + \frac{2}{k^2} s_2 + \ldots + \frac{2}{k^l} s_l. \tag{33}$$

[To establish (33), we use the result that the coefficients $a_0, a_1, \ldots, a_n$ of (6) are real, and, therefore, the imaginary part of s_m is zero.]

The right-hand side of (33) is, by Lemma 3, less than

$$\frac{2}{k} \frac{1+\alpha}{\alpha} + \frac{2}{k^2} \left(\frac{1+\alpha}{\alpha}\right)^2 + \ldots + \frac{2}{k^l} \left(\frac{1+\alpha}{\alpha}\right)^l = T\left(\frac{1+\alpha}{\alpha}\right) < Cn^{\gamma} \log n$$

in virtue of (28). We estimate the left-hand side of (33) from below, using (29), (30), (31), and Lemma 1. We obtain

$$P_n^* \frac{\log n}{C \log\log n} - Q_n C n^\gamma \log n - (n - P_n^* - Q_n)2 < C n^\gamma \log n,$$

and hence

$$P_n^* \left(\frac{\log n}{C \log\log n} + 2\right) < C(1+Q_n) n^\gamma \log n + 2(n - Q_n)$$

$$< C(1 + B \log n) n^\gamma \log n + 2n;$$

we have used (19) for the last line. With the aid of (18) and (19), it follows that, for sufficiently large n,

$$R_n < 2P_n^* + 2Q_n < \frac{4C \log\log n}{\log n} \{n + BCn^\gamma (\log n)^2\} + 2B \log n.$$

This proves the theorem expressed by (9) if we take account of (32).

3. We now proceed to prove the first part of (5).

LEMMA 5. *If $n \geqq 2$, there is a polynomial of the form* (1) *for which the point $x = 1$ is a root whose multiplicity is not less than*

$$\left[\sqrt{\left(\frac{2n \log 2}{\log n}\right)}\right] - 1.$$

Consider the k linear forms

$$\begin{aligned}
y_1 &= x_0 + x_1 + x_2 + \ldots + x_n, \\
y_2 &= 1x_1 + 2x_2 + \ldots + nx_n, \\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
y_k &= \binom{0}{k-1} x_0 + \binom{1}{k-1} x_1 + \binom{2}{k-1} x_2 + \ldots + \binom{n}{k-1} x_n.
\end{aligned}$$

Their coefficients are non-negative integers (binomial coefficients), and they depend on $n+1$ variables $x_0, x_1, x_2, \ldots, x_n$. If each of these variables takes the values 0 and 1 independently of the others, we obtain 2^{n+1} systems of integers $y_1, y_2, \ldots, y_k$, or, as it is perhaps preferable to say, 2^{n+1} lattice points in the k-dimensional space, situated in a portion of this space which is defined by the inequalities

$$0 \leqslant y_1 \leqslant \binom{n+1}{1}, \quad 0 \leqslant y_2 \leqslant \binom{n+1}{2}, \quad \ldots, \quad 0 \leqslant y_k \leqslant \binom{n+1}{k}.$$

If the total number of lattice points in this portion of space is less than the number of systems $x_0, x_1, x_2, \ldots, x_n$, that is if

$$\left[\binom{n+1}{1}+1\right]\left[\binom{n+1}{2}+1\right]\ldots\left[\binom{n+1}{k}+1\right] < 2^{n+1}, \tag{34}$$

there are certainly two different systems $x_0', x_1', \ldots, x_n'$ and $x_0'', x_1'', \ldots, x_n''$ which give rise to the same lattice point; that is, putting

$$x_0''-x_0' = \xi_0, \quad x_1''-x_1' = \xi_1, \quad \ldots, \quad x_n''-x_n' = \xi_n,$$

we have

$$\begin{aligned} \xi_0+ \quad & \xi_1+ \quad & \xi_2+\ldots+ \quad & \xi_n = 0, \\ & 1\xi_1+ \quad & 2\xi_2+\ldots+ \quad & n\xi_n = 0, \\ \ldots \quad \ldots \quad & \ldots \quad \ldots \quad & \ldots \quad \ldots \quad & \ldots \quad \ldots \\ \binom{0}{k-1}\xi_0+ & \binom{1}{k-1}\xi_1+ & \binom{2}{k-1}\xi_2+\ldots+ & \binom{n}{k-1}\xi_n = 0, \end{aligned} \tag{35}$$

where the ξ_ν are equal to 1 or 0 or -1, but *not all equal to* 0. This fact may be expressed by saying that the polynomial

$$\phi(x) = \xi_0+\xi_1 x+\xi_2 x^2+\ldots+\xi_n x^n$$

does not vanish identically, and equations (35) may be written as

$$\phi(1) = \phi'(1) = \phi''(1) = \ldots = \phi^{(k-1)}(1) = 0.$$

Dividing $\phi(x)$ by $\xi_r x^r$, where ξ_r is the first non-vanishing number in the sequence $\xi_0, \xi_1, \xi_2, \ldots$ we obtain a polynomial of the form (1) that has not less than k zeros concentrated at the point $x = 1$.

All that remains to do is to discuss the inequality (34). If $n \geqslant 2$,

$$\frac{n+2}{2} \leqslant n, \quad \frac{n^2+n+2}{2} \leqslant n^2, \quad \binom{n+1}{3}+1 < n^3, \quad \ldots,$$

and so (34) is certainly true if

$$n \cdot n^2 \cdot n^3 \ldots n^k < 2^n,$$

$$\frac{k(k+1)}{2} < \frac{n \log 2}{\log n},$$

and it is *a fortiori* true if

$$(k+1)^2 \leqslant \frac{2n \log 2}{\log n}.$$

This last inequality is satisfied by

$$k = \left[\sqrt{\left(\frac{2n \log 2}{\log n}\right)}\right] - 1.$$

This proves Lemma 5.

By Lemma 5 we obtained a polynomial having a great number of real roots concentrated at the right-hand end-point of the interval $0 < x < 1$. If the ξ_ν could vary continuously, it would be very easy to shift these roots from the end-point into the interior of the interval; but since they cannot, we need a less obvious transformation that succeeds in shifting the roots.

LEMMA 6. *Assume that the point $x = 1$ is a root of the polynomial*

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_\mu x^\mu + \ldots + a_m x^m,$$

of multiplicity not less than $2q$, and form with the same real coefficients $a_0, a_1, \ldots, a_m$ the polynomial

$$F(x) = a_0 + a_1 x + a_2 x^4 + \ldots + a_\mu x^{\mu^2} + \ldots + a_m x^{m^2}.$$

Then the point $x = 1$ is a root of $F(x)$ with a multiplicity not less than q, and, moreover, $F(x)$ possesses in the interval $0 < x < 1$ at least q different roots of odd order.

(I) That the point $x = 1$ is a root of multiplicity greater than or equal to $2q$, for $f(x)$, may be expressed by the equation

$$P(0)a_0 + P(1)a_1 + P(2)a_2 + \ldots + P(n)a_n = 0,$$

where $P(x)$ is an *arbitrary* polynomial of degree not exceeding $2q-1$. Thus, in particular, if $Q(x)$ is an arbitrary polynomial of degree not exceeding $q-1$, we have

$$Q(0)a_0 + Q(1)a_1 + Q(4)a_2 + \ldots + Q(n^2)a_n = 0,$$

i.e. the point $x = 1$ is a root for $F(x)$ with a multiplicity not less than q.

From this it follows that $F(e^{-1/t}) = O(t^{-q})$ as t tends to ∞, and therefore

$$\int_0^\infty t^{m-\frac{1}{2}} F(e^{-1/t})\, dt \tag{36}$$

converges for $m = 0, 1, 2, \ldots, q-1$.

(II) Using the well known integral formula†

$$\int_0^\infty e^{-at-b/t}t^{-\frac{1}{2}}dt = \sqrt{(\pi/a)}e^{-2\sqrt{(ab)}},$$

we obtain, for $x>0$,

$$\int_0^\infty e^{-xt}\sum_{\nu=0}^{n} a_\nu e^{-\nu^2/t}t^{-\frac{1}{2}}dt = \sqrt{(\pi/x)}\sum_{\nu=0}^{n} a_\nu e^{-2\nu\sqrt{x}},$$

that is

$$\int_0^\infty e^{-xt}F(e^{-1/t})t^{-\frac{1}{2}}dt = \sqrt{(\pi/x)}f(e^{-2\sqrt{x}}). \tag{37}$$

Now, by hypothesis, the expansion of the right-hand side in ascending powers of $\sqrt{x}$ contains no smaller power than $x^{q-\frac{1}{2}}$. Differentiating (37) $q-1$ times and putting $x=0$, we find that the integral (36) vanishes for $m=0, 1, 2, \ldots, q-1$. That is to say, the first q moments of the function $F(e^{-1/t})t^{-\frac{1}{2}}$ in the interval $0<t<\infty$ vanish; from this it follows, by a classical argument‡, that the function in question has at least q different zeros of odd order in the interval§.

In order to prove the first inequality (5), consider n as given, and choose the integer m such that

$$m^2 \leqslant n < (m+1)^2.$$

Applying Lemma 5 to m (instead of n) we find a polynomial $1+\epsilon_1 x+\ldots+\epsilon_m x^m$ to which we may apply Lemma 6. The resulting polynomial

$$1+\epsilon_1 x+\epsilon_2 x^4+\epsilon_3 x^9+\ldots+\epsilon_m x^{m^2}$$

is of the required form (1), and possesses at least

$$n^{\frac{1}{4}}\sqrt{\left(\frac{\log 2}{\log n}\right)}-O(1),$$

different zeros of odd order in the interval $0<x<1$. Thus we have proved the first part of the inequality (5), even if we count in P_n the roots of odd multiplicity just once and do not count at all the roots of even multiplicity.

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† See, *e.g.*, C. Jordan, *Cours d'Analyse*, 3 ed., 2 (Paris, 1913), 198.

‡ J. Liouville, *Journal de Math.*, 2 (1837), 1–2.

§ It may be proved that, if $f(x)$ has p roots in the interval $0<x<1$, $F(x)$ has there at least $p+q$ different roots of odd order. This proposition depends on a suitable extension of a theorem of Laguerre.

ON PLANE-FORMS IN FOUR DIMENSIONS

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In this paper is initiated the study of threefolds by the functional method which has already been applied to surfaces†. A plane-form is the locus generated by ∞^1 planes, and in four dimensions is the dual construct of a scroll. An arbitrary space section is a scroll of general type in ordinary space, the enumerative properties of which have been exhaustively treated by James‡. Further, James's discussion§ of scrolls in [4], which is confined to the consideration of multiple secants and tangents, is in its dual form complementary to the present work; taken together, the two papers give a complete account both of scrolls and of plane-forms in four dimensions.

After describing the general features of a plane-form, with reference to some important special cases, we proceed to obtain the characters of its multiple loci. These possess novel elements of considerable interest; thus in general the double surface on the form contains a *triple curve*, with a number of quadruple points—a type of surface which does not appear to have been noticed before. The apparent contour of the form—the section of the tangent cone from an external point—is a developable surface of general character whose properties, expressed by the Cayley-Plücker and Salmon-Cremona equations, may be deduced from the formulae obtained.

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† The following papers by the author will be referred to by the numbers shown: I. "Some formulae for surfaces in higher space", *Proc. Camb. Phil. Soc.*, 25 (1929), 390, additions to which will be found in *Proc. London Math. Soc.* (2), 32 (1931), 154. II. "Enumerative formulae for surfaces", *Proc. Camb. Phil. Soc.*, 26 (1930), 43. III. "Multiple tangents to forms in four dimensions", *ibid.*, 285. Thus, in the sequel, a reference III (20) is to equation (20) of paper III.

‡ James, *Trans. Camb. Phil. Soc.*, 23 (1925), 201 (referred to as J 1).

§ James, *Proc. London Math. Soc.* (2), 27 (1928), 513 (referred to as J 2).

The rest of the discussion is concerned with the multiple tangents to the form and the loci which they generate; it follows closely the sequence of results obtained for general forms† in four dimensions, with the addition of formulae which arise only in the present case.

There is, however, another subject in which this investigation continues the work already mentioned: the estimation of redundancies due to exceptional elements in the loci which we have to consider. Some of the problems to which these results give rise may be disposed of readily enough; in other cases the explanation seems to lie very deep, and the present paper can only be regarded as a first step towards their solution.

Introduction.

1.1. A form F which is generated by ∞^1 planes is specified by two characters‡ (n, a); in [4] the order n is the number of generating planes which meet a given line, and the rank a is the class of a plane section. Alternatively, the form may be specified by the order n and the genus p, where

$$a = 2(n+p-1).$$

Through a point of F there passes in general a single generating plane; but there is a double surface Γ of points through each of which pass two generating planes, and a triple curve T on Γ through every point of which there pass three such planes. The curve T possesses a number of quadruple points at which concur four planes. Further, Γ contains a cuspidal curve J of pinch-points for which the two generating planes coincide.

An arbitrary space section of F is a scroll of order n and rank a in [3], whence‖ it follows that Γ is of order

$$b = \binom{n}{2} - \tfrac{1}{2}a,$$

that T is of order

$$t = \binom{n}{3} + \tfrac{1}{2}an + 2a - n,$$

and that the cuspidal curve is of order

$$j = 2(a-n).$$

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† For these results see III.

‡ The characters of a variety generated by ∞^1 spaces $[k]$ are given by Severi, *Memorie di Torino* (2), 52 (1903), § 4.

‖ For the details see J 1.

1.2. *The dual relations with scrolls.* The generating planes of F reciprocate into the generators of a scroll f of characters (n, a) in [4]. Multiple tangent planes to F correspond to multiple secants of f, and multiple tangent planes to f correspond to multiple tangents to F. An r-ple point of F reciprocates into a space containing r generators. An r-ple directrix curve of F, through each point of which pass r generating planes, corresponds to a developable of spaces, each containing r generators of f. In particular, an r-ple directrix line on F corresponds to a pencil of spaces through a plane directrix of f, each space containing r generators. To the improper nodes of f correspond

$$d = \binom{n}{2} - \tfrac{3}{2}a + n$$

bitangent spaces, each containing two generating planes, whose line of intersection lies on the double surface.

1.3. *Some particular cases.* The plane-forms of the first few orders have various exceptional features, of which some are already known†, and others will be deduced from the formulae about to be given. The plane-regulus (2, 2) generates a quadric cone containing a conjugate regulus; the cubic form (3, 4), discussed by Segre‡, contains a double plane and a residual congruence of directrix lines, one such line passing through each point of the form and one lying in a given space. The form (4, 6), which has been briefly considered by James§, has for double surface a cubic scroll and possesses a regulus of directrices. The two quintic forms are known in their dual connection, discussed by Severi. Since the scroll (5, 8) contains a rational plane quartic whose three nodes are improper nodes of the scroll, the form (5, 8) must contain a directrix line, the intersection of its three bitangent spaces. The scroll (5, 10) lies on a plane-form (5, 10) whose generating planes meet it in cubic curves; hence the form has for its double surface an elliptic quintic scroll, and its generating planes meet the scroll in cubic curves.

The elliptic sextic scroll, which is required later, does not appear to have been considered before. It may be obtained as the residual intersection‖ of two cubic forms which have in common three arbitrary planes; the three intersections of the planes are improper nodes of the scroll,

† Severi, *Rend. di Palermo*, 15 (1901), 33; also J 2, 514, *et. seq.*

‡ Segre, *Memorie di Torino* (2), 39 (1889), 43.

§ James, *Proc. Camb. Phil. Soc.*, 22 (1924), 204.

‖ The necessary formulae are described in I, 391, 402.

and each plane meets the scroll in a quartic having nodes at the two improper nodes in its plane. These facts may be verified by James's formulae for the quadrisecants and bitangents of a scroll. The scroll also contains two directrix plane cubics, as follows from the formulae (5) and (6). This may be established by considering in [5] the residual intersection of three quadric forms containing two arbitrary planes; this intersection is an elliptic sextic scroll which meets the planes in cubic curves.

1.4. *The functional process.* The method by which most of our results are obtained is due to Cayley, and depends upon the association of two forms F, F', of characters (n, a) and (n', a') respectively, in arbitrary position such that their complete intersection is to be regarded as part of the double surface on the composite form $(n+n', a+a')$, in addition to the surfaces Γ, Γ'. For any enumerative formula $f(n, a)$ connected with F we thus derive a functional equation which takes one of the forms:

(i) $$f(n+n', a+a')-f(n, a)-f(n', a') = \phi(n, n', a, a'),$$

or $$\Delta f(n, a) = \phi(n, n', a, a');$$

(ii) $$f(n+n', a+a')-f(n, a) = \phi(n, n', a, a'),$$

or $$\Delta^* f(n, a) = \phi(n, n', a, a').$$

The solution to (i), which in James's notation is denoted by $\int \phi(n, n', a, a')$, consists of a particular solution together with the linear terms $Aa+Nn$, whose coefficients must be found from two known special cases. The solution of (ii), denoted by $\int^* \phi(n, n', a, a')$, contains an added constant C which is determined by one known special case.

For the sake of brevity, details of calculation and, whenever they resemble previous work, the functional equations themselves, will not be given: only the results are stated. In this connection it is convenient to remark upon the considerable number of equations in which the function ϕ is zero; in case (i) the corresponding solution is $f = Aa+Nn$, and in (ii) it is $f = C$.

1.5. *The principle of redundancy.* There are various ways in which the exceptional features of a form, as indicated in § 1.3, give rise to redundant solutions (finite or infinite in number) in an enumerative problem. If the given form F contains ∞^r lines, these will occur redundantly in problems concerning ∞^r lines or the points of contact

of multiple tangents which have freedom ∞^{r+1}; in such cases the number of redundant solutions is finite. If, however, F contains ∞^s lines ($s > r$), the redundant solutions are infinite in number.

In the present discussion we consider only plane-forms, and for these the redundancies are contributed by directrix lines, which meet all the generating planes. The form may contain a congruence of directrices, *e.g.*, (3, 4), or a scroll, which may be simple or double : (4, 6) possesses scrolls of each type, (5, 10) of the latter type only. There may be a simple directrix [(5, 8)] or a number of triple directrices†, as for (5, 8), (6, 12), and (7, 16).

When the directrices are simple and the redundancy is finite, it may easily be determined by means of a principle established in previous work. If each of the lines counts k times, suppose, in a given problem, then k is a constant for the problem and is independent of the form, which may be plane-ruled or otherwise. To find the line constant k, it is therefore possible to employ the formulae of III in the appropriate cases; the process is as follows.

(i) The general cubic form f_3 contains a congruence of lines, six passing through each point of f_3 and 27 lying in each space section. The value of k for a formula giving the order of a certain congruence is obtained by putting $n = 3$ in the known formula‡ : the result, in cases where the formula should vanish, is $18k$. The corresponding formula applied to (3, 4) gives the value $3k$, since one directrix passes through each point of the form. Similarly for the class of each such congruence.

(ii) The quartic form f_4 contains a scroll of order 320; hence the value of k for the order of a scroll of tangents, or of the surface of contact of ∞^2 tangents, is found by substituting $n = 4$ in a formula which should vanish, but which gives $320k$. From this we may determine the redundancy of the double scroll on (4, 6). Suppose that each of its generators counts $d^{(4)}$ times in the formula; those of the directrix regulus count k times, as already stated. The total contribution of these scrolls to the formula is thus $3d^{(4)} + 2k$, and in a problem which has no genuine solutions the value of $d^{(4)}$ is thereby obtained.

(iii) The quintic form f_5 contains 23.125 lines, as was shown by Schubert§. To find k for a formula giving the order of the curve of

† See §2.7. In general, the double surface always contains d lines, but these are not directrices and do not occur redundantly in the formulae.

‡ III, (15)–(23).

§ Schubert, *Math. Annalen*, 26 (1886), 52.

contact of ∞^1 tangents, or the number of tangents satisfying six conditions, we put $n=5$ in the appropriate formula; k then appears multiplied by the above number of lines. The application of these results to plane-forms will be considered later.

The multiple loci.

2.1. *The double surface.* As remarked in §1.1, the surface Γ is of order

$$b=\binom{n}{2}-\tfrac{1}{2}a. \tag{1}$$

We have now to determine its remaining characters; this is effected by means of the Severi intersection formulae† for the surface common to two forms of orders n, n', ranks a, a', classes m, m', and containing cuspidal curves of orders j, j' respectively.

Thus the rank of the surface is

$$\Delta\rho_1=an'+a'n.$$

Its class is

$$\Delta\rho_2=mn'+aa'+m'n,$$

and the number of apparent pinch-points‡ is

$$\Delta\omega_2=jn'+aa'+j'n.$$

Since the three characters are additive, it follows that the corresponding characters of Γ are the solutions ρ_1, ρ_2, and ω_2 of these equations. For a plane-form F we have $m=n$ and $j=2(a-n)$. Substituting and using special cases we obtain

$$\rho_1=an-6a+4n, \tag{2}$$

$$\rho_2=n^2+\binom{a}{2}-12a+11n, \tag{3}$$

$$\omega_2=2(a-n)n+\tfrac{1}{2}a^2-20a+22n, \tag{4}$$

from (3, 4), (4, 6), (5, 10).

2.11. Γ meets each generating plane in a curve of order $\int^* n'=n-2$, and rank $\int^* a'=a-4$, from the case (3, 4).

† Severi, *op. cit.* (in §1.1).

‡ For these characters see I, §§1–2.

2.2. *The triple curve.* The order of T is given by $t = \int bn' + b'n$, whence

$$t = \binom{n}{3} - \tfrac{1}{2}an + 2a - n. \tag{5}$$

To find its rank we use the Severi formula for the intersection of a surface and a form : thus the rank t_1 is given by

$$\Delta t_1 = ab' + a'b + \rho_1 n' + \rho_1' n,$$

whence
$$t_1 = a\binom{n}{2} - 6an + 4n^2 - \tfrac{1}{2}a^2 + 29a - 28n, \tag{6}$$

vanishing for (3, 4), (4, 6), (5, 10). The number of points in which T meets a generating plane is

$$\tau = \int^* b' + n'(n-2) = \binom{n}{2} - \tfrac{1}{2}a - 2n + 5, \tag{7}$$

vanishing as above†.

2.21. The class of immersion x of T in F is given by

$$x = \int ab' + a'b + nz' + n'z,$$

where $z = an - 5a + 3n$ is the class of immersion of the double curve on a space section‡. With the above special cases we obtain

$$x = a\binom{n}{2} - \tfrac{1}{2}a^2 + n(3n - 5a) + 23a - 21n. \tag{8}$$

2.22. The number of quadruple points, on T as well as F, is

$$\begin{aligned}\chi &= \int tn' + t'n + bb' \\ &= \binom{n}{4} - \tfrac{1}{2}a\binom{n}{2} + \tfrac{1}{8}a^2 + n(2a - n) - \tfrac{29}{4}a + 6n.\end{aligned} \tag{9}$$

This vanishes for (3, 4), (4, 6), (5, 8), (5, 10), (6, 10), (6, 12), (7, 14), (7, 16), (8, 18), and has the value 1 for (7, 12). In its dual significance for scrolls, the formula was first given by James§.

2.3. *The cuspidal curve.* This is the locus J of points of Γ at which the two generating planes coincide; it is cuspidal on F, but simple on Γ.

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† These τ points are nodes on the curve in which Γ meets a generating plane.

‡ J1, 206.

§ J2, 521. Note that $x = (n-3)t - 4\chi$, as follows from consideration of the third polar of an arbitrary point.

As remarked in § 1 . 1, its order is

$$j = 2(a-n). \tag{10}$$

Its rank ρ must be linear in the characters, and, since J is a conic for (3, 4) and a rational quartic for (4, 6), we have

$$\rho = 5a-6n. \tag{11}$$

This number is also the order of its scroll of tangents, which place it in birational correspondence with the generating planes of F. Hence J must be of genus p, as is verified from the above values of j and ρ.

2 . 31. The cuspidal curve meets each generating plane in a single point [from the case (3, 4)], and touches it there. It meets the triple curve T in σ points which are stationary on T ; these are given by

$$\sigma = \int jn'+j'n = 2(a-n)n-12a+14n, \tag{12}$$

vanishing for (3, 4), (4, 6), and (5, 10).

We note, finally, that the coincident generating planes at a point of J have a limiting line of intersection, which is *the tangent to J at the point in question.*

2 . 4. *The intersection formulae.* A surface on F is characterized by two numbers (α, β), α being its order and β the order of the curve in which it meets an arbitrary generating plane. Similarly, a curve on F is typified by numbers (α, β), where α is the order and β is the number of points in which it meets an arbitrary generating plane.

From the known correspondence theory† it follows that two surfaces (α, β) and (α', β') meet in a curve of order

$$\alpha\beta'+\alpha'\beta-n\beta\beta', \tag{13}$$

and that a surface (α, β) and a curve (α', β') meet in

$$\alpha\beta'+\alpha'\beta-n\beta\beta' \tag{13}'$$

points.

2 . 41. These results require modification when the curve lies on Γ, or when the surface (α', β') is Γ itself. In this case, since through an arbitrary point of Γ there pass two generating planes, the number of

† See Segre, *Encyk. der Math. Wiss.*, III C 7, 954.

coincidences of the correspondence is

$$\alpha(n-2)+2\left\{\binom{n}{2}-\tfrac{1}{2}a\right\}\beta-n(n-2)\beta,$$

or
$$\alpha(n-2)-(a-n)\beta. \tag{14}$$

This formula gives the order of the curve of intersection of Γ and a surface (α, β), or the number of points in which a curve (α, β) meets Γ.

2.42. Suppose now that (α', β') is an arbitrary curve on Γ. Since through a point of the curve there pass two generating planes, the number of points in which it meets a given surface (α, β) is

$$a\beta'+2a'\beta-n\beta\beta'. \tag{15}$$

Exception must be made in the case of the cuspidal and triple curves. For the former meets each generating plane in two consecutive points, and through each point of the curve there passes one generating plane; hence the above formula is replaced by

$$2a+j\beta-2n\beta. \tag{16}$$

The triple curve meets each generating plane in τ points, where (§ 2.2)

$$\tau=\binom{n}{2}-\tfrac{1}{2}a-2n+5.$$

Through an arbitrary point of the curve there pass three generating planes. Hence it meets a surface (α, β) in $\tau\alpha+3t\beta-n\tau\beta$ points, reducing to

$$\left\{\binom{n}{2}-\tfrac{1}{2}a-2n+5\right\}\alpha+\{n^2-an+6a-7n\}\beta. \tag{17}$$

2.43. We may apply this result to find the class of immersion of the triple curve in F. The tangent cone to F from an arbitrary point has a surface of contact for which $\alpha=a$, $\beta=1$. Hence the number of points in which it meets T, excluding the σ stationary points, is

$$x=ab+3n^2-5an+23a-21n,$$

as in § 2.21.

2.5. *Class of immersion of the cuspidal curve.* We first find the class of immersion of J in Γ, and for this purpose we require two of

James's results† concerning curves on a scroll. A curve (μ, ν) of order μ, which meets each generator of the scroll (n, a) in ν points, is of rank

$$R = 2\mu\nu - n\nu^2 + \nu(a-n), \tag{18}$$

and has class of immersion

$$\omega = \mu + \nu(a-n) \tag{19}$$

in the scroll.

Now the class ζ of immersion of J in Γ must be linear in the characters of F, and must vanish for the form (3, 4). For the form (4, 6) the curve J lies on a cubic scroll. We have $\mu = 4$, $R = 6$, and it has been shown‡ that for this curve $\nu = 2$. Substituting in the above formula we obtain $\omega = 6$. Hence the class of immersion of J in the double surface is

$$\zeta = 3(3a-4n). \tag{20}$$

We obtain a verification in the case (5, 10), for which the double surface is a scroll (5, 10) and the cuspidal curve is an elliptic decimic, so that $\mu = 10$, $R = 20$. The formula gives $\zeta = 30$, whence, from (19), we have $\nu = 4$. These results agree with (18).

2.51. With the same special cases we determine the class Z of immersion of J in F, which, as before, is linear in the characters. We note first that the tangent cone to F from a given point P contains the cuspidal curve, every secant of which is an improper tangent to F. Hence, by § 2.41, its residual intersection γ with Γ is of order

$$z = (n-2)a - (a-n) - j = an - 5a + 3n,$$

as already stated in § 2.21. The curve γ meets J at the Z points in question.

For (3, 4), $z = 1$, whence it follows that $Z = 2$. Thus the number of points in which γ meets a generating plane is

$$\kappa = \int^* a' = a - 3,$$

from the case (3, 4), since the generating plane meets Γ in a line.

Hence, for (4, 6), $z = 6$, $\kappa = 3$; and, since every generating plane of the form is a conic plane of its double scroll, it follows that γ has

† J1, 205, 226.

‡ James, *loc. cit.* (§ 1.3), 205.

characters $\mu = 6$, $\nu = 3$. For (5, 10), $z = 15$, $\kappa = 7$; and, since each generating plane contains a directrix curve of the double scroll, in this case γ has characters $\mu = 15$, $\nu = 4$. From the intersection formula (13) applied to the scrolls it appears that $Z = 6$ for (4, 6) and $Z = 20$ for (5, 10). These three special cases agree with the result

$$Z = 5a - 6n. \tag{21}$$

2.6. *The singular generators.* The tangent space to F at an arbitrary point meets F in a generating plane π and a surface of order $n-1$, which intersects π in the curve c_{n-2} lying on Γ (§ 2.11) and a line l along which the space touches F. The pencil of spaces through π gives rise to a pencil of lines in π, whose vertex is the point of contact of J with π. These lines l, which we call "singular generators", form a congruence, of which one line passes through a general point of F, and j lie in a general space section; they are, in fact, the reciprocals of the tangent planes to the scroll f (§ 1.2).

Through a point of Γ there pass in general two singular generators, one in each generating plane; but at a point of J these coincide with *the tangent to J at the given point.* These "cuspidal generators" are therefore the tangents to the cuspidal curve, and, by § 2.3, generate a scroll of order

$$\rho = 5a - 6n,$$

which we call the "pinch-scroll".

2.61. One cuspidal generator lies in each generating plane and touches the corresponding c_{n-2} at its intersection with J; for otherwise the plane would contain two tangents to Γ at a point of J, and would therefore be the tangent plane to Γ at that point; which is not the case.

It follows that a generator of the pinch-scroll meets Γ in $n-4$ points other than at J; the order α of the residual intersection of the pinch-scroll and Γ is obviously given by

$$\alpha = \int \rho n' + \rho' n = (5a - 6n)n - 24a + 30n, \tag{22}$$

vanishing for (3, 4) and (4, 6). The correspondence $(n-4, 1)$ between this curve and J has for united points the σ stationary points on the cuspidal curve. Thus

$$\sigma = (n-4)j + \alpha - (n-4)\rho,$$

giving

$$\sigma = jn - 12a + 14n,$$

as in § 2.31.

2.62. Among the cuspidal generators there is a finite number ξ of *stationary cuspidal generators* which have stationary contact with F at a point of J. In fact, the surface Γ has a scroll of inflexional tangents whose curve of contact meets J at the ξ points in question. To determine this number we use the two cases (§ 2.5) for which Γ is a scroll. It has been shown by James† that, for a scroll (n, a), the above curve of contact is of order $4a-5n$, and meets each generator in a single point. For the cubic scroll this curve is the directrix which, as may be seen from § 2.5, does not meet J. Hence ξ vanishes for (4, 6). For the scroll (5, 10) the curve is of order 15 and, from the formula (13), meets J in $4.15+1.10-4.5 = 50$ points. It follows that the number of stationary cuspidal generators is

$$\xi = 10(2a-3n). \tag{23}$$

2.7. *Some further special cases.* In the light of these results we now analyse the singularities of some plane-forms which are to be used later.

(5, 8) : this contains a triple line, since $t = 1$, as well as an ordinary directrix line (§ 1.3). Its double surface has characters

$$b = 6, \quad \rho_1 = \rho_2 = \omega_2 = 12,$$

and is therefore a del Pezzo surface; since $d = 3$, the surface contains three ordinary lines, and the triple line thus has the effect of absorbing three of the lines in the general cases, without altering the characters of the surface.

(5, 10) : as already stated, the double surface is an elliptic quintic scroll. There are no higher singularities.

(6, 10) : this, it will appear, contains no directrix lines. The double surface has characters $b = 10$, $\rho_1 = 24$, $\rho_2 = 27$, $\omega_2 = 30$, $d = 6$, and contains a rational quartic as triple curve. It is thus a sub-species of the rational surface which is represented by means of plane quartics passing through six base-points.

(6, 12) : since $t = 2$, $t_1 = 0$, the triple curve must be a pair of non-intersecting lines. The double surface has characters

$$b = 9, \quad \rho_1 = 24, \quad \rho_2 = 24, \quad \omega_2 = 36, \quad \text{and} \quad d = 3,$$

† J 2, 518.

so that it contains three ordinary lines, as follows from consideration of the dual case (§ 1 . 3).

(7, 12) : the locus in [9] of which this is the projection has been considered by J. G. Semple† ; it is represented by means of cubic scrolls with a common nodal line. The double surface has characters

$$b = 15, \quad \rho_1 = 40, \quad \rho_2 = 48, \quad \omega_2 = 56, \quad d = 10,$$

and contains a triple decimic of rank 24, with one quadruple point. There are no directrix lines. The same remark is true of (7, 14).

(7, 16) : from the formulae for t and t_1 it appears that the form contains four triple lines in general position. Besides these the double surface has four ordinary lines ($d = 4$).

The various formulae obtained later in §§ 6–7 provide criteria for determining the forms of higher order which do not contain linear directrices ; for this purpose the most powerful formula is (§ 7 . 5), giving the number of sextuple tangents. This should vanish for $n \leqslant 11$; we thus derive, in addition to the rational and elliptic types, the following forms without singular features :

(8, 18), (9, 20), (10, 22), (10, 24), (11, 24), (11, 26).

Systems of multiple tangents.

3 . 1. There are two hypercongruences or systems of freedom four, formed by the bitangents and inflexional tangents. The orders and classes are given by‡

$$\mu_{2-2} = \nu_{2-2} = \binom{a}{2} - \tfrac{9}{2}a + 4n. \tag{24}$$

$$\mu_3 = \nu_3 = 3(a-n), \tag{25}$$

There are three complexes, formed by the 4-point tangents, the 3—2 tangents, and the tritangents ; the classes§ of these systems are

† Semple, *Proc. Camb. Phil. Soc.*, 25 (1929), 163.

‡ All the subsequent results which may be derived from the properties of a scroll in [3] are due to James : J 1, 207–13. The extension to surfaces is given in II, (8)–(34).

§ The notation is that of III.

known to be

$$\nu_4 = 4(3a-4n), \tag{26}$$

$$\nu_{3-2} = 6\left\{2\binom{a}{2} - an - 9a + 12n\right\}, \tag{27}$$

$${}_3\nu_2 = 2\binom{a}{3} - 18\binom{a}{2} + 8an + 49a - 64n. \tag{28}$$

In each case the corresponding order μ is found to be connected with its class ν by the relation

$$2\mu = \nu. \tag{29}$$

3.11. *The apparent contour.* These results have an important application to curves and developables in ordinary space. The tangent cone to F from an arbitrary point P has a surface of contact of order a, whose projection from P on a given [3] is a developable of class n and rank a. Its double and cuspidal curves are the projections of the cones of bitangents and inflexional tangents passing through P; their orders are therefore given by (24) and (25). The double curve contains ${}_3\mu_2$ triple points, the cuspidal curve contains μ_4 stationary points, and the two curves meet in μ_{3-2} points, which are stationary on the double curve. The formulae thus obtained are the Cayley-Plücker and Salmon-Cremona equations for a general curve of order n and rank a.

3.2. We consider now the modified orders of the complexes when the tangents are drawn from an ordinary, double, triple, or quadruple point of F, using the symbols t, δ, τ, and χ in the four cases.

Thus the number of 4-point tangents in each case is

$$t_4 = \mu_4 - 4, \quad \delta_4 = \mu_4 - 8, \quad \tau_4 = \mu_4 - 12, \quad \chi_4 = \mu_4 - 16, \tag{30}$$

the constants being found from suitable special cases.

The numbers of 3—2 tangents are†

$$\begin{aligned} t_{3-2} &= \int^* \mu'_{3-2} + 3(a-4)(a'-n') + 3a'(a-n-1) \\ &= \mu_{3-2} - 3(3a-2n-10), \end{aligned} \tag{31}$$

vanishing for (5, 8), (5, 10).

† In this section, James's results for analogous problems in [3] are used throughout.

Similarly,

$$\left.\begin{aligned} \delta_{3-2} &= \mu_{3-2}-6(3a-2n-12), \\ \tau_{3-2} &= \mu_{3-2}-9(3a-2n-14), \\ \chi_{3-2} &= \mu_{3-2}-12(3a-2n-16). \end{aligned}\right\} \qquad (31)'$$

The numbers of tritangents are found to be

$$\left.\begin{aligned} {}_3t_2 &= {}_3\mu_2-2\mu_{2-2}+2(4a-20), \quad {}_3\delta_2 = {}_3\mu_2-4\mu_{2-2}+4(5a-28), \\ {}_3\tau_2 &= {}_3\mu_2-6\mu_{2-2}+6(6a-36), \quad {}_3\chi_2 = {}_3\mu_2-8\mu_{2-2}+8(7a-44). \end{aligned}\right\} \qquad (32)$$

3.3. The number of 4-point tangents at a point of F must be a constant (§ 1.4), and is found from one special case. Thus, at an ordinary point, $t_{4:}=1$, since, for (3, 4), the tangent must lie on the form, and it is known that one line of the congruence passes through each point (§ 1.3).

This may also be determined directly. At an ordinary point of F there is a plane pencil of inflexional tangents lying in the tangent space, and therefore meeting the generating plane at the point in a line; this line is the singular generator at the given point (§ 2.6). Of the pencil of 3-point tangents there is one ray which meets a fourth consecutive generating plane, and this is the required 4-point tangent. At a general m-ple point of F there are m pencils of 3-point tangents, and there is one 4-point tangent in each pencil. At a point of the cuspidal curve, however, the two 4-point tangents coincide with the cuspidal generator.

3.31. Thus there are ∞^2 3-point tangents which lie on F; these are the singular generators. There is a scroll of 4-point tangents, formed by the cuspidal generators, which lie on F; and there is a finite number of 5-point tangents lying on F—the stationary cuspidal generators, $20a-30n$ in number (§ 2.62).

3.4. The numbers of inflexional tangents at a point, which are elsewhere tangent, are

$$t_{3:2} = a-6, \quad \delta_{3:2} = a-8, \quad \tau_{3:2} = a-10, \quad \chi_{3:2} = a-12. \qquad (33)$$

The numbers of tangents at a point, which are elsewhere 3-point tangent, are

$$\left.\begin{aligned} t_{2:3} &= 3(a-n-2), \quad \delta_{2:3} = 3(a-n-3), \\ \tau_{2:3} &= 3(a-n-4), \quad \chi_{2:3} = 3(a-n-5). \end{aligned}\right\} \qquad (34)$$

The numbers of tangents at a point, which are elsewhere bitangent, are

$$\left.\begin{aligned} t_{2:2-2} &= \mu_{2-2}-4(a-5), \quad \text{vanishing for } (4,6),\ (5,8),\ (5,10).\\ \delta_{2:2-2} &= \mu_{2-2}-6(a-6), \quad \text{vanishing}^{\dagger} \text{ for } (5,8),\ (6,10),\ (6,12).\\ \tau_{2:2-2} &= \mu_{2-2}-8(a-7), \quad \text{vanishing}^{\ddagger} \text{ for } (6,10),\ (7,12),\ (7,14).\\ \chi_{2:2-2} &= \mu_{2-2}-10(a-8), \quad \text{vanishing for } (7,12),\ (8,14),\ (8,16). \end{aligned}\right\} \tag{35}$$

3.5. *Congruences of tangents.* There are five types of congruences, each defined by two numbers: the order μ of the form generated by the tangents, and the class ν, *i.e.* the number of tangents in a given space. The formulae for the classes are known to be as follows.

The 5-point tangents are of class

$$\nu_5 = 10(2a-3n). \tag{36}$$

The 4—2 tangents are of class

$$\nu_{4-2} = 24\binom{a}{2}-16an-124a+192n. \tag{37}$$

The 3—3 tangents are of class

$$\nu_{3-3} = 9\left\{2\binom{a}{2}-2an+2\binom{n}{2}-7a+12n\right\}. \tag{38}$$

The 3—2—2 tangents are of class

$$\nu_{3-2-2} = 3\left\{6\binom{a}{3}-54\binom{a}{2}-2n\binom{a}{2}-16\binom{n}{2}+41an+147a-240n\right\}. \tag{39}$$

The quadritangents are of class

$$\begin{aligned} {}_4\nu_2 = {} & 2\binom{a}{4}-27\binom{a}{3}+\tfrac{277}{2}\binom{a}{2}+8n\binom{a}{2}\\ & +32\binom{n}{2}-100an-\tfrac{1173}{4}a+480n. \end{aligned} \tag{40}$$

In each case *the corresponding value of μ is found to be given by the relation*

$$2\mu = 3\nu. \tag{41}$$

† The double surface on (5, 10) is ruled.

‡ This does not vanish for (6, 12) or (7, 16) on account of the triple lines.

This result has an interesting application to the theory of redundancy. It has been shown† that, for the form (3, 4), ν has the value $2k$, where k is the line constant for the problem in question. Hence we obtain the relation $\mu = 3k$ in each of the five cases. Now one directrix line passes through each ordinary point of (3, 4). Hence it follows that the contribution of these lines to the redundancy of the formulae is the same for the *order* as for the *class* of each congruence. The same conclusion is arrived at by calculating the values of k by means of the known results for the general form for f_n (§ 1 . 5).

Surfaces of contact.

4 . 1. We begin by determining the order of the surface M_5 of contact of the form F_5 generated by the 5-point tangents.

The 4-point tangents at points of a plane section ϖ generate a scroll S of order m_4+n, where‡ $m_4 = 6a-7n$ is the order of the curve of contact of the scroll generated by the 4-point tangents in a space section. The scroll S meets F again in a curve of order

$$\gamma = n(n+m_4)-4n. \tag{i}$$

The correspondence $(n-4, 1)$ between ϖ and γ has the following united points :

(i) the m_5 points in which the required surface meets ϖ ;

(ii) the points in which Γ meets ϖ, counted λ times, say ;

(iii) the ρ points in which the pinch-scroll meets ϖ (§ 3 . 31), counted μ times.

Hence we have

$$(n-4)n+\gamma-(n-4)(n+m_4) = m_5+\lambda\left\{\binom{n}{2}-\tfrac{1}{2}a\right\}+\mu(5a-6n). \tag{ii}$$

Since m_5 must be linear in the characters, $\lambda = 2$. To find the value of μ we use the case (5, 8) for which M_5 is the complete intersection, counted five times, of F_5 and F. Since $\mu_5 = 15$, $m_5 = 15$ also, and we obtain, on substitution, $\mu = 3$. Hence

$$m_5 = 10a-13n. \tag{42}$$

† See II, 71.

‡ J 1, 211 ; or II (32). The usual specialization argument is employed here.

The order c_5 of the curve in which M_5 meets an arbitrary generating plane must be a constant (§ 1 . 4). From the case (5, 8) we therefore have

$$c_5 = 3.$$

The same result may be obtained from (4, 6), for which M_5 consists of the double scroll, counted twice, and the directrix regulus (§ 8 . 2); the former meets each generating plane in a conic, the latter in a line.

4 . 11. The 4-point tangent at a point of the cuspidal curve is tangent to the curve and meets a consecutive tangent. It follows that each point of J is a point of contact of a 5-point tangent, so that *the cuspidal curve lies on* M_5.

4 . 2. *Surfaces of contact of* F_{4-2}. For the surface $M_{4:2}$ of 4-point contacts we require

LEMMA 1. *The 4-point tangents of* F *which belong to an arbitrary* $\{5\}$ *of order* N *touch* F *along a surface of order*† $N(nt_{4:}+m_4)$.

For the surface $M_{2:4}$ of ordinary contacts we require

LEMMA 2. *The tangents belonging to a complex*‡ (α, β) *have a surface of contact of order* $a\alpha+n\beta$.

These, like all similar results, are obtained very easily by specialization. We thus derive the formulae

$$m_{4:2} = 6a(a-n)+Aa+Nn,$$

$$m_{2:4} = 2(a+2n)(3a-4n)+A'a+N'n,$$

where the four constants are to be determined.

These formulae must vanish for (5, 8); and, since the equations expressing this condition are identical, there is a presumption, which is confirmed later, that $A = A'$ and $N = N'$. A second condition is obtained from the fact that, for (6 . 10), $4m_{4:2}+2m_{2:4} = 6\mu_{4-2}$: the form (6, 12) does not give an independent equation.

† These and subsequent formulae are stated quite generally so as to be applicable, as they stand, to later work.

‡ Here, as elsewhere, the characters (α, β) represent the order and class of the given line-system.

Hence we have

$$m_{4:2} = 6a(a-n)-68a+80n, \tag{43}$$

$$m_{2:4} = 2(a+2n)(3a-4n)-68a+80n. \tag{44}$$

To find the orders of the curves in which these surfaces meet a generating plane π, consider a generator g lying in π; the tangents to F at points of g form a complex (1, 1), so that, for $M_{2:4}$,

$$c_{2:4} = \int^* \mu_4' + \nu_4' = 6(3a-4n-4),$$

vanishing for (5, 8).

The 4-point tangents at points of g generate a cubic scroll† ; thus

$$c_{4:2} = \int^* 3a' = 3(a-8).$$

Then, for (6, 10) and (6, 12), we have the verification $4c_{4:2}+2c_{2:4} = \mu_{4-2}$.

4.3. *The surface* M_{3-3}. For this we require

LEMMA 3. *The 3-point tangents belonging to a* $\{4\}$ (α, β) *have a contact surface of order* $\alpha(t_{3:}+\nu_3)+\beta t_{3:}$.

Here $t_{3:}$ represents the order of the cone of inflexional tangents at a point of F; in the present case, $t_{3:} = 1$. Hence

$$m_{3-2} = 3\{(a-n)(3a-n)-24a+27n\}, \tag{45}$$

which vanishes for (5, 8) and for (6, 10), (6, 12) gives

$$3m_{3-3} = 6\mu_{3-3}.$$

Also
$$c_{3-3} = \int^* 2\mu_3' + \nu_3' = 9(a-n-3).$$

4.4. *The surface* $M_{3:2-2}$. We use Lemma 3 and

LEMMA 4. *The* 3—2 *tangents belonging to a* $\{5\}$ *of order* N *have a surface of 3-point contacts of order* $N(nt_{3:2}+m_{3:2})$, where‡

$$m_{3:2} = 3a^2-2an-30a+30n.$$

† The 4-point tangents at points of π form a congruence (4, 2).

‡ The results quoted in the present section are given in J 1, 208–13.

We thus obtain

$$m_{3:2-2} = 9\binom{a}{3} - n\binom{a}{2} + \tfrac{81}{2}an - 81\binom{a}{2} - 8\binom{n}{2} + \tfrac{441}{2}a - 280n. \quad (46)$$

This vanishes for (5, 8), (6, 10), and (6, 12).

To find the order $c_{3:2-2}$ of the curve of section by a generating plane π, we note that the 3 : 2 tangents at points of a generator g form a scroll of order $t_{3:2}+a_{3:2} = 3(a-6)$, and that the 3-point tangents at points of g form a congruence (2, 1). Hence

$$c_{3:2-2} = \int^* 3(a-6)\,a' + 3\mu'_{2-2} = 3(\tfrac{1}{2}a^2 - 11a + 4n + 36),$$

vanishing as above.

4.5. *The surface $M_{2:2-3}$.* In addition to Lemma 2 we use

LEMMA 5. *The 2—3 tangents belonging to a $\{5\}$ of order N have a 2-point contact surface of order $N(nt_{2:3}+m_{2:3})$, where*

$$m_{2:3} = 3(a^2-n^2) - 30a + 30n.$$

LEMMA 6. *The bitangents belonging to a $\{4\}$ (α, β) touch F along a surface of order $\alpha(2\nu_{2-2}+nt_{2:2})+\beta nt_{2:2}$, where*

$$t_{2:2} = \mu_{2-2} - 2(a-4).$$

From these results we have

$$m_{2:2-3} = 6\Big\{3\binom{a}{3} + n\binom{a}{2} - 27\binom{a}{2} - 2a\binom{n}{2} + \tfrac{3}{2}an + 24\binom{n}{2} + \tfrac{147}{2}a - 80n\Big\}. \quad (47)$$

The above special cases apply.

To find $c_{2:2-3}$ we note the following facts :

The 2 : 2 tangents at points of g generate a congruence $(2a-8, a-4)$; the 2 : 3 tangents generate a scroll of order $t_{2:3}+a_{2:3} = 9(a-n-2)$; and the tangents at points of g form a complex (1, 1). Hence

$$c_{2:2-3} = 9(a^2 - an - 16a + 16n + 24),$$

vanishing as above.

4.6. *The surface* ${}_4M_2$. For this we employ Lemmas 2, 6, and

LEMMA 7. *The tritangents belonging to a* $\{5\}$ *of order* N *have a contact surface of order* $N({}_{nt_{2\,2-2}}+{}_3m_2)$, *where*

$${}_3m_2 = 3\binom{a}{3} - 27\binom{a}{2} + n\binom{a}{2} + \tfrac{7}{2}an + 8\binom{n}{2} + \tfrac{147}{2}a - 72n.$$

Hence we have

$${}_4m_2 = 4\binom{a}{4} - 54\binom{a}{3} + 2n\binom{a}{3} - 10n\binom{a}{2} + 16a\binom{n}{2} + 227\binom{a}{2}$$

$$-67an - 128\binom{n}{2} - \tfrac{1173}{2}a + 640n. \quad (48)$$

This vanishes for (5, 8), (6, 10), (6, 12), (7, 12), (7, 14), and (7, 16). To obtain ${}_4c_2$ we observe that the $2:2-2$ tangents at points of g generate a scroll of order

$$t_{2:2-2} + a_{2:2-2} = 3\left\{\binom{a}{2} - \tfrac{17}{2}a + 4n + 20\right\},$$

whence

$${}_4c_2 = 3\left\{\binom{a}{3} - 13\binom{a}{2} + 4an + \tfrac{125}{2}a - 48n - 112\right\}.$$

The same special cases apply.

Scrolls of multiple tangents.

5.1. *The scroll* S_6. The scroll of 6-point tangents is generated by lines which meet six consecutive generating planes of F; since five lines meet six given planes in [4], it follows that S_6 touches each generating plane at five points. And since, for (6, 10), the curve of contact, counted six times, is the complete intersection of S_6 and F, its order ν_6 is in this case 30. Further, it must vanish for (5, 8), so that

$$\nu_6 = 15(5a - 8n) \quad (49)$$

5.2. The remaining cases present no special difficulty; all the formulae vanish for (5, 8), (6, 10), and (6, 12), and for forms of higher order, according to the multiplicities of the suffixes.

Thus the scroll of $5-2$ tangents is of order

$$\nu_{5-2} = 25\{a(2a - 3n) - 26a + 40n\}. \quad (50)$$

The scroll of 4—3 tangents is of order

$$\nu_{4-3} = 30\{(a-n)(3a-4n)-24a+36n\}. \tag{51}$$

Both these formulae vanish for (5, 8), (6, 10), and (6, 12).

The scroll of 4—2—2 tangents is of order

$$\nu_{4-2-2} = 10\left\{9\binom{a}{3}-89\binom{a}{2}-4n\binom{a}{2}+78an-32\binom{n}{2}+\tfrac{537}{2}a-480n\right\}. \tag{52}$$

The scroll of 3—3—2 tangents is of order

$$\nu_{3-3-2} = 45\left\{3\binom{a}{3}-25\binom{a}{2}-2n\binom{a}{2}+a\binom{n}{2}\right.$$
$$\left.+27an-24\binom{n}{2}+\tfrac{1233}{2}a-120n\right\}. \tag{53}$$

These results vanish for (5, 8), (6, 10), (6, 12), (7, 12), (7, 14), (7, 16).

The tritangents which are elsewhere inflexionally tangent generate a scroll of order

$${}_3\nu_{2-3} = 15\left\{4\binom{a}{4}-54\binom{a}{3}-n\binom{a}{3}-8a\binom{n}{2}+29n\binom{a}{2}+277\binom{a}{2}\right.$$
$$\left.+160\binom{n}{2}-\tfrac{5333}{2}an-\tfrac{11733}{2}a+1120n\right\}, \tag{54}$$

vanishing for (5, 8), (6, 10), ..., (8, 14), (8, 16), (8, 18).

The quintangents generate a scroll of order

$${}_5\nu_2 = 5\left\{\binom{a}{5}-18\binom{a}{4}+\tfrac{5377}{4}\binom{a}{3}+4n\binom{a}{3}+16a\binom{n}{2}-68n\binom{a}{2}\right.$$
$$\left.-\tfrac{20555}{4}\binom{a}{2}+482an-256\binom{n}{2}+\tfrac{74311}{8}a-1792n\right\}, \tag{55}$$

vanishing for (5, 8), (6, 10), ..., (9, 20).

The curves of contact.

6.1. From the results of § 4 we build up the functional equations which give the orders of the curves of contact of the various scrolls.

The order m_6 of the curve C_6 of 6-point contacts is obtained indirectly as follows. The formula must be linear in the characters of F, so that

$$m_6 = Aa+Nn, \text{ suppose.}$$

For (6, 10) this curve must form the complete intersection with S_6; hence (§ 5 . 1) we have

$$10A+6N=30. \qquad \text{(i)}$$

We now determine the number $\nu_{6:1}$ of 6-point tangents which are secant to the double surface. Evidently this is given by

$$\Delta\nu_{6:1}=\nu_6 b'+\nu_6' b+\{nn'\nu_6-6\,.\,n'm_6\}+\{nn'\nu_6'-6\,.\,nm_6'\}.$$

Hence

$$\nu_{6:1}=15\left\{5a\binom{n}{2}-24\binom{n}{3}-8n^2-\tfrac{5}{2}a^2+4an\right\}-6nm_6+A'a+N'n,$$

where A' and N' are to be found from special cases.

Since the formula must vanish for (6, 10), (7, 12), and (7, 14) we obtain, with (i), four equations for the four unknown constants. We thus derive the results

$$\nu_{6:1}=15\left\{5a\binom{n}{2}-24\binom{n}{3}+20n^2-\tfrac{5}{2}a^2-14an+58a-80n\right\}, \qquad (56)$$

and

$$m_6=5(9a-14n). \qquad (57)$$

6 . 11. The number of points in which this curve meets a generating plane is constant, and, from the case (6, 10), is equal to 5. The curve also meets J at certain points, since both curves lie on M_5; these intersections are the points of contact of the ξ stationary cuspidal generators (§ 2 . 62).

6 . 2. *Curves of contact of* S_{5-2}. For these we employ

LEMMA 8. *The 5-point tangents belonging to a* $\{5\}$ *of order* N *touch* F *along a curve of order* $N(\nu_5+m_5)$.

LEMMA 9. *The tangents belonging to a* $\{2\}$ (α, β) *have a curve of contact of order* $a\alpha+n\beta$.

Thus the orders of the curves are given by

$$m_{5:2}=(30a-43n)a+Aa+Nn,$$

$$m_{2:5}=5(2a-3n)(3a+2n)+A'a+N'n.$$

The results must vanish† for (6, 10); and, as before, this condition

† In order to establish the fact that (6, 10) contains no directrix lines, it is sufficient to observe that the formulae given later in this section vanish for (6, 10), (7, 12) and (7, 14).

leads to the same equation for A and N as for A' and N'. Assuming that $A = A'$ and $N = N'$ and using the fact that, for (7, 12),

$$5m_{5:2} + 2m_{2:5} = 7\nu_{5-2},$$

we obtain

$$m_{5:2} = (30a - 43n)a - 390a + 580n. \tag{58}$$

$$m_{2:5} = 5(2a - 3n)(3a + 2n) - 390a + 580n. \tag{59}$$

6.3. In discussing the curves of contact of S_{4-3} we proceed in a similar manner, using

LEMMA 10. *The 4-point tangents of a* $\{4\}$ (α, β) *have a contact curve of order* $\alpha(\nu_4 + nt_{4:}) + \beta m_4$.

LEMMA 11. *The 3-point tangents of a* $\{3\}$ (α, β) *have a contact curve of order* $\alpha\mu_3 + \beta(\nu_3 + nt_{3:})$.

From the same special cases we obtain

$$m_{4:3} = 6(a - n)(9a - 11n) - 432a + 624n. \tag{60}$$

$$m_{3:4} = 2(3a - 4n)(9a - 7n) - 432a + 624n. \tag{61}$$

6.4. The remaining formulae all vanish for (6, 10), (7, 12), and (7, 14). First we require

LEMMA 12. *The* $4-2$ *tangents belonging to a* $\{5\}$ *of order* N *have a curve of 4-point contacts of order* $N(\nu_{4-2} + m_{4:2})$.

From this and Lemma 10 we obtain

$$m_{4:2-2} = 2\left\{27\binom{a}{3} - 11n\binom{a}{2} - 267\binom{a}{2} + \tfrac{443}{2}an - 44n^2 + \tfrac{1611}{2}a - 1336n\right\}. \tag{62}$$

For the curve of ordinary contacts we use

LEMMA 13. *The* $4-2$ *tangents of a* $\{5\}$ *of order* N *have a curve of 2-point contacts of order* $N(\nu_{4-2} + m_{2:4})$.

LEMMA 14. *The bitangents of a* $\{3\}$ (α, β) *have a contact curve of order* $2\alpha\mu_{2-2} + \beta(2\nu_{2-2} + nt_{2:2})$.

Hence we have

$$m_{2;4-2} = 4\left\{27\binom{a}{3} - 6n\binom{a}{2} - 8a\binom{n}{2} - 267\binom{a}{2} + 183an + 16n^2 + \tfrac{1611}{2}a - 1336n\right\}. \quad (63)$$

6.5. Next consider the curves of contact of S_{3-3-2}. For the curve of 3-point contacts we use Lemma 11 and

LEMMA 15. *The* 3—2 *tangents of a* $\{4\}$ (α, β) *have a curve of* 3*-point contacts of order* $\alpha(\nu_{3-2} + nt_{3;2}) + \beta m_{3;2}$.

LEMMA 16. *The* 3—3 *tangents belonging to a* $\{5\}$ *of order* N *have a contact curve of order* $N(2\nu_{3-2} + m_{3-3})$.

We thus obtain

$$m_{3;3-2} = 6\left\{27\binom{a}{3} - 16n\binom{a}{2} - 225\binom{a}{2} + 7a\binom{n}{2} + 224an - 90n^2 + \tfrac{1107}{2}a - 930n\right\}. \quad (64)$$

For the curve of ordinary contacts we employ Lemma 9 and

LEMMA 17. *The* 3—2 *tangents of a* $\{4\}$ (α, β) *have a curve of* 2*-point contacts of order* $\alpha(\nu_{3-2} + nt_{2;3}) + \beta m_{2;3}$.

Hence we have

$$m_{2;3-3} = 9\left\{9\binom{a}{3} - 4n\binom{a}{2} - 75\binom{a}{2} - a\binom{n}{2} + 6\binom{n}{3} + 68an - 18n^2 + \tfrac{369}{2}a - 312n\right\}. \quad (65)$$

6.6. The curves of contact of ${}_3S_{2-3}$ are discussed as follows. For the curve of 3-point contact we require Lemmas 11 and 15, and also

LEMMA 18. *The* 3—2—2 *tangents belonging to a* $\{5\}$ *of order* N *have a curve of* 3*-point contacts of order* $N(\nu_{3-2-2} + m_{3;2-2})$.

Thus

$$m_{3;2-2-2} = 3\left\{12\binom{a}{4} - 162\binom{a}{3} - \tfrac{7}{3}n\binom{a}{3} \tfrac{77}{2}na^2 - \tfrac{56}{3}a\binom{n}{2} + 831\binom{a}{2} - 777an + \tfrac{608}{3}n^2 - \tfrac{3519}{2}a + \tfrac{8912}{3}n\right\}. \quad (66)$$

For the curve of 2-point contacts we use Lemmas 9, 14, 17, and

LEMMA 19. *The* 3—2—2 *tangents belonging to a* {5} *of order N have a curve of* 2*-point contacts of order* $N(2\nu_{3-2-2}+m_{2:2-3})$.

LEMMA 20. *The tritangents of a* {4} (α, β) *have a contact curve of order* $\alpha(3_3\nu_2+nt_{2:2-2})+\beta_3 m_2$.

Hence

$$\tfrac{1}{3}m_{2:2-2-3} = 36\binom{a}{4}-486\binom{a}{3}-3n\binom{a}{3}-4\binom{n}{2}\binom{a}{2}+26a\binom{n}{2}-48\binom{n}{3}$$
$$+185n\binom{a}{2}+1246\tfrac{1}{2}a^2-2024\tfrac{1}{2}an+312n^2$$
$$-6525a+8928n. \tag{67}$$

6 . 7. Finally, we obtain the order of the contact curve of ${}_5S_2$ by means of Lemmas 9, 14, 20, and

LEMMA 21. *The quadritangents belonging to a* {5} *of order N have a contact curve of order* $N(4_4\nu_2+{}_4m_2)$.

Thus we have

$${}_5m_2 = 15\binom{a}{5}-270\binom{a}{4}+2n\binom{a}{4}+16\binom{n}{2}\binom{a}{2}+25n\binom{a}{3}+2013\tfrac{3}{4}\binom{a}{3}$$
$$-761\tfrac{1}{2}n\binom{a}{2}-24a\binom{n}{2}+96\binom{n}{3}-7706\tfrac{1}{4}\binom{a}{2}+6204\tfrac{3}{4}an$$
$$-992n^2+\tfrac{111465}{8}a-23648n. \tag{68}$$

Multiple tangents satisfying six conditions.

7 . 1. With the exception of ν_7, the number of 7-point tangents, all the formulae of this section vanish for (6, 10), (7, 12) and (7, 14), by means of which the functional equations have been solved and checked. Some of the results vanish for forms of higher orders, as indicated below.

The number of 6—2 tangents is

$$\nu_{6-2} = 15\{(5a-8n)a-74a+120n\}. \tag{69}$$

The number of 5—3 tangents is

$$\nu_{5-3} = 75\{(a-n)(2a-3n)-17a+27n\}. \tag{70}$$

The number of 4—4 tangents is

$$\nu_{4-4} = 10\{(3a-4n)^2-66a+104n\}. \tag{71}$$

7.2. These vanish only for the three forms stated; the next three formulae vanish also for (8, 14), (8, 16), (8, 18).

The number of 4—3—2 tangents is

$$\nu_{4-3-2} = 60\left\{9\binom{a}{3}-7n\binom{a}{2}+4a\binom{n}{2}-79\binom{a}{2}+\tfrac{191}{2}an-96\binom{n}{2}+\tfrac{399}{2}a-420n\right\} \tag{72}$$

The number of 3—3—3 tangents is

$${}_3\nu_3 = 135\left\{\binom{a}{3}-\binom{n}{3}-n\binom{a}{2}+a\binom{n}{2}-7\binom{a}{2}+\tfrac{19}{2}an-12\binom{n}{2}+\tfrac{31}{2}a-35n\right\}. \tag{73}$$

The number of 5—2—2 tangents is

$$\nu_{5-2-2} = 25\left\{6\binom{a}{3}-3n\binom{a}{2}-66\binom{a}{2}+\tfrac{1223}{2}an-24\binom{n}{2}+225a-420n\right\}. \tag{74}$$

7.3. The two following results vanish also for (9, 16), (9, 18), (9, 20).

The number of bitangents which are elsewhere inflexionally bitangent is

$$\nu_{3-3-2-2} = 45\left\{6\binom{a}{4}-78\binom{a}{3}-3n\binom{a}{3}+\binom{n}{2}\binom{a}{2}+61n\binom{a}{2}-\tfrac{73}{2}a\binom{n}{2}+12\binom{n}{3}+\tfrac{7654}{4}a^2-\tfrac{9225}{2}an+432\binom{n}{2}-\tfrac{19177}{2}a+1680n\right\}. \tag{75}$$

The number of bitangents which are elsewhere 4-point tangent is

$${}_3\nu_{2-4} = 20\left\{6\binom{a}{4}-87\binom{a}{3}-2n\binom{a}{3}+54n\binom{a}{2}-16a\binom{n}{2}+\tfrac{2634}{4}a^2-507an+160n^2-\tfrac{26917}{2}a+2080n\right\}. \tag{76}$$

7.4. The number of quadritangents which are elsewhere 3-point tangent is

$$
\begin{aligned}
{}_4\nu_{2-3} = 15\Big\{ & 5\binom{a}{5} - 90\binom{a}{4} - n\binom{a}{4} - 8\binom{n}{2}\binom{a}{2} + \tfrac{75}{2}n\binom{a}{3} + \tfrac{2685}{4}\binom{a}{3} \\
& - \tfrac{18771}{4}n\binom{a}{2} - 48\binom{n}{3} + 212a\binom{n}{2} - 1104n^2 - 1284a^2 \\
& + 2922an + \tfrac{1}{8}a(5n-3a) + \tfrac{237151}{4}a - 8976n\Big\}. \qquad (77)
\end{aligned}
$$

This vanishes, in addition, for (10, 18), (10, 20), (10, 22), (10, 24).

7.5. The number of tangents having six contacts is

$$
\begin{aligned}
{}_6\nu_2 = 5\Big\{ & \binom{a}{6} - \tfrac{45}{2}\binom{a}{5} + \tfrac{439}{2}\binom{a}{4} + 4n\binom{a}{4} - \tfrac{23851}{2}\binom{a}{3} - 86n\binom{a}{3} \\
& + 16\binom{n}{2}\binom{a}{2} - 328a\binom{n}{2} + 64\binom{n}{3} + 805n\binom{a}{2} + 1888a^2 + \tfrac{13}{16}a^2 \\
& - 4242\tfrac{1}{2}an + 1472n^2 - 8001\tfrac{5}{8}a + 11968n\Big\}. \qquad (78)
\end{aligned}
$$

This vanishes for the above-mentioned cases and for (11, 20), (11, 22), (11, 24), (11, 26).

7.6. A further check on some of these results is obtained by considering the correspondence on the residual intersection of S_6 and F, of order

$$m_6' = n\nu_6 - 6m_6.$$

The united points arise from the 6—2 tangents and the 6-point tangents which meet Γ (counted twice). Hence we have

$$\nu_{6-2} + 2\nu_{6:1} = 2(n-7)\,m_6' - (n-6)(n-7)\,\nu_6,$$

which verifies the formulae of §§ 6.1, 7.1.

7.7. *The 7-point tangents.* To determine the number ν_7 of these tangents we first find the number $\nu_{6:}$ of 6-point tangents in points of Γ. By (14) the number of points in which the curve C_6 meets Γ is $(n-2)m_6 - 5(a-n)$. But this (§ 6.11) includes the ξ points, counted

λ times, say, in which C_6 meets J. Hence we have

$$\nu_{6:} = (n-2)\,m_6 - 5(a-n) - \lambda \,.\, 10(2a-3n).$$

The constant λ is found from the fact that $\nu_{6:}$, like ν_7, must vanish for (6, 10). Hence $\lambda = 5$ and

$$\nu_{6:} = nm_6 - 195a + 295n. \tag{79}$$

Now consider the correspondence between C_6 and the residual intersection with F of S_6, which is of order

$$m_6' = n\nu_6 - 6m_6.$$

The united points are the points ν_7 of contact of the 7-point tangents and the points $\nu_{6:}$ determined above. Thus

$$\nu_7 + \nu_{6:} = (n-6)\,m_6 + m_6' - (n-6)\,\nu_6,$$

which gives

$$\nu_7 = 35(3a-5n). \tag{80}$$

8.1. *Analysis of redundancies.* The following tables give the ordinary and multiple line constants for the formulae previously considered. The values of k are determined as in § 1.5; for the orders μ and classes ν of the congruences they are then verified by means of the results for (3, 4), (§ 3.5). We thus have

	μ_5 ν_5	μ_{4-2} ν_{4-2}	μ_{3-3} ν_{3-3}	μ_{3-2-2} ν_{3-2-2}	${}_4\mu_2$ ${}_4\nu_2$
k	−5	16	9	−36	16

8.2. *Surfaces of contact.* For (4, 6) the contribution $3d^{(4)}$ of the double scroll is found from the fact that the total contribution is $2k+3d^{(4)}$ (§ 1.5). For (5, 10) the contribution $5d^{(5)}$ of the double scroll is the value of the appropriate formula, save in the case of m_5, for which $d^{(5)}$ is found by the method of § 4.1. The results are

m	m_5	$m_{4:2}$	$m_{2:4}$	m_{3-3}	$m_{3:2-2}$	$m_{2:2-3}$	${}_4m_2$
k	1	−2	−4	−3	4	12	−8
$d^{(4)}$	2	−4	−8	−6	8	24	−16
$d^{(5)}$	−8	4	24	12	−8	−36	16

8.3. *Scrolls of tangents.* The process is as above; we obtain

ν	ν_6	ν_{5-2}	ν_{4-3}	ν_{4-2-2}	ν_{3-3-2}	$_3\nu_{2-3}$	$_5\nu_2$
k	−6	20	24	−48	−54	96	−32
$d^{(4)}$	−6	20	24	−48	−54	96	−32
$d^{(5)}$	30	−50	−60	80	90	−180	32

8.4. *Curves of contact.* Since (5, 8) contains an ordinary as well as a triple directrix, the total redundancy if $k+t^{(5)}$, suppose. It is found that *in each case* $t^{(5)}$ *and* k *are connected by the relation*

$$t^{(5)} = 9k,$$

the value of the formula being $10k$; k has the values

m	m_6	$m_{5:2}$	$m_{2:5}$	$m_{4:3}$	$m_{3:4}$	$m_{4:2-2}$	$m_{2:2-4}$	$m_{3\ 3-2}$	$m_{2:3-3}$	$_3m_{3:2}$	$_3m_{2:2-3}$	$_5m_2$
k	1	−2	−5	−3	−4	4	16	12	9	−8	−36	16

8.5. *Multiple tangents.* In this case $t^{(5)}$ and k are found to be connected by the relation

$$t^{(5)} = 4k,$$

where k takes the values

ν	ν_7	ν_{6-2}	ν_{5-3}	ν_{4-4}	ν_{4-3-2}	$_3\nu_3$	ν_{5-2-2}	$\nu_{3-3-2-2}$	$_3\nu_{2-4}$	$_4\nu_{2-3}$	$_6\nu_2$
k	−7	24	30	16	−144	−27	−60	216	128	−240	64

In the same way we may determine the redundancies of the triple lines on (6, 12) and (7, 16); but the results do not appear to be of interest, at any rate, for the present.

THE OPERATIONAL CALCULUS AND THE EVALUATION OF KAPTEYN INTEGRALS

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1. *Introduction.*

A number of papers† have recently been published on integrals of Kapteyn's and Bateman's types involving Bessel functions. It does not appear to be generally known that these integrals can be simply evaluated by the use of Heaviside's operational calculus, though Dr. van der Pol‡ has discussed one or two of the simpler cases by this means.

In its present-day form, the operational calculus is a convenient shorthand for a systematic use of Laplace's transformation. We call $f(p)$ the symbolic representation of $F(x)$ if it is defined by

$$f(p) = p\int_0^\infty e^{-px}F(x)\,dx\,; \tag{1.1}$$

for brevity, we write $f(p)\doteqdot F(x)$ or $F(x)\doteqdot f(p)$. A trivial example is $x^\alpha \doteqdot \Gamma(1+\alpha)p^{-\alpha}$, when $R(\alpha) > -1$.

In the present note no attempt is made at an exhaustive treatment; the examples are chosen as illustrative of a simple method which often enables us to replace heavy work involving generalized hypergeometric functions by elementary algebra. For clarity of exposition, all discussion of convergence of series or integrals is omitted; it can easily be supplied by the reader.

† W. N. Bailey, *Proc. London Math. Soc.* (2), 30 (1930), 422–424; 31 (1930), 200–208; L. J. Mordell, *Journal London Math. Soc.*, 5 (1930), 203–208.

‡ B. van der Pol, *Phil. Mag.*, 8 (1929), 861–898. It is, perhaps, of interest to draw attention to a recent paper by van der Pol and Niessen [*Ann. der Phys.* (5), 6 (1930), 273–294], which illustrates the great power of the method in physical problems.

Some earlier work by Prof. G. H. Hardy† covers the same ground as part of this paper, but uses infinite integrals of the type (1 . 1) instead of the equivalent operators. An example of this method is worked out at the end of chapter 12 of Watson's *Bessel functions*.

2. *Operational formulæ.*

The formulæ which we shall use are enumerated below without proofs, which can be found in van der Pol's paper or in an earlier and more rigorous discussion by Doetsch‡.

We are dealing with functions $F(x)$, continuous in any finite interval $0 < x < x_0$, such that

$$\lim_{\epsilon \to 0} \int_{\epsilon}^{x_0} |F(x)|\,dx$$

exists. The integrals which we propose to evaluate are all of the form

$$\int_0^x F_1(t)F_2(x-t)\,dt,$$

which Volterra calls a *produit de composition* and denotes by $F_1 * F_2(x)$; such a product can easily be shown to be continuous in any finite interval. The class of functions $F(x)$ is also such that

$$f(p) = p\int_0^\infty e^{-px}F(x)\,dx$$

is absolutely convergent in at least a half plane.

The fundamental theorem involved is Lerch's theorem§, which here takes the form that, if $f_1(p) \doteqdot F_1(x)$, $f_2(p) \doteqdot F_2(x)$, then $f_1(p) \equiv f_2(p)$ implies $F_1(x) \equiv F_2(x)$.

The theorem which enables us to evaluate simply Kapteyn integrals is the well-known one‖

$$f_1(p)f_2(p)/p \doteqdot F_1 * F_2(x).$$

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† *Messenger of Math.*, 50 (1920), 165–171.

‡ *Math. Annalen*, 89 (1923), 192–207. The notation used by Doetsch is different from van der Pol's, which is more convenient for our purpose. This paper is the first of a series, mainly in the *Math. Zeitschrift*, by Doetsch and Bernstein, in which the method is applied to the solution of problems in the theory of the conduction of heat. Doetsch's most recent paper [*Math. Zeitschrift*, 32 (1930), 587–599] deals with its application to the Hermite polynomials.

§ *Acta Math.*, 27 (1903), 339–351.

‖ *Cf.* Horn, *Jahresber. d. D.M.V.*, 25 (1917), 301–325; or Doetsch, *loc. cit.*, 198 (I).

Other theorems which we shall need are that

$$f(p) \doteqdot F(x)$$

implies

$$f(p/c) \doteqdot F(cx) \quad (c \text{ constant}), \tag{2.1}$$

$$pf(p) \doteqdot dF/dx \quad [\text{if } F(0) = 0], \tag{2.2}$$

$$p^{-1}f(p) \doteqdot \int_0^x F(x)\,dx, \tag{2.3}$$

$$-p\frac{d}{dp}\left\{\frac{f(p)}{p}\right\} \doteqdot xF(x), \tag{2.4}$$

$$\int_p^\infty \frac{f(p)}{p}\,dp \doteqdot \int_0^x \frac{F(x)}{x}\,dx, \tag{2.5}$$

$$\int_0^p \frac{f(p)}{p}\,dp \doteqdot \int_x^\infty \frac{F(x)}{x}\,dx, \tag{2.6}$$

$$\int_0^\infty \frac{f(p)}{p}\,dp = \int_0^\infty \frac{F(x)}{x}\,dx. \tag{2.7}$$

3. *Elementary examples.*

By the aid of the general theorems of § 2 and the following particular results

$$p/(1+p^2) \doteqdot \sin x, \tag{3.1}$$

$$p^2/(1+p^2) \doteqdot \cos x, \tag{3.2}$$

$$p\alpha^\mu/(1+p^2)^{\frac{1}{2}} \doteqdot J_\mu(x), \tag{3.3}$$

$$p\alpha^\mu \doteqdot \mu J_\mu(x)/x, \tag{3.4}$$

where

$$\alpha = (1+p^2)^{\frac{1}{2}} - p,$$

we can at once derive a number of the more simple integrals. For instance, Bateman's integral is immediately obtained thus:

$$\begin{aligned}\int_0^x J_\mu(t)J_\nu(x-t)\,\frac{dt}{t} &\doteqdot \frac{1}{p}\,\frac{p\alpha^\mu}{\mu}\,\frac{p\alpha^\nu}{(1+p^2)^{\frac{1}{2}}} \\ &= \frac{p\alpha^{\mu+\nu}}{\mu(1+p^2)^{\frac{1}{2}}} \\ &\doteqdot \frac{J_{\mu+\nu}(x)}{\mu},\end{aligned}$$

and hence, by Lerch's theorem,

$$\int_0^x J_\mu(t)\,J_\nu(x-t)\,\frac{dt}{t} = \frac{J_{\mu+\nu}(x)}{\mu}.$$

Again, we can quite simply evaluate the following integral† :

$$\int_0^x J_\mu(t)\,J_\nu(x-t)\,dt \doteqdot \frac{p\alpha^{\mu+\nu}}{(1+p^2)}$$

$$= \frac{p\alpha^{\mu+\nu}}{(1+p^2)^{\frac{1}{2}}}\,\frac{2\alpha}{1+\alpha^2}$$

$$= \frac{2p}{(1+p^2)^{\frac{1}{2}}} \sum_{r=0}^{\infty}{}' (-1)^r \alpha^{\mu+\nu+2r+1}$$

$$\doteqdot 2 \sum_{r=0}^{\infty}{}' (-1)^r J_{\mu+\nu+2r+1}(x),$$

i.e.
$$\int_0^x J_\mu(t)\,J_\nu(x-t)\,dt = 2 \sum_{r=0}^{\infty}{}' (-1)^r J_{\mu+\nu+2r+1}(x).$$

By the use of (2.2) and (2.5), we deduce from

$$\mu J_\mu(x)/x \doteqdot p\alpha^\mu$$

that
$$\mu J_\mu(x)/x^2 \doteqdot p \int_p^\infty \alpha^\mu\,dp$$

$$= p \int_0^\alpha \alpha^\mu\,\frac{1+\alpha^2}{2\alpha^2}\,d\alpha = \tfrac{1}{2}p\left(\frac{\alpha^{\mu-1}}{\mu-1} + \frac{\alpha^{\mu+1}}{\mu+1}\right);$$

hence we see that

$$\int_0^x J_\mu(t)\,J_\nu(x-t)\,\frac{dt}{t^2} \doteqdot \frac{1}{2\mu}\,p\,\left(\frac{\alpha^{\mu-1}}{\mu-1} + \frac{\alpha^{\mu+1}}{\mu+1}\right)\frac{\alpha^\nu}{(1+p^2)^{\frac{1}{2}}}$$

$$\doteqdot \frac{1}{2\mu}\left\{\frac{J_{\mu+\nu-1}(x)}{\mu-1} + \frac{J_{\mu+\nu+1}(x)}{\mu+1}\right\},$$

a result due to Bailey.

Lastly we have

$$\int_0^x J_\mu(t)\sin(x-t)\,dt \doteqdot p\alpha^\mu(1+p^2)^{-\frac{3}{2}}$$

$$= -2\mu\,\frac{p}{(1+p^2)^{\frac{1}{2}}} \sum_0^\infty{}' (-1)^r \alpha^{\mu+1+2r} - p\,\frac{d}{dp}\left\{\frac{\alpha^{\mu+1}}{(1+p^2)^{\frac{1}{2}}}\right\}$$

$$\doteqdot -2\mu \sum_0^\infty{}' (-1)^r J_{\mu+1+2r}(x) + x J_{\mu+1}(x),$$

[by the use of (2.4) and (3.3)], a result again due to Bailey.

† *Cf.* Watson, *Bessel functions*, § 12.2, where the proof involves the use of an expansion due to Bateman.

Other integrals, such as

$$\int_0^x \frac{J_\mu(t)}{t}\frac{J_\nu(x-t)}{x-t}\,dt, \quad \int_0^x \frac{J_\mu(t)}{t^2}\frac{J_\nu(x-t)}{x-t}\,dt, \quad \int_0^x J_\mu(t)\begin{matrix}\cos\\ \sin\end{matrix}(x-t)\,\frac{dt}{t},$$

etc., can be evaluated in the same way, which is solely dependent on the symbolical representation (3.1), (3.2), (3.3), and (3.4) of the trigonometric functions and Bessel's function $J_\mu(x)$. We now proceed with other representations of the Bessel functions.

4. *The symbolic representation of* $x^\lambda J_\mu(x)$.

By the use of Hankel's generalization of the integral of Lipschitz†, we find that

$$x^\lambda J_\mu(x) \doteq p\int_0^\infty e^{-px}x^\lambda J_\mu(x)\,dx$$

$$= \frac{\Gamma(\lambda+\mu+1)}{2^\lambda p^{\lambda+\mu}\Gamma(\mu+1)}\,{}_2F_1\left(\frac{\lambda+\mu+1}{2}, \frac{\lambda+\mu+2}{2};\ \mu+1;\ -\frac{1}{p^2}\right);$$

a particular case of this is

$$x^\mu J_\mu(x) \doteq \frac{\Gamma(\mu+\frac{1}{2})2^\mu p}{\Gamma(\frac{1}{2})(1+p^2)^{\mu+\frac{1}{2}}}.$$

An immediate deduction from this particular case is that

$$\int_0^x t^\mu(x-t)^\nu J_\mu(t)J_\nu(x-t)\,dt \doteq \frac{\Gamma(\mu+\frac{1}{2})\Gamma(\nu+\frac{1}{2})2^{\mu+\nu}p}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})(1+p^2)^{\mu+\nu+1}}$$

$$\doteq \frac{\Gamma(\mu+\frac{1}{2})\Gamma(\nu+\frac{1}{2})}{(2\pi)^{\frac{1}{2}}\Gamma(\mu+\nu+\frac{1}{2})}\,x^{\mu+\nu+\frac{1}{2}}J_{\mu+\nu+\frac{1}{2}}(x).$$

Similarly we see that

$$\int_0^x t^\mu(x-t)^{-\mu-1}J_\mu(t)J_\nu(x-t)\,dt$$

$$\doteq \frac{2^{\mu-\nu}\Gamma(\mu+\frac{1}{2})\Gamma(\nu-\mu)}{\Gamma(\frac{1}{2})\Gamma(\nu+1)p^{\nu+\mu}}\left(1+\frac{1}{p^2}\right)^{-\mu-\frac{1}{2}}{}_2F_1\left(\frac{\nu-\mu}{2}, \frac{\nu-\mu+1}{2};\ \nu+1;\ -\frac{1}{p^2}\right)$$

$$= \frac{2^{\mu-\nu}\Gamma(\mu+\frac{1}{2})\Gamma(\nu-\mu)}{\Gamma(\frac{1}{2})\Gamma(\nu+1)p^{\nu+\mu}}\,{}_2F_1\left(\frac{\mu+\nu+1}{2}, \frac{\mu+\nu+2}{2};\ \nu+1;\ -\frac{1}{p^2}\right)$$

$$\doteq \frac{2^\mu\Gamma(\mu+\frac{1}{2})\Gamma(\nu-\mu)}{\Gamma(\frac{1}{2})\Gamma(\mu+\nu+1)}\,x^\mu J_\nu(x).$$

† *Cf.* Watson, *op. cit.*, §13.2, (2).

Lastly, we have

$$\int_0^x t^{-1}(x-t)^{\kappa-1}J_\mu(t)\,J_\nu(x-t)\,dt$$

$$\doteq \frac{\Gamma(\kappa+\nu)\alpha^\mu}{\Gamma(\nu+1)2^\nu\mu p^{\kappa+\nu+1}}\,{}_2F_1\left(\frac{\kappa+\nu}{2},\ \frac{\kappa+\nu+1}{2};\ \nu+1;\ -\frac{1}{p^2}\right)$$

$$= 2^\kappa\frac{p\,\Gamma(\kappa+\nu)}{\mu\,\Gamma(\nu+1)}\,\alpha^{\kappa+\mu+\nu}(1-\alpha^2)^{-\kappa-\nu}\,{}_2F_1\left(\frac{\kappa+\nu}{2},\ \frac{\kappa+\nu+1}{2};\ \nu+1;\ -\frac{4\alpha^2}{(1-\alpha^2)^2}\right)$$

$$= 2^\kappa\frac{p\,\Gamma(\kappa+\nu)}{\mu\,\Gamma(\nu+1)}\,\alpha^{\kappa+\mu+\nu}\,{}_2F_1(\kappa+\nu,\ \kappa;\ \nu+1;\ -\alpha^2),\quad \text{by a result of Gauss}\dagger,$$

$$= \frac{2^\kappa p}{\mu}\sum_0^\infty(-1)^n\frac{\Gamma(\kappa+\nu+n)}{\Gamma(\nu+1+n)}\frac{(\kappa)_n}{n!}\,\alpha^{\kappa+\mu+\nu+2n}$$

$$\doteq \frac{2^\kappa}{\mu x}\sum_0^\infty(-1)^n\frac{\Gamma(\kappa+\nu+n)}{\Gamma(\nu+1+n)}\frac{(\kappa)_n}{n!}\,(\kappa+\mu+\nu+2n)\,J_{\kappa+\mu+\nu+2n}(x),$$

a result again due to Bailey. The three examples of this section are quite straightforward typical applications of the symbolic representation of $x^\lambda J_\mu(x)$.

5. *The symbolic representation of* $x^\lambda J_\mu(2x^{\frac12})$.

Van der Pol makes use of the following formulae:

$$x^\mu J_{2\mu}(2x^{\frac12}) \doteq p^{-2\mu}e^{-1/p}, \tag{5.1}$$

$$x^\mu I_{2\mu}(2x^{\frac12}) \doteq p^{-2\mu}e^{1/p}. \tag{5.2}$$

These are particular cases of

$$x^\lambda J_\mu(2x^{\frac12}) \doteq \frac{\Gamma(\lambda+\frac12\mu+1)}{\Gamma(\mu+1)\,p^{\lambda+\frac12\mu}}\,{}_1F_1\left(\lambda+\tfrac12\mu+1;\ \mu+1;\ -\frac{1}{p}\right), \tag{5.3}$$

$$x^\lambda I_\mu(2x^{\frac12}) \doteq \frac{\Gamma(\lambda+\frac12\mu+1)}{\Gamma(\mu+1)\,p^{\lambda+\frac12\mu}}\,{}_1F_1\left(\lambda+\tfrac12\mu+1;\ \mu+1;\ \frac{1}{p}\right), \tag{5.4}$$

which are immediate consequences of Hankel's generalization‡ of Weber's first exponential integral.

By the aid of these formulae, a number of integrals involving Bessel functions can be quickly evaluated. For example,

$$\int_0^x t^{\frac12\mu}J_\mu(2t^{\frac12})(x-t)^{\frac12\nu}I_\nu\{2(x-t)^{\frac12}\}\,dt \doteq p^{-\mu-\nu-1} \doteq \frac{x^{\mu+\nu+1}}{\Gamma(\mu+\nu+2)},$$

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† *Cf.* Forsyth, *Differential equations* (1914), 241, Ex. 5 (i).

‡ Watson, *op. cit.*, §13.3, (2).

a generalization of a known formula†. Similarly Sonine's first and second finite integrals can be very quickly evaluated. A slightly more difficult integral to handle is

$$\int_0^{\frac{1}{2}\pi} J_\mu(z\sin\theta)\,I_\nu(z\cos\theta)\tan^{\mu+1}\theta\,d\theta$$

$$= \tfrac{1}{2}\int_0^x J_\mu\{2(x-t)^{\frac{1}{2}}\}\,I_\nu(2t^{\frac{1}{2}})\,(x-t)^{\frac{1}{2}\mu}\,t^{-\frac{1}{2}\mu-1}\,dt \quad (z=2x^{\frac{1}{2}})$$

$$\doteq \tfrac{1}{2}\frac{\Gamma(\frac{1}{2}\nu-\frac{1}{2}\mu)}{\Gamma(\nu+1)}\,p^{-\frac{1}{2}(\nu+\mu)}e^{-1/p}\,{}_1F_1\left\{\tfrac{1}{2}(\nu-\mu);\ \nu+1;\ \frac{1}{p}\right\}$$

$$= \tfrac{1}{2}\frac{\Gamma(\frac{1}{2}\nu-\frac{1}{2}\mu)}{\Gamma(\nu+1)}\,p^{-\frac{1}{2}(\nu+\mu)}\,{}_1F_1\left\{\tfrac{1}{2}(\nu+\mu)+1;\ \nu+1;\ -\frac{1}{p}\right\}\ddagger$$

$$\doteq \tfrac{1}{2}\frac{\Gamma(\frac{1}{2}\nu-\frac{1}{2}\mu)}{\Gamma(\frac{1}{2}\nu+\frac{1}{2}\mu+1)}\,x^{\frac{1}{2}\mu}J_\nu(2x^{\frac{1}{2}})$$

$$= \tfrac{1}{2}\frac{\Gamma(\frac{1}{2}\nu-\frac{1}{2}\mu)}{\Gamma(\frac{1}{2}\nu+\frac{1}{2}\mu+1)}\left(\frac{z}{2}\right)^\mu J_\nu(z),$$

a result due to Rutgers§.

6. *Other symbolic representations of Bessel functions.*

An interesting particular case of (5 . 3) is

$$J_\mu(2x^{\frac{1}{2}}) \doteq \frac{\Gamma(\frac{1}{2}\mu+1)}{\Gamma(\mu+1)p^{\frac{1}{2}\mu}}\,{}_1F_1\left(\tfrac{1}{2}\mu+1;\ \mu+1;\ -\frac{1}{p}\right)$$

$$= \sqrt{\left(\frac{\pi}{4p}\right)}\,e^{-1/2p}\left\{I_{\frac{1}{2}(\mu-1)}\left(\frac{1}{2p}\right)-I_{\frac{1}{2}(\mu+1)}\left(\frac{1}{2p}\right)\right\}.$$

When we use (2 . 1), we get

$$J_\mu(4x^{\frac{1}{2}}) \doteq \sqrt{\left(\frac{\pi}{p}\right)}\,e^{-2/p}\left\{I_{\frac{1}{2}(\mu-1)}\left(\frac{2}{p}\right)-I_{\frac{1}{2}(\mu+1)}\left(\frac{2}{p}\right)\right\}, \qquad (6.1)$$

a form which suggests a connection with Weber's second exponential

† *Ibid.*, 376 (2).

‡ By Kümmer's formula, ${}_1F_1(\alpha;\ \rho;\ \zeta) = e^\zeta\,{}_1F_1(\rho-\alpha;\ \rho;\ -\zeta)$.

§ *Cf.* Watson, *op. cit.*, § 12.1 (4), where the factor $\frac{1}{2}$ is omitted.

integral, which shows that

$$J_\nu^2(2x^{\frac{1}{2}}) \doteqdot p\int_0^\infty e^{-px}J_\nu^2(2x^{\frac{1}{2}})\,dx$$

$$= e^{-2/p}I_\nu\left(\frac{2}{p}\right). \tag{6.2}$$

Hence we have

$$\int_0^x J_\nu^2(2t^{\frac{1}{2}})(x-t)^{-\frac{1}{2}}dt \doteqdot \sqrt{\left(\frac{\pi}{p}\right)}e^{-2/p}I_\nu\left(\frac{2}{p}\right)$$

$$\doteqdot \sum_{n=0}^{\infty} J_{2\nu+2n+1}(4x^{\frac{1}{2}}),$$

a result which Bailey† found by the use of Bateman's expansion.

Our last example will be the evaluation of‡

$$\int_0^x \frac{J_\nu^2(2t^{\frac{1}{2}})}{(x-t)^{\frac{1}{2}}}\frac{dt}{t}.$$

This involves a knowledge of the symbolic representation of $J_\nu^2(2x^{\frac{1}{2}})/x$, which we deduce from (6.2). For, by (2.5), we have

$$\int_0^x \frac{J_\nu^2(2x^{\frac{1}{2}})}{x}\,dx \doteqdot \int_p^\infty e^{-2/p}I_\nu\left(\frac{2}{p}\right)\frac{dp}{p},$$

and hence, by (2.2), we obtain

$$\frac{J_\nu^2(2x^{\frac{1}{2}})}{x} \doteqdot p\int_0^\infty e^{-2/p}I_\nu\left(\frac{2}{p}\right)\frac{dp}{p}$$

$$= \frac{p}{\nu}e^{-2/p}\left[I_\nu\left(\frac{2}{p}\right)+2\sum_{r=1}^{\infty} I_{\nu+r}\left(\frac{2}{p}\right)\right],$$

after integration by parts and the use of the recurrence formulae for $I_\nu(u)$. We have, in fact, obtained symbolically the value of

$$\int_0^\infty e^{-px}J_\nu^2(2x^{\frac{1}{2}})\frac{dx}{x}.$$

† (5.5) of the first paper quoted.

‡ *Ibid.* (5.7).

It follows from this that

$$\int_0^x \frac{J_\nu^2(2t^{\frac{1}{2}})}{(x-t)^{\frac{1}{2}}}\frac{dt}{t} \doteqdot \frac{\pi^{\frac{1}{2}}p^{\frac{1}{2}}}{\nu}e^{-2/p}\left[I_\nu\left(\frac{2}{p}\right)+2\sum_{r=1}^{\infty}I_{\nu+r}\left(\frac{2}{p}\right)\right]$$

$$\doteqdot \frac{1}{\nu}\frac{d}{dx}\left[\sum_{n=0}^{\infty}J_{2\nu+2n+1}(4x^{\frac{1}{2}})+2\sum_{r=1}^{\infty}\sum_{n=0}^{\infty}J_{2\nu+2n+2r+1}(4x^{\frac{1}{2}})\right]$$

$$=\frac{2}{\nu x^{\frac{1}{2}}}\left[\sum_{n=0}^{\infty}J'_{2\nu+2n+1}(4x^{\frac{1}{2}})+2\sum_{r=1}^{\infty}\sum_{n=0}^{\infty}J'_{2\nu+2n+2r+1}(4x^{\frac{1}{2}})\right]$$

$$=\frac{1}{\nu x^{\frac{1}{2}}}\left[J_{2\nu}(4x^{\frac{1}{2}})+2\sum_{r=1}^{\infty}J_{2\nu+2r}(4x^{\frac{1}{2}})\right]$$

$$=\frac{1}{2\nu x}\sum_{n=0}^{\infty}(2\nu+2n+1)J_{2\nu+2n+1}(4x^{\frac{1}{2}}).$$

AN EXTENSION OF MEISSEL'S EXPANSIONS IN KAPTEYN SERIES, AND SOME SIMILAR EXPANSIONS

By W. N. BAILEY.

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1. The expansions*

$$(1.1)\quad 2\sum_{n=1}^{\infty}\frac{J_{2n}(2nz)}{n^2+\zeta^2} = \frac{z^2}{1^2+\zeta^2}+\frac{z^4\zeta^2}{(1^2+\zeta^2)(2^2+\zeta^2)}+\frac{z^6\zeta^4}{(1^2+\zeta^2)(2^2+\zeta^2)(3^2+\zeta^2)}+\dots,$$

$$(1.2)\quad 2\sum_{n=1}^{\infty}\frac{J_{2n-1}\{(2n-1)z\}}{(2n-1)^2+\zeta^2} = \frac{z}{1^2+\zeta^2}+\frac{z^3\zeta^2}{(1^2+\zeta^2)(3^2+\zeta^2)}+\frac{z^5\zeta^4}{(1^2+\zeta^2)(3^2+\zeta^2)(5^2+\zeta^2)}+\dots,$$

are valid when ζ is a complex variable which is unrestricted except that ζi must not be an integer in (1 . 1) nor an odd integer in (1 . 2), and z lies in the domain K defined by the inequality

$$\left|\frac{z\exp\sqrt{(1-z^2)}}{1+\sqrt{(1-z^2)}}\right|<1.$$

In the theory of Kapteyn series as developed in Watson's treatise, these expansions form the basis of the whole discussion. One important deduction from the expansions is Kapteyn's expansion of z^n as a Kapteyn series, from which can be deduced the formula†

$$(1.3)\qquad (\tfrac{1}{2}z)^{\nu}=\nu^2\sum_{m=0}^{\infty}\frac{\Gamma(\nu+m)}{(\nu+2m)^{\nu+1}\,m!}J_{\nu+2m}\{(\nu+2m)z\},$$

which is valid throughout the domain K.

2. So far as I am aware, it has not been noticed that the expansions (1 . 1), (1 . 2), and (1 . 3) are all particular cases of a slightly more general

* G. N. Watson, *Theory of Bessel functions*, §§ 17 . 23 and 17 . 31.

† *Loc. cit.*, §§ 17 . 32, 17 . 33, and 17 . 5.

formula, viz.

$$(2.1)\quad \sum_{r=1}^{\infty} \frac{z^{2a+2r-2}\,\xi^{2r-2}}{\{a^2+\xi^2\}\{(a+1)^2+\xi^2\}\ldots\{(a+r-1)^2+\xi^2\}} = 2\sum_{n=0}^{\infty} \frac{\Gamma(2a+n)}{n!\,(a+n)^{2a-1}} \frac{J_{2a+2n}\{2z(a+n)\}}{(a+n)^2+\xi^2},$$

which is valid if ξ is an unrestricted complex variable except that the terms must be finite and z lies in the domain K. This formula reduces to (1.1) when $a=1$, to (1.2) when $a=\frac{1}{2}$, and to (1.3) when $\xi=0$, $a=\frac{1}{2}\nu$. Another interesting expansion is obtained by multiplying by ξ^2 and letting ξ tend to infinity, when we derive the formula

$$(2.11)\quad \frac{z^{2a}}{1-z^2} = 2\sum_{n=0}^{\infty} \frac{\Gamma(2a+n)}{n!\,(a+n)^{2a-1}} J_{2a+2n}\{2z(a+n)\}.$$

This gives known results when $a=1$ and when $a=\frac{1}{2}$*.

To prove (2.1), we note that, by partial fractions,

$$\frac{x^{r-1}}{\{a^2+x\}\{(a+1)^2+x\}\ldots\{(a+r-1)^2+x\}} = \sum_{n=1}^{r} \frac{(-1)^{r-n}\,2(a+n-1)^{2r-1}\,\Gamma(2a+n-1)}{\{(a+n-1)^2+x\}(n-1)!\,(r-n)!\,\Gamma(2a+n+r-1)}.$$

We therefore have, replacing x by ξ^2,

$$\begin{aligned}
&\sum_{r=1}^{\infty} \frac{z^{2a+2r-2}\,\xi^{2r-2}}{\{a^2+\xi^2\}\{(a+1)^2+\xi^2\}\ldots\{(a+r-1)^2+\xi^2\}} \\
&\quad= \sum_{r=1}^{\infty}\sum_{n=1}^{r} \frac{(-1)^{r-n}\,2(a+n-1)^{2r-1}\,\Gamma(2a+n-1)\,z^{2a+2r-2}}{\{(a+n-1)^2+\xi^2\}(n-1)!\,(r-n)!\,\Gamma(2a+n+r-1)} \\
&\quad= \sum_{n=1}^{\infty}\sum_{r=n}^{\infty} \frac{(-1)^{r-n}\,2(a+n-1)^{2r-1}\,\Gamma(2a+n-1)\,z^{2a+2r-2}}{\{(a+n-1)^2+\xi^2\}(n-1)!\,(r-n)!\,\Gamma(2a+n+r-1)} \\
&\quad= \sum_{n=1}^{\infty}\sum_{s=0}^{\infty} \frac{(-1)^{s}\,2(a+n-1)^{2n+2s-1}\,\Gamma(2a+n-1)\,z^{2a+2n+2s-2}}{\{(a+n-1)^2+\xi^2\}(n-1)!\,s!\,\Gamma(2a+2n+s-1)} \\
&\quad= \sum_{n=1}^{\infty} \frac{2\Gamma(2a+n-1)}{(n-1)!\,(a+n-1)^{2a-1}} \frac{J_{2a+2n-2}\{2z(a+n-1)\}}{(a+n-1)^2+\xi^2},
\end{aligned}$$

and the truth of (2.1) is evident.

The interchange in the order of summation can be justified† only if $|z|<0{\cdot}6627\ldots$, and the validity of the expansion throughout the

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* *Loc. cit.*, § 17.33.

† *Loc. cit.*, § 17.3.

remainder of the domain K has to be inferred by the theory of analytic continuation. It would be interesting to have a direct proof of the validity of the expansion (2.1) throughout the domain K. Such a proof would, of course, include a direct proof of (1.3), and at present no such proof is known*.

3. By differentiating (2.1) any number of times with regard to ζ^2 and then putting $\zeta = 0$, we deduce that, if m is a positive integer,

$$\frac{1}{z^{2a}} \sum_{n=0}^{\infty} \frac{\Gamma(2a+n)}{n!\,(a+n)^{2m+2a-1}} J_{2a+2n}\{2z(a+n)\}$$

is an even polynomial in z of degree $2m-2$. This is a generalization of the known results† that

$$\sum_{n=1}^{\infty} \frac{J_{2n}(2nz)}{n^{2m}}, \quad \sum_{n=1}^{\infty} \frac{J_{2n-1}\{(2n-1)z\}}{(2n-1)^{2m}}$$

are polynomials in z, the former an even polynomial of degree $2m$, and the latter an odd polynomial of degree $2m-1$. These follow from the more general formula by taking $a = 1$ and $a = \frac{1}{2}$ respectively.

For small integral values of m we derive from (2.1) the following results:

$$(3.1) \quad \sum_{n=0}^{\infty} \frac{\Gamma(2a+n)}{n!\,(a+n)^{2a+1}} J_{2a+2n}\{2z(a+n)\} = \frac{z^{2a}}{2a^2},$$

$$(3.2) \quad \sum_{n=0}^{\infty} \frac{\Gamma(2a+n)}{n!\,(a+n)^{2a+3}} J_{2a+2n}\{2z(a+n)\} = \frac{z^{2a}}{2a^2}\left\{\frac{1}{a^2} - \frac{z^2}{(a+1)^2}\right\},$$

$$(3.3) \quad \sum_{n=0}^{\infty} \frac{\Gamma(2a+n)}{n!\,(a+n)^{2a+5}} J_{2a+2n}\{2z(a+n)\}$$
$$= \frac{z^{2a}}{2a^2}\left[\frac{1}{a^4} - \frac{z^2}{(a+1)^2}\left\{\frac{1}{a^2} + \frac{1}{(a+1)^2}\right\} + \frac{z^4}{(a+1)^2(a+2)^2}\right].$$

When $a = 1$ and when $a = \frac{1}{2}$ these reduce to results given by Meissel†.

4. An argument similar to that used in § 2 shows that

$$(4.1) \quad \sum_{r=1}^{\infty} \frac{(a_1)_r (a_2)_r z^{2r} \zeta^{2r-2}}{(\beta_1)_r (\beta_2)_r \{a^2+\zeta^2\}\{(a+1)^2+\zeta^2\}\ldots\{(a+r-1)^2+\zeta^2\}}$$
$$= \sum_{n=1}^{\infty} \frac{2(a_1)_n (a_2)_n \Gamma(2a+n-1)(a+n-1)^{2n-1} z^{2n}}{(\beta_1)_n (\beta_2)_n (n-1)!\,\Gamma(2a+2n-1)\{(a+n-1)^2+\zeta^2\}}$$
$$\times {}_2F_3\left[\begin{matrix} a_1+n,\ a_2+n; \\ 2a+2n-1,\ \beta_1+n,\ \beta_2+n \end{matrix}\ -(a+n-1)^2 z^2\right].$$

* *Loc. cit.*, § 17.5.

† *Loc. cit.*, § 17.23.

In particular, taking $\alpha_1 = \frac{1}{2}(\mu+\nu+1)$, $\alpha_2 = \frac{1}{2}(\mu+\nu)+1$, $\beta_1 = \mu+1$, $\beta_2 = \nu+1$, where $\mu+\nu = 2a-2$, we find that

$$(4.2)\quad \sum_{r=1}^{\infty} \frac{\Gamma(2a+2r-1)}{\Gamma(\mu+r+1)\,\Gamma(\nu+r+1)} \frac{(\frac{1}{2}z)^{\mu+\nu+2r} z^{2r-2}}{\{a^2+\zeta^2\}\{(a+1)^2+\zeta^2\}\ldots\{(a+r-1)^2+\zeta^2\}}$$

$$= 2\sum_{n=0}^{\infty} \frac{\Gamma(2a+n)}{n!\,(a+n)^{2a-1}\{(a+n)^2+\zeta^2\}} J_{\mu+n+1}\{z(a+n)\}\, J_{\nu+n+1}\{z(a+n)\},$$

provided that $\mu+\nu = 2a-2$.

For example, if $\mu = \nu = a-1$, we derive the result

$$(4.3)\quad \sum_{r=1}^{\infty} \frac{\Gamma(2a+2r-1)}{\{\Gamma(a+r)\}^2} \frac{(\frac{1}{2}z)^{2a+2r-2}\,\zeta^{2r-2}}{\{a^2+\zeta^2\}\{(a+1)^2+\zeta^2\}\ldots\{(a+r-1)^2+\zeta^2\}}$$

$$= 2\sum_{n=0}^{\infty} \frac{\Gamma(2a+n)}{n!\,(a+n)^{2a-1}\{(a+n)^2+\zeta^2\}} J^2_{a+n}\{z(a+n)\}.$$

When $a = 1$, this reduces to the known result*

$$(4.31)\qquad 2\sum_{n=1}^{\infty} \frac{J_n^2(nz)}{n^2+\zeta^2} = \tfrac{1}{2}\frac{z^2}{1^2+\zeta^2} + \frac{1.3}{2.4}\frac{z^4\zeta^2}{(1^2+\zeta^2)(2^2+\zeta^2)} + \ldots,$$

and when $\zeta = 0$ we derive from (4.3) the expansion†

$$(4.32)\qquad (\tfrac{1}{2}z)^{2a} = \frac{2a^2\{\Gamma(a+1)\}^2}{\Gamma(2a+1)} \sum_{n=0}^{\infty} \frac{\Gamma(2a+n)}{n!\,(a+n)^{2a+1}} J^2_{a+n}\{z(a+n)\}.$$

The expansion (4.3) can also be obtained‡ from (2.1) by replacing z by $z\sin\theta$, and integrating with respect to θ from 0 to $\frac{1}{2}\pi$.

From (4.3) we infer as in § 3 that, if m is a positive integer,

$$\frac{1}{z^{2a}} \sum_{n=0}^{\infty} \frac{\Gamma(2a+n)}{n!\,(a+n)^{2m+2a-1}} J^2_{a+n}\{z(a+n)\}$$

is an even polynomial in z of degree $2m-2$.

5. Similar methods can be used to obtain some other expansions of a type similar to Meissel's, except that, instead of Kapteyn series, we obtain Neumann series. In this case there is no difficulty with regard to the interchange in the order of summation.

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* *Loc. cit.*, § 17.6 (1).

† *Loc. cit.*, § 17.6 (4). In the formula given in Watson's treatise, the factor $2\nu\{\Gamma(\nu+1)\}^2$ on the right should be replaced by $2\nu^2\{\Gamma(\nu+1)\}^2$.

‡ *Cf.* Watson, *loc. cit.*, § 17.6.

By partial fractions we have

$$\frac{1}{\{a^2+x\}\{(a+1)^2+x\}\dots\{(a+r-1)^2+x\}}$$

$$=\sum_{n=0}^{r-1}\frac{(-1)^n\,2(a+n)\,\Gamma(2a+n)}{\{(a+n)^2+x\}\,n!\,(r-n-1)!\,\Gamma(2a+n+r)},$$

and hence, as before,

$$(5.1)\qquad \sum_{r=1}^{\infty}\frac{(-1)^{r-1}(\frac{1}{2}z)^{2a+2r-2}}{\{a^2+\zeta^2\}\{(a+1)^2+\zeta^2\}\dots\{(a+r-1)^2+\zeta^2\}}$$

$$=2\sum_{n=0}^{\infty}\frac{(a+n)\,\Gamma(2a+n)}{n!\,\{(a+n)^2+\zeta^2\}}\,J_{2a+2n}(z).$$

In particular, if we multiply by ζ^2 and let ζ tend to infinity, we derive the well known result

$$(5.11)\qquad (\tfrac{1}{2}z)^{2a}=2\sum_{n=0}^{\infty}\frac{(a+n)\,\Gamma(2a+n)}{n!}\,J_{2a+2n}(z).$$

For $a=0$ and general values of ζ, (5.1) becomes

$$(5.12)\qquad \frac{J_0(z)}{\zeta^2}+2\sum_{n=1}^{\infty}\frac{J_{2n}(z)}{n^2+\zeta^2}=\sum_{r=1}^{\infty}\frac{(-1)^{r-1}(\frac{1}{2}z)^{2r-2}}{\zeta^2(1^2+\zeta^2)\dots\{(r-1)^2+\zeta^2\}}.$$

Similarly, we find that

$$(5.2)\qquad \sum_{r=1}^{\infty}\frac{(-1)^{r-1}(\alpha_1)_r(\alpha_2)_r z^{2r}}{(\beta_1)_r(\beta_2)_r\{a^2+\zeta^2\}\{(a+1)^2+\zeta^2\}\dots\{(a+r-1)^2+\zeta^2\}}$$

$$=\sum_{n=1}^{\infty}\frac{2(\alpha_1)_n(\alpha_2)_n(a+n-1)\,\Gamma(2a+n-1)\,z^{2n}}{(\beta_1)_n(\beta_2)_n(n-1)!\,\Gamma(2a+2n-1)\{(a+n-1)^2+\zeta^2\}}$$

$$\times{}_2F_3\left[\begin{matrix}\alpha_1+n,\ \alpha_2+n\,;\ -z^2\\ 2a+2n-1,\ \beta_1+n,\ \beta_2+n\end{matrix}\right].$$

Choosing α_1, α_2, β_1, β_2, so that the series ${}_2F_3$ is a product of Bessel functions, we deduce that

$$(5.3)\qquad \sum_{r=1}^{\infty}\frac{(-1)^{r-1}\,\Gamma(2a+2r-1)}{\Gamma(\mu+r+1)\,\Gamma(\nu+r+1)}\,\frac{(\frac{1}{2}z)^{2a+2r-2}}{\{a^2+\zeta^2\}\{(a+1)^2+\zeta^2\}\dots\{(a+r-1)^2+\zeta^2\}}$$

$$=2\sum_{n=0}^{\infty}\frac{(a+n)\,\Gamma(2a+n)}{n!\,\{(a+n)^2+\zeta^2\}}\,J_{\mu+n+1}(z)\,J_{\nu+n+1}(z),$$

where $\mu+\nu=2a-2$.

In particular, if $\mu = \nu = a-1$, we derive the expansion

$$(5.4)\quad \sum_{r=1}^{\infty} \frac{(-1)^{r-1}\,\Gamma(2a+2r-1)}{\{\Gamma(a+r)\}^2} \frac{(\frac{1}{2}z)^{2a+2r-2}}{\{a^2+\zeta^2\}\{(a+1)^2+\zeta^2\}\ldots\{(a+r-1)^2+\zeta^2\}}$$
$$= 2\sum_{n=0}^{\infty} \frac{(a+n)\,\Gamma(2a+n)}{n!\,\{(a+n)^2+\zeta^2\}}\, J^2_{a+n}(z).$$

For example, multiplying by ζ^2 and letting ζ tend to infinity, we derive the known formula*

$$(5.41)\qquad (\tfrac{1}{2}z)^{2a} = \frac{2\{\Gamma(a+1)\}^2}{\Gamma(2a+1)} \sum_{n=0}^{\infty} \frac{(a+n)\,\Gamma(2a+n)}{n!}\, J^2_{a+n}(z).$$

6. The formulae given above are capable of further generalization. Thus, by the methods of § 2, it is easily shown that

$$(6.1)\quad \sum_{r=1}^{\infty} \frac{(-1)^{r-1} z^{a+b+2r-2}\,\zeta^{2r-2}}{(a+\zeta)_r\,(b-\zeta)_r}$$
$$= \sum_{n=0}^{\infty} \frac{\Gamma(a+b+n)}{n!}\left[\frac{J_{a+b+2n}\{2z(a+n)\}}{(a+n)^{a+b}\,(a+\zeta+n)} + \frac{J_{a+b+2n}\{2z(b+n)\}}{(b+n)^{a+b}\,(b-\zeta+n)}\right],$$

where ζ is any complex number such that neither $a+\zeta$ nor $b-\zeta$ is zero or a negative integer, and z lies in the domain K. This formula reduces to (2.1) when $b = a$ and ζ is replaced by ζi.

If $\zeta = 0$, we deduce that

$$(6.2)\quad \frac{z^{a+b}}{ab} = \sum_{n=0}^{\infty} \frac{\Gamma(a+b+n)}{n!}\left[\frac{J_{a+b+2n}\{2z(a+n)\}}{(a+n)^{a+b+1}} + \frac{J_{a+b+2n}\{2z(b+n)\}}{(b+n)^{a+b+1}}\right],$$

and this is a generalization of (1.3).

If we multiply (6.1) by ζ and let ζ tend to infinity, we find that

$$(6.3)\quad \sum_{n=0}^{\infty} \frac{\Gamma(a+b+n)}{n!}\,\frac{J_{a+b+2n}\{2z(a+n)\}}{(a+n)^{a+b}}$$
$$= \sum_{n=0}^{\infty} \frac{\Gamma(a+b+n)}{n!}\,\frac{J_{a+b+2n}\{2z(b+n)\}}{(b+n)^{a+b}}.$$

Formulae of this kind are, however, not so interesting as those given above, and so the more general results have not been given completely.

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* *Loc. cit.*, § 5.5.

TEN-FIGURE TABLE OF THE COMPLETE ELLIPTIC INTEGRALS K, K', E, E', AND A TABLE OF $\frac{1}{\vartheta_3^2(0|\tau)}$, $\frac{1}{\vartheta_3^2(0|\tau')}$

By L. M. MILNE-THOMSON.

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The tables proceed by intervals of $0 \cdot 01$ of the parameter $m(=k^2)$. A table of q has already been given*.

By means of the values of q, the functions

$$\vartheta_1'/(2q^{\frac{1}{4}}) = 1-3q^2+5q^6-\ldots,$$

$$\vartheta_2/(2q^{\frac{1}{4}}) = 1+q^2+q^6+\ldots,$$

$$\vartheta_3 \quad = 1+2q+2q^4+\ldots,$$

$$\vartheta_4 \quad = 1-2q+2q^4+\ldots,$$

were calculated and checked by substitution in the identity $\vartheta_1' = \vartheta_2\vartheta_3\vartheta_4$. This ensured a correct table of all four functions. Corresponding tables of the reciprocals of all four were then made and checked by the same process.

To obtain a correct table of $1/\vartheta_3^2$ the square of the reciprocal of ϑ_3 was compared with the reciprocal of ϑ_3^2. The table of $1/\vartheta_3^2$ can be used for obtaining the argument $x = u/\vartheta_3^2$ for the expression of theta functions or Fourier series of the elliptic functions of u.

The values of K and K' were formed by multiplying ϑ_3^2 by $\pi/2$.

$$-\vartheta_2''/2q^{\frac{1}{4}} = 1+3^2q^2+5^2q^6+\ldots$$

was found, and the values of E and E' were then obtained from the formula

$$E = -\frac{\vartheta_2''}{\vartheta_2} \cdot \tfrac{1}{2}\pi \cdot \frac{1}{\vartheta_3^2}.$$

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* *Journal London Math. Soc.*, 5 (1930), 148.

The symmetry about $m = \frac{1}{2}$ of functions of the parameter m yields K' and E' from the same calculation.

Finally a check on the whole process was made by using Legendre's relation

$$KE' + K'E - KK' = \tfrac{1}{2}\pi.$$

The results indicate that the error should rarely exceed one unit in the last figure.

The formula just given for E can be obtained as follows* :

Since $$Z(u) = E(u) - \frac{E}{K}u,$$

we have
$$\begin{aligned} \mathrm{dn}^2 u &= \frac{d}{du} Z(u) + \frac{E}{K} \\ &= \frac{d^2}{du^2} \log \Theta(u) + \frac{E}{K} \\ &= \frac{d^2}{du^2} \log \vartheta_4\left(\frac{u}{\vartheta_3^2}\right) + \frac{E}{K} \\ &= \frac{1}{\vartheta_3^4} \frac{d^2}{dx^2} \log \vartheta_4(x) + \frac{E}{K}, \end{aligned}$$

where $x = u/\vartheta_3^2$.

Putting $x = 0$, we get

$$\frac{E}{K} = 1 - \frac{\vartheta_4''}{\vartheta_3^4 \vartheta_4},$$

since $\vartheta_4' = 0$.

Again $$\vartheta_2 \vartheta_4(x)\, \mathrm{cn}(x\vartheta_3^2) = \vartheta_4 \vartheta_2(x).$$

Differentiating twice and then putting $x = 0$, we get

$$\vartheta_2(\vartheta_4'' - \vartheta_3^4 \vartheta_4) = \vartheta_4 \vartheta_2''$$

or
$$1 - \frac{\vartheta_4''}{\vartheta_3^4 \vartheta_4} = -\frac{\vartheta_2''}{\vartheta_3^4 \vartheta_2},$$

so that
$$\frac{E}{K} = -\frac{\vartheta_2''}{\vartheta_3^4 \vartheta_2}$$

or
$$E = -\frac{\vartheta_2''}{\vartheta_2} \cdot \tfrac{1}{2}\pi \cdot \frac{1}{\vartheta_3^2},$$

since $K = \frac{1}{2}\pi \vartheta_3^2$.

* The simple proof here given was kindly supplied by a Referee. My own method of arriving at the formula was much less direct.

TABLE OF THE COMPLETE ELLIPTIC INTEGRALS K AND K'.

m	K	Δ	Δ	K'	m_1
0·00	1·5707 96327			∞	1·00
		39 49235	——		
·01	1·5747 45562	39 94351	3414 95918	3·6956 37365	0·99
·02	1·5787 39913	40 40429	1982 66499	3·3541 41447	·98
·03	1·5827 80342	40 87506	1397 62456	3·1558 74948	·97
·04	1·5868 67848	41 35605	1077 75243	3·0161 12492	0·96
0·05	1·5910 03453	41 84768	875 84752	2·9083 37249	·95
·06	1·5951 88221	42 35024	736 79492	2·8207 52497	·94
·07	1·5994 23245	42 86409	635 21599	2·7470 73005	·93
·08	1·6037 09654	43 38966	557 78073	2·6835 51406	·92
·09	1·6080 48620	43 92728	496 81220	2·6277 73333	0·91
0·10	1·6124 41348	44 47743	447 57567	2·5780 92113	·90
·11	1·6168 89091	45 04047	406 99223	2·5333 34546	·89
·12	1·6213 93138	45 61691	372 97294	2·4926 35323	·88
·13	1·6259 54829	46 20720	344 05069	2·4553 38029	·87
·14	1·6305 75549	46 81184	319 16473	2·4209 32960	0·86
0·15	1·6352 56733	47 43133	297 52932	2·3890 16487	·85
·16	1·6399 99866	48 06625	278 54986	2·3592 63555	·84
·17	1·6448 06491	48 71715	261 76833	2·3314 08569	·83
·18	1·6496 78206	49 38461	246 82598	2·3052 31736	·82
·19	1·6546 16667	50 06931	233 43811	2·2805 49138	0·81
0·20	1·6596 23598	50 77189	221 37571	2·2572 05327	·80
·21	1·6647 00787	51 49300	210 45257	2·2350 67756	·79
·22	1·6698 50087	52 23342	200 51573	2·2140 22499	·78
·23	1·6750 73429	52 99394	191 43836	2·1939 70926	·77
·24	1·6803 72823	53 77532	183 11442	2·1748 27090	0·76
0·25	1·6857 50355	54 57845	175 45463	2·1565 15648	·75
·26	1·6912 08200	55 40420	168 38321	2·1389 70185	·74
·27	1·6967 48620	56 25358	161 83544	2·1221 31864	·73
·28	1·7023 73978	57 12753	155 75573	2·1059 48320	·72
·29	1·7080 86731	58 02717	150 09611	2·0903 72747	0·71
0·30	1·7138 89448	58 95360	144 81489	2·0753 63136	·70
·31	1·7197 84808	59 90802	139 87569	2·0608 81647	·69
·32	1·7257 75610	60 89169	135 24669	2·0468 94078	·68
·33	1·7318 64779	61 90594	130 89980	2·0333 69409	·67
·34	1·7380 55373	62 95224	126 81031	2·0202 79429	0·66
0·35	1·7443 50597	64 03207	122 95620	2·0075 98398	·65
·36	1·7507 53804	65 14702	119 31797	1·9953 02778	·64
·37	1·7572 68506	66 29883	115 87818	1·9833 70981	·63
·38	1·7638 98389	67 48935	112 62118	1·9717 83163	·62
·39	1·7706 47324	68 72048	109 53295	1·9605 21045	0·61
0·40	1·7775 19372	69 99433	106 60085	1·9495 67750	·60
·41	1·7845 18805	71 31311	103 81346	1·9389 07665	·59
·42	1·7916 50116	72 67924	101 16049	1·9285 26319	·58
·43	1·7989 18040	74 09520	98 63254	1·9184 10270	·57
·44	1·8063 27560	75 56377	96 22106	1·9085 47016	0·56
0·45	1·8138 83937	77 08790	93 91830	1·8989 24910	·55
·46	1·8215 92727	78 67071	91 71720	1·8895 33080	·54
·47	1·8294 59798	80 31566	89 61119	1·8803 61360	·53
·48	1·8374 91364	82 02634	87 59438	1·8714 00241	·52
·49	1·8456 93998	83 80679	85 66126	1·8626 40803	0·51
0·50	1·8540 74677			1·8540 74677	0·50
m_1	K'	Δ	Δ	K	m

TABLE OF THE COMPLETE ELLIPTIC INTEGRALS E AND E'.

m	E	Δ	Δ	E'	m_1
0·00	1·5707 96327			1·0000 00000	1·00
·01	1·5668 61942	39 34385	159 93546	1·0159 93546	0·99
·02	1·5629 12645	39 49297	126 00974	1·0285 94520	·98
·03	1·5589 48244	39 64401	113 52341	1·0399 46861	·97
·04	1·5549 68546	39 79698	105 55366	1·0505 02227	0·96
		39 95194	99 71501		
0·05	1·5509 73352			1·0604 73728	·95
·06	1·5469 62456	40 10896	95 12402	1·0699 86130	·94
·07	1·5429 35653	40 26803	91 35277	1·0791 21407	·93
·08	1·5388 92730	40 42923	88 16096	1·0879 37503	·92
·09	1·5348 33465	40 59265	85 40014	1·0964 77517	0·91
		40 75828	82 97216		
0·10	1·5307 57637			1·1047 74733	·90
·11	1·5266 65017	40 92620	80 80874	1·1128 55607	·89
·12	1·5225 55369	41 09648	78 86054	1·1207 41661	·88
·13	1·5184 28454	41 26915	77 09074	1·1284 50735	·87
·14	1·5142 84027	41 44427	75 47108	1·1359 97843	0·86
		41 62196	73 97949		
0·15	1·5101 21831			1·1433 95792	·85
·16	1·5059 41612	41 80219	72 59837	1·1506 55629	·84
·17	1·5017 43101	41 98511	71 31350	1·1577 86979	·83
·18	1·4975 26026	42 17075	70 11314	1·1647 98293	·82
·19	1·4932 90109	42 35917	68 98760	1·1716 97053	0·81
		42 55051	67 92871		
0·20	1·4890 35058			1·1784 89924	·80
·21	1·4847 60581	42 74477	66 92959	1·1851 82883	·79
·22	1·4804 66375	42 94206	65 98428	1·1917 81311	·78
·23	1·4761 52126	43 14249	65 08776	1·1982 90087	·77
·24	1·4718 17514	43 34612	64 23554	1·2047 13641	0·76
		43 55305	63 42387		
0·25	1·4674 62209			1·2110 56028	·75
·26	1·4630 85873	43 76336	62 64927	1·2173 20955	·74
·27	1·4586 88155	43 97718	61 90884	1·2235 11839	·73
·28	1·4542 68698	44 19457	61 19989	1·2296 31828	·72
·29	1·4498 27128	44 41570	60 52008	1·2356 83836	0·71
		44 64064	59 86731		
0·30	1·4453 63064			1·2416 70567	·70
·31	1·4408 76115	44 86949	59 23971	1·2475 94538	·69
·32	1·4363 65871	45 10244	58 63555	1·2534 58093	·68
·33	1·4318 31919	45 33952	58 05328	1·2592 63421	·67
·34	1·4272 73821	45 58098	57 49155	1·2650 12576	0·66
		45 82688	56 94904		
0·35	1·4226 91133			1·2707 07480	·65
·36	1·4180 83394	46 07739	56 42463	1·2763 49943	·64
·37	1·4134 50127	46 33267	55 91725	1·2819 41668	·63
·38	1·4087 90839	46 59288	55 42594	1·2874 84262	·62
·39	1·4041 05019	46 85820	54 94977	1·2929 79239	0·61
		47 12880	54 48795		
0·40	1·3993 92139			1·2984 28034	·60
·41	1·3946 51652	47 40487	54 03974	1·3038 32008	·59
·42	1·3898 82992	47 68660	53 60440	1·3091 92448	·58
·43	1·3850 85568	47 97424	53 18128	1·3145 10576	·57
·44	1·3802 58774	48 26794	52 76981	1·3197 87557	0·56
		48 56802	52 36941		
0·45	1·3754 01972			1·3250 24498	·55
·46	1·3705 14505	48 87467	51 97955	1·3302 22453	·54
·47	1·3655 95691	49 18814	51 59977	1·3353 82430	·53
·48	1·3606 44814	49 50877	51 22958	1·3405 05388	·52
·49	1·3556 61135	49 83679	50 86857	1·3455 92245	0·51
		50 17254	50 51636		
0·50	1·3506 43881			1·3506 43881	0·50
m_1	E'	Δ	Δ	E	m

TABLE OF THE FUNCTIONS $\frac{1}{\vartheta_3^2(0|\tau)}$ AND $\frac{1}{\vartheta_3^2(0|\tau')}$.

m	$\frac{1}{\vartheta_3^2(0\|\tau)}$	Δ	Δ	$\frac{1}{\vartheta_3^2(0\|\tau')}$	m_1
0·00	1·00000 00000			0·00000 00000	**1·00**
		250 78557	42504 07094		
·01	0·99749 21443			0·42504 07094	**0·99**
		252 37428	4327 47604		
·02	0·99496 84015			0·46831 54698	·98
		253 98980	2942 17196		
·03	0·99242 85035			0·49773 71894	·97
		255 63290	2306 44487		
·04	0·98987 21745			0·52080 16381	**0·96**
		257 30436	1929 95237		
0·05	0·98729 91309			0·54010 11618	·95
		259 00504	1677 02151		
·06	0·98470 90805			0·55687 13769	·94
		260 73559	1493 58974		
·07	0·98210 17246			0·57180 72743	·93
		262 49707	1353 50907		
·08	0·97947 67539			0·58534 23650	·92
		264 29024	1242 46904		
·09	0·97683 38515			0·59776 70554	**0·91**
		266 11611	1151 92919		
0·10	0·97417 26904			0·60928 63473	·90
		267 97562	1076 45375		
·11	0·97149 29342			0·62005 08848	·89
		269 86978	1012 40599		
·12	0·96879 42364			0·63017 49447	·88
		271 79968	957 25396		
·13	0·96607 62396			0·63974 74843	·87
		273 76640	909 17659		
·14	0·96333 85756			0·64883 92502	**0·86**
		275 77105	866 82789		
0·15	0·96058 08651			0·65750 75291	·85
		277 81485	829 18996		
·16	0·95780 27166			0·66579 94287	·84
		279 89908	795 47766		
·17	0·95500 37258			0·67375 42053	·83
		282 02504	765 07492		
·18	0·95218 34754			0·68140 49545	·82
		284 19406	737 49102		
·19	0·94934 15348			0·68877 98647	**0·81**
		286 40755	712 32985		
0·20	0·94647 74593			0·69590 31632	·80
		288 66712	689 26797		
·21	0·94359 07881			0·70279 58429	·79
		290 97415	668 03835		
·22	0·94068 10466			0·70947 62264	·78
		293 33042	648 41852		
·23	0·93774 77424			0·71596 04116	·77
		295 73760	630 22150		
·24	0·93479 03664			0·72226 26266	**0·76**
		298 19748	613 28890		
0·25	0·93180 83916			0·72839 55156	·75
		300 71196	597 48551		
·26	0·92880 12720			0·73437 03707	·74
		303 28299	582 69536		
·27	0·92576 84421			0·74019 73243	·73
		305 91269	568 81810		
·28	0·92270 93152			0·74588 55053	·72
		308 60325	555 76664		
·29	0·91962 32827			0·75144 31717	**0·71**
		311 35694	543 46488		
0·30	0·91650 97133			0·75687 78205	·70
		314 17621	531 84605		
·31	0·91336 79512			0·76219 62810	·69
		317 06359	520 85126		
·32	0·91019 73153			0·76740 47936	·68
		320 02180	510 42841		
·33	0·90699 70973			0·77250 90777	·67
		323 05369	500 53119		
·34	0·90376 65604			0·77751 43896	**0·66**
		326 16222	491 11832		
0·35	0·90050 49382			0·78242 55728	·65
		329 35060	482 15279		
·36	0·89721 14322			0·78724 71007	·64
		332 62221	473 60144		
·37	0·89388 52101			0·79198 31151	·63
		335 98054	465 43435		
·38	0·89052 54047			0·79663 74586	·62
		339 42939	457 62450		
·39	0·88713 11108			0·80121 37036	**0·61**
		342 97279	450 14744		
0·40	0·88370 13829			0·80571 51780	·60
		346 61494	442 98099		
·41	0·88023 52335			0·81014 49879	·59
		350 36037	436 10482		
·42	0·87673 16298			0·81450 60361	·58
		354 21389	429 50056		
·43	0·87318 94909			0·81880 10417	·57
		358 18061	423 15132		
·44	0·86960 76848			0·82303 25549	**0·56**
		362 26596	417 04157		
0·45	0·86598 50252			0·82720 29706	·55
		366 47579	411 15718		
·46	0·86232 02673			0·83131 45424	·54
		370 81632	405 48502		
·47	0·85861 21041			0·83536 93926	·53
		375 29424	400 01305		
·48	0·85485 91617			0·83936 95231	·52
		379 91658	394 73015		
·49	0·85105 99959			0·84331 68246	**0·51**
0·50	0·84721 30848	384 69111	389 62602	0·84721 30848	**0·50**
m_1	$\frac{1}{\vartheta_3^2(0\|\tau')}$	Δ	Δ	$\frac{1}{\vartheta_3^2(0\|\tau)}$	m

A SPECIAL TYPE OF QUINTIC SYMMETROID

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Introduction.

In a series of interesting papers* Mr. L. Roth has developed the geometrical properties of the quintic symmetroid Γ, defined as the locus of the point $P \equiv (x, y, z, t)$ of three-dimensional space for which the hyperquadric

$$\Sigma \equiv xX+yY+zZ+tT = 0, \tag{1}$$

where X, Y, Z, T are four linearly independent hyperquadrics in four-dimensional space, is a hypercone. The object of the present paper is to discuss the particular case in which the linear system (or web) (1) contains a hyperquadric T, which consists of two quadritangent hyperplanes of the curve C common to the three hyperquadrics X, Y, Z.

The paper is divided into two parts, the first of which deals with the more general case in which T consists of any pair of hyperplanes, and the second with the applications of the results of Part I to the case in which these hyperplanes are quadritangent hyperplanes of C.

PART I.

The general case of a pair of hyperplanes.

1. Roth has shown† geometrically that, if the web (1) contains a pair of hyperplanes π, π', then the corresponding point P is a triple point on the symmetroid Γ, which will therefore have an equation of the form

$$-\Gamma \equiv u_3t^2+2u_4t+u_5 = 0, \tag{2}$$

where the triple point is taken as the vertex $D \equiv (0, 0, 0, 1)$ of the tetrahedron of reference, and u_r $(r = 3, 4, 5)$ is a homogeneous polynomial

* L. Roth, *Proc. Camb. Phil. Soc.*, 25 (1929), 268–271; *Proc. London Math. Soc.* (2), 30 (1930), 118–126; 425–432. I shall refer to these three papers as R I, R II, R III.

† R III, 431.

of degree r in x, y, z. The enveloping cone from the triple point to Γ is

$$\Phi \equiv u_4^2 - u_3 u_5 = 0, \tag{3}$$

which is of order eight. We easily deduce the following results :

The trinodal cone u_3 of Γ at the triple point D is a contact cone of the enveloping cone Φ from D, i.e. touches Φ along twelve generators which are the common generators of u_3 and a quartic cone u_4.

Any plane ϖ intersects the enveloping cone Φ in a curve of order eight which touches the quintic curve in which ϖ meets Γ at the twenty points in which this quintic curve is met by the quartic curve ϖu_4. This quartic curve meets the curve of order eight again in its twelve points of contact with the cubic curve ϖu_3.

2. Roth has also remarked* that, if the hyperquadric Σ touches a given hyperplane π, the locus of the corresponding point P is a contact quartic surface of Γ, *i.e.* touches it along a curve of order ten.

Consider the case in which π is one of the two hyperplanes π, π', which together constitute a hyperquadric of the web. The pencil of the web corresponding to a line through the triple point D consists of hyperquadrics through the two quadric surfaces in which π and π' meet any hyperquadric of the pencil. If a hyperquadric Σ' of the pencil touches π, the quadric $\pi\Sigma'$ is a cone; and, since every hyperquadric of the pencil contains this cone, it follows that every hyperquadric of the pencil touches π. Thus the contact quartic surface of Γ corresponding to π must be a cone with vertex D. Hence we deduce the following result :

The enveloping cone to Γ from the triple point D breaks up into two quartic cones, which we may call v_4, w_4, the contact quartic surfaces of Γ, corresponding respectively to π, π'.

We thus have

$$\Phi \equiv u_4^2 - u_3 u_5 \equiv v_4 w_4 = 0. \tag{4}$$

3. Viewed from another standpoint, the enveloping cone in question consists of lines through D which touch Γ elsewhere. Consider such a line l, touching Γ at Q; to Q corresponds a cone of the web with vertex Q', say. Then† Q' lies on the hyperquadric T corresponding to D, *i.e.* on π or π'. If Q' lies on π, it is the vertex of a cone of the net of quadric

* R III, 427.

† R III, 426.

surfaces in π, of which the base points are the eight points in which π meets the curve C; it is a well known result* that the locus of Q' is a sextic curve τ of genus three. A similar result holds if Q' lies on π'.

Thus, *the locus of points of contact Q of lines through D consists of two curves of order ten and genus three. Each of these curves has a sextuple point at D, the twelve tangents there being the twelve lines through D which meet Γ in five coincident points there, i.e. the twelve common generators of the cones u_3, u_4.*

4. Roth† has shown that Γ has sixteen double points (nodes) besides the triple point D. These correspond to the sixteen line-hypercones of the web of hyperquadrics. From (4) it is clear that the enveloping cone Φ has sixteen double generators, the common generators of v_4 and w_4. These are clearly the joins of D to the nodes of Γ.

If a pencil of hyperquadrics contains a pair of hyperplanes it contains, in general, two hypercones besides. If one of these becomes coincident with the pair of hyperplanes π, π', then every hyperquadric of the pencil meets the plane of intersection of π and π' in a line-pair. Thus a generator of the trinodal cone u_3, which meets Γ in four coincident points at D, corresponds to a pencil of the web every hyperquadric of which touches the plane $\pi\pi'$ at the same point.

5. Roth‡ has shown that the locus of vertices of hypercones of the web, in general of order ten, breaks up in the case which we are considering into the plane $\pi\,\pi'$ and a surface J of order nine, which meets the plane $\pi\pi'$ in a cubic curve σ, the locus of vertices of hypercones of the web which correspond to the points in which the generators of u_3 meet Γ again (besides in D). Thus π (and similarly in the case of π') meets J in a curve of order nine, which breaks up into this plane cubic curve σ and the sextic curve τ obtained in § 3 as the locus of the points Q' in π.

To a line through D there corresponds a pencil of hyperquadrics; each of these meets π in the same quadric surface, which passes through the eight points in which π meets the curve C. Conversely, a quadric surface through these eight points determines in the web a pencil containing the degenerate hyperquadric π, π'. Hence we have a one-one correspondence between the lines through D and the quadrics of the net in π determined by these eight points. To the generators of v_4 correspond the cones of the net. To the generators of u_3 correspond the quadrics of the net that

* See, for instance, Hesse, *Gesammelte Werke* (1897), 345–404 (349).

† R I, 271.

‡ R III, 431.

touch the plane $\pi\pi'$. Hence, applying Hesse's* theorems on the relationship between the plane quartic curve and a net of quadrics in three dimensions, we deduce that

The trinodal cone u_3 is a contact cone of v_4 (and similarly of w_4) such that the six generators of contact do not lie on a quadric cone.

Clearly, in the correspondence explained above, the six generators of contact correspond to the six points of intersection of the plane cubic σ with the twisted sextic τ.

PART II.

The case of a pair of quadritangent hyperplanes.

6. Now suppose that π, π' are quadritangent hyperplanes of the curve C. The base points of the net of quadrics in π then coincide in pairs in the four points of contact of π with C, and the sextic curve τ, the locus of vertices of cones of this net, breaks up into two twisted cubics through these four points. Hence v_4 (and similarly w_4) breaks up into two quadric cones, *i.e.*

$$v_4 \equiv v_2 v_2', \quad w_4 \equiv w_2 w_2', \tag{5}$$

and we have

$$\Phi \equiv u_4^2 - u_3 u_5 \equiv v_2 v_2' w_2 w_2'. \tag{6}$$

Now Roth† has shown that Γ has a node not only at a point corresponding to a line-hypercone of the web, but also at a point corresponding to a hypercone whose vertex is at one of the base points common to all hyperquadrics of the web, this case occurring when two of the base points coincide. Thus, in our particular case, the symmetroid Γ has eight further nodes, lying on the generators common to v_2, v_2' and to w_2, w_2'. Applying the results of Part I, we deduce that

The four quadric cones v_2, v_2', w_2, w_2' are contact cones of u_3 and of u_5, the generators of contact all lying on the quartic cone u_4.

Taking a plane section, we have a cubic curve u_3 with four contact conics v_2, v_2', w_2, w_2'. Now a plane cubic has three distinct systems of contact conics‡, the six points of contact of any two conics of the same

* Hesse, *loc. cit.*, 360.

† R III, 426.

‡ See, for instance, Pascal, *Repertorium der höheren Mathematik*, Aufl. 2, 2 (1910), 393, where the result is obtained by means of elliptic functions, or, for a geometrical account, Cremona, *Opere*, 1 (1914), 461.

system lying on a conic. We have shown (§ 5) that v_2, v_2' do not belong to the same system, neither do w_2, w_2'. Hence v_2 must belong to the same system as one of w_2, w_2', say w_2. Since all twelve points of contact lie on a quartic, it easily follows that v_2' must belong to the same system as w_2'. Hence, in the three-dimensional figure, the six generators of contact of v_2, w_2 with u_3 lie on a quadric cone p, and the six generators of contact of v_2', w_2' with u_3 lie on a quadric cone p_2'. Hence

$$p_2^2 - v_2 w_2 \equiv u_3 l_1, \quad p_2'^2 - v_2' w_2' \equiv u_3 l_1, \tag{7}$$

where l_1, l_1' are linear in x, y, z.

Clearly, l_1 is a tangent plane of the quadric cones v_2 and w_2, and p_2 passes through the generators of contact. Similarly for l_2'.

Also u_4 contains the twelve generators which p_2, p_2' have in common with u_3, so that we have

$$u_4 \equiv p_2 p_2' + u_3 m_1, \tag{8}$$

where m_1 is linear in x, y, z.

Substituting in (6), we get

$$u_5 \equiv u_3(m_1^2 - l_1 l_1') + p_2^2 l_1 + 2 p_2 p_2' m_1 + p_2'^2 l_1. \tag{9}$$

The equation (2) of Γ thus becomes

$$-\Gamma \equiv u_3(t+m_1)^2 + 2 p_2 p_2'(t+m_1) + p_2^2 l_1' + p_2'^2 l_1 - u_3 l_1 l_1' = 0,$$

or, writing t for $-(t+m_1)$, which is equivalent to taking the plane $t+m_1=0$ as the face opposite to D of the new tetrahedron of reference,

$$\Gamma \equiv \begin{vmatrix} l_1 & t & p_2 \\ t & l_1' & p_2' \\ p_2 & p_2' & u_3 \end{vmatrix} = 0. \tag{10}$$

This is a fundamental result.

We may summarize the results so far obtained as follows:

The quintic symmetroid of the web defined by any three hyperquadrics and the degenerate hyperquadric, consisting of two quadritangent hyperplanes of their curve of intersection, can have its equation expressed in the form (10). *The enveloping cone from the trinode breaks up into four quadric cones,*

$$\begin{matrix} v_2 & v_2' \\ w_2 & w_2'. \end{matrix}$$

The symmetroid has twenty-four nodes, in addition to the trinode. Sixteen of these lie on the common generators of a cone of the first row with a cone of the second row, and correspond to line-hypercones of the web. The remaining eight nodes lie on the common generators of cones in the same row, and correspond to cones of the web with vertices at the eight points of contact of the four quadritangent hyperplanes. Any one of the four quadric cones contains eight nodes of the first type and four of the second.

7. By equating to zero certain of the first minors of the determinant in (10) we obtain important surfaces. There are, in addition to the quadric cones v_2, v_2', w_2, w_2', two cubic surfaces

$$S_3 \equiv p_2 t - p_2' l_1 = 0, \quad S_3 \equiv p_2' t - p_2 l_1' = 0, \tag{11}$$

and a quadric cone

$$k_2 \equiv l_1 l_1' - t^2 = 0. \tag{12}$$

It is easily seen that S_3 has a node at D, the six lines through D consisting of the four generators common to p_2 and p_2', and the two generators in which l_1 cuts p_2. It contains the curves of contact of the two quadric cones v_2, w_2 with Γ. Similar results hold for S_3'.

The quadric cone k_2 is a contact cone of Γ, the curve of contact lying on the cubic surfaces S_3 and S_3'. It touches the two planes l_1, l_1', the plane $t = 0$ being the plane of contact.

Let the contact curves with Γ of the cones k_2, v_2, w_2, v_2', w_2' be called K, V, W, V', W'; they are all quintic curves. V, W, V', W' have triple points at D and are rational (being birationally related to twisted cubic curves, § 6). The curves V, W lie on the quadric cone p_2, and also on a cubic surface S_3 with a node at D, the six lines of this cubic surface which meet at D lying on p_2. Four of them lie also on p_2', the plane of the other two, viz. l_1, touching the cones v_2, w_2. The cubic surface S_3 meets Γ again in a quintic curve K, along which Γ is touched by the quadric cone k_2, which also touches the plane l_1. The same curve K is obtained similarly from the curves V', W'.

8. Let us now consider the relationship of k_2 with the nodes of Γ. It is easily seen that $v_2 w_2$ intersects $v_2' w_2'$ in sixteen generators which meet Γ in sixteen nodes that lie on k_2. Thus

k_2 passes through all the eight nodes of the second type and through the eight nodes of the first type defined by the two pairs of contact cones from D whose generators of contact with u_3 do not lie on a quadric cone.

The positions of these nodes with regard to the curves V, W, V', W' are indicated in the diagram

The curve V meets the quadric cone w_2' in six coincident points at D, and therefore in four further points, which are nodes of the first type. Hence V meets W', besides at D, in four nodes of the first type which lie on k_2. Similarly V' meets W in four nodes of the first type on k_2. The curves V and V' meet in four nodes of the second type : similarly for W and W'.

The plane l_1 is the tangent plane of the cones k_2, v_2, w_2; hence k_2 touches both v_2 and w_2, the points of contact, from equations (7) and (12), being at the two points in which the line $l_1 = t = 0$ meets the cone $p_2 = 0$. These points clearly lie on Γ, hence V and W both touch k_2. The ten points of intersection of V and k_2 are thus all accounted for, viz. two coincident points at the point of contact and eight nodes of Γ common to V, V', and to V, W'. It follows that

The remaining eight nodes, viz. four on V and W, and four on V' and W', do not lie on k_2.

Finally, V meets K in nine points, viz. (i) four nodes of Γ of the first type, (ii) four nodes of Γ of the second type, (iii) a point on the line $l_1 t$.

We recall also that the six tangents to V and W at D lie on the quadric cone p_2.

9. The symmetroid Γ possesses also another interesting family of contact quartic surfaces in addition to those of § 2. Consider the quartic surface

$$\Psi \equiv \begin{vmatrix} l_1 & t & p_2 & \lambda \\ t & l_1' & p_2' & \mu \\ p_2 & p_2' & u_3 & \nu \\ \lambda & \mu & \nu & 0 \end{vmatrix} = 0, \tag{13}$$

where λ, μ are constants, and ν is linear in x, y, z, t. In virtue of the

identity

$$\begin{vmatrix} l_1 & t & p_2 & \lambda_1 \\ t & l_1' & p_2' & \mu_1 \\ p_2 & p_2' & u_3 & \nu_1 \\ \lambda_1 & \mu_1 & \nu_1 & 0 \end{vmatrix} \times \begin{vmatrix} l_1 & t & p_2 & \lambda_2 \\ t & l_1' & p_2' & \mu_2 \\ p_2 & p_2' & u_3 & \nu_2 \\ \lambda_2 & \mu_2 & \nu_2 & 0 \end{vmatrix} - \begin{vmatrix} l_1 & t & p_2 & \lambda_1 \\ t & l_1' & p_2' & \mu_1 \\ p_2 & p_2' & u_3 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 & 0 \end{vmatrix}^2$$

$$\equiv \begin{vmatrix} l_1 & t & p_2 \\ t & l_1' & p_2' \\ p_2 & p_2' & u_3 \end{vmatrix} \times \begin{vmatrix} l_1 & t & p_2 & \lambda_1 & \lambda_2 \\ t & l_1' & p_2' & \mu_1 & \mu_2 \\ p_2 & p_2' & u_3 & \nu_1 & \nu_2 \\ \lambda_1 & \mu_1 & \nu_1 & 0 & 0 \\ \lambda_2 & \mu_2 & \nu_2 & 0 & 0 \end{vmatrix},$$

which we may write

$$\Psi_1\Psi_2 - \Psi_{12}^2 \equiv \Gamma\Xi, \tag{14}$$

we have the following result :

The quartic surfaces Ψ_1 *and* Ψ_2 *are contact surfaces of* Γ, *and the surface* Ψ_{12} *passes through their curves of contact with* Γ. *These curves of contact are of the tenth order, and* Ψ_{12} *meets* Ψ_1 (*and* Ψ_2) *in a residual sextic curve, the curve of contact of a cubic surface* Ξ *with* Ψ_1 (*and* Ψ_2).

If we take $\lambda_2 = \mu_2 = 0$, Ψ_2 breaks up into the quadric cone k_2 and the plane ν_2 taken twice over. Also Ψ_{12} breaks up into ν_2 and the cubic surface

$$\lambda_1(tp_2' - l_1'p_2) + \mu_1(tp_2 - l_1p_2') + \nu_1(l_1l_1' - t^2) = 0,$$

which is the general cubic surface containing the curve K.

Thus *any cubic surface through the curve* K *meets* Γ *in a residual curve of order ten along which* Γ *is touched by a quartic surface* Ψ_1.

For $\lambda_2 = \mu_2 = 0$, Ξ breaks up into the plane ν_2 counted twice, and the plane $\lambda_1^2 l_1' - 2\lambda_1\mu_1 t + \mu_1^2 l_1 = 0$, which is a general tangent plane of k_2; *this plane touches* Ψ_1 *along a conic.* The quartic surface Ψ_1 is therefore not general, but possesses a trope.

A cubic surface N through K intersects Γ in a residual curve of order ten, along which Γ is touched by Ψ_1. Ψ_{12} intersects Ψ_1 in this curve of order ten and in a residual curve of order six, along which Ψ_1 is touched by Ξ. Ψ_1 is intersected by the above cubic surface N in the curve of order ten and in a residual conic, viz. the conic of the trope of Ψ_1. Hence

the conic of the trope and the sextic of contact of Ξ and Ψ_1 are coresidual and lie on a quadric. *The curve of contact of* Ξ *and* Ψ_1 *is a quadri-cubic curve, and the quadric containing this sextic passes through the conic of the trope of* Ψ_1.

10. Let us now examine the Jacobian surface J of § 5, the surface of order nine which is the locus of vertices of hypercones of the web (1), in our particular case. In the more general case in which π, π' are any two hyperplanes, we have in the plane $\pi\pi'$ a cubic curve σ, lying on J, which is the locus of the point of intersection of line-pairs of the *net* of conics in which the *web* (1) is met by the plane $\pi\pi'$. In π we have a sextic curve τ, the locus of vertices of cones of the *net* of quadrics in which the web (1) is met by π. The curve τ lies on J, and hence the six points in which it meets the plane $\pi\pi'$ lie on σ. Similarly for the sextic curve τ' in π'.

When π, π' are quadritangent hyperplanes of the curve C, τ breaks up into two twisted cubics, which we may call V_0, V_0', since the cones of which they are the vertices correspond to the points of the curves V, V' on Γ. V_0, V_0' meet in the four points of contact of π with C. Similarly we have two twisted cubics W_0, W_0' in π'.

We have seen (§ 4) that the generators of the cubic cone u_3 are in one-one correspondence with the points of the cubic curve σ. The argument of § 6 therefore proves that the six points in which the curves V_0, W_0 meet σ lie on a conic.

To a point of intersection of V and W (other than D) there corresponds a line-hypercone of the web : its axis must therefore meet both V_0 and W_0. Hence

Four axes of line-hypercones of the web (1) *meet both the twisted cubics* V_0 *and* W_0, *and similarly for the pairs* V_0 *and* W_0', V_0' *and* W_0, V_0' *and* W_0'.

11. Roth* has shown that to a point of the curve of intersection of a hyperquadric of the web and the Jacobian surface J there corresponds a point on the curve of intersection of the symmetroid Γ and the polar quartic surface with respect to Γ of the point corresponding to the given hyperquadric. Take the hypercone Ω corresponding to the vertex O of the cone k_2. The polar quartic surface of O meets Γ in K and a residual curve K' of order fifteen. Since every polar quartic surface passes through

* R III, 427.

the nodes of Γ, it follows that K' has a double point at each of the eight nodes of Γ which do not lie on K, and passes simply through the sixteen nodes of Γ which lie on K. If we denote the curves on J which correspond to K, K' on Γ by K_0, K_0' respectively, we have the following results :

K_0 and K_0' each pass through the points of contact of π and π' with the curve C. The axes of the line-hypercones of the web which meet V_0 and W_0 (or V_0' and W_0') are chords of K_0'. The axes of the line-hypercones which meet V_0 and W_0' (or V_0' and W_0) meet both K_0 and K'.

The curve V meets the polar quartic surface of O in twenty points. Six of these coincide at the trinode D, since the polar surface has a node and the curve a triple point there. Twelve other points of intersection lie at the twelve nodes of Γ which lie on V, leaving two further points, one of which lies at the point in which V touches k_2 (§ 8). Hence V meets K' in one further point. Thus *the twisted cubics V_0, V_0', W_0, W_0' each meet K_0 and K_0' in one point, other than the points of contact of π, π' with C.*

The curve K' has a sextuple point at D, the six tangents being the six generators common to u_3 and the polar cone of O with respect to u_3. Hence

K_0' meets the cubic curve σ in six points which lie on the conic in which the hypercone Ω meets the plane $\pi\pi'$.

The curve K' meets the quadric cone k_2 in thirty points, of which eight are at nodes of Γ of the first type, and eight at nodes of Γ of the second type, and two coincide at O. There remain twelve points, which consist of six coincident pairs, since K' lies on Γ and k_2 is a contact cone. Thus K' meets K in six points other than O and nodes of Γ. They are plainly the six points such that generators of k_2 touch K, since the polar surface of O touches Γ where K' meets K. Hence

K_0' meets K_0 at the vertex of Ω and at six further points H, besides the eight points of contact of π, π' with C.

Consider a generator r of k_2. It meets the quintic curve K in two points k, k' other than O. If we project K from one of these points on to a plane we obtain a quartic curve with one double-point. Hence K is of genus two. Hence K_0 is also of genus two, and contains a linear series of pairs of points, corresponding to the pairs of points in which the generators of k_2 meet K. Let k_0, k_0' on K_0 correspond to k, k on K.

Plainly the hyperquadrics of a pencil of the web (1) touch Ω at k_0, k_0'. Let α, α' be the hypercones whose vertices are k_0, k_0'. Then Ω belongs to the pencil determined by α and α', and contains their vertices. Also α contains the vertex of α' and conversely, but neither contains the vertex of Ω; these results following from the fact that the generator r meets Γ in one point at O, but in two coincident points at each of k, k'. The polar hyperplane of the vertex O_0 of Ω with respect to the pencil α, α' will therefore meet α, α' in cones with vertices k_0, k_0', each passing through the vertex of the other. The cones therefore meet in the line $k_0 k_0'$ and in a twisted cubic through $k_0 k_0'$. Both the line and the twisted cubic must clearly lie on the hyperquadric Ω.

Since K' meets K at the six points where a generator of k_2 touches K, it is evident that K_0' meets K_0 at the six points H where k_0, k_0' become coincident. In this case α, α' coincide, and the polar hyperplane of O_0 with respect to α meets the surface $\alpha\Omega$ in the tangent to K_0 at k_0 and a residual twisted cubic touching K_0 at k_0. Hence

The six points H in which K_0' meets K_0 are such that the tangents to K_0 there lie wholly in Ω. Also, if α be the hypercone of the web (1) *whose vertex is at one of these points H, the polar hyperplane of the vertex of Ω with respect to α meets the surface $\alpha\Omega$ in the tangent to K_0 at H and a twisted cubic touching K_0 at H.*

In general only one hyperquadric of the web contains a given line, but the line k_0, k' lies on a pencil of hyperquadrics of the web.

To find the order of K_0, we note that the points in which it meets an arbitrary hyperplane correspond to the points in which K meets the curve of contact with Γ of the contact quartic surface corresponding to this solid. All such curves of contact plainly pass through the nodes of Γ which correspond to line-hypercones; hence eight points of intersection have to be left out of account. We are left with twelve points, which are to be regarded as six coincident pairs, since we are considering the intersections of a curve on Γ with a contact surface. Hence K_0 *is of order six and genus two.* Also K_0, K_0' form the complete intersection of Ω with the Jacobian surface; hence K_0' *is of order twelve.*

The fact that K_0' is of order twelve may easily be verified directly, since every contact quartic surface corresponding to a hyperplane has a node at D, and so meets K' in twelve coincident points at D. Also, there are eight nodes of Γ which are double points on K', and eight which are simple points on K'. We are left with $15 \times 4 - 12 - 16 - 8 = 24$ points of intersection, and the order of K_0' is twelve.

12. It may be remarked, in conclusion, that Γ belongs to a more general class of surfaces defined as follows :

Let Ψ be a quartic surface with a trope ; through the conic of contact draw a cubic surface which will meet Ψ in a residual curve of order ten, the curve of contact with Ψ of a quintic surface. The equation of Ψ will be of the form $L_1 U_3 - S_2^2 = 0$, where the suffix denotes the degree in the coordinates. A cubic surface through the conic $L_1 = 0$, $S_2 = 0$ has an equation of the form $L_1 S_2' - L_1' S_2 = 0$. It is clear that the quintic surface

$$M_1(L_1 U_3 - S_2^2) - L_1 S_2'^2 + 2 L_1' S_2 S_2' - L_1'^2 U_3 = 0$$

touches Ψ along the residual curve of intersection of Ψ and the cubic surface. The equation of this quintic surface may be written in the form

$$\begin{vmatrix} L_1 & L_1' & S_2 \\ L_1' & M_1 & S_2' \\ S_2 & S_2' & U_3 \end{vmatrix} = 0.$$

QUADRICS AND QUADRIC CONES OF A SET OF THREE ASSOCIATED STEINER TRIHEDRAL PAIRS

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1. *Introduction and notation.*

A set of three associated double sixes on the cubic surface and quadrics associated with them have been investigated* by means of a natural group notation and analysis. In this paper we consider a larger group, consisting of three sets of three associated double sixes derived from a set of three associated Steiner trihedral pairs.

We use Dixon's notation throughout, with certain extensions, except that, for convenience, we replace the S_r, T_r of his earlier paper by $2S_r^{(1)}$, $2T_r^{(1)}$. We suppose the cubic surface to have the equation $UVW = U'V'W'$, wherein $U=0$, ..., $U'=0$, ... are triple tangent planes. Then the six planes are connected by the identities

$$A_rU+B_rV+C_rW = A_r'U'+B_r'V'+C_r'W' \quad (r = 1, 2, 3),$$

where

$$A_1+A_2+A_3 = 0, \quad \ldots, \quad A_1'+A_2'+A_3' = 0, \quad A_rB_rC_r = A_r'B_r'C_r'.$$

In order to emphasize the arrangement of the three sets to be considered, we now put

$$A_rU-A_r'U' = x_r, \quad B_rV-B_r'V' = y_r, \quad C_rW-C_r'W' = z_r,$$

$$C_rW-B_r'V' = \xi_r, \quad A_rU-C_r'W' = \eta_r, \quad B_rV-A_r'U' = \zeta_r,$$

$$B_rV-C_r'W' = X_r, \quad C_rW-A_r'U' = Y_r, \quad A_rU-B_r'V' = Z_r.$$

* A. L. Dixon (1) *Journal London Math. Soc.*, 1 (1926), 170–175; (2) *Proc. London Math. Soc.* (2), 26 (1927), 351–362.

The sum of each row and each column is thus zero. The twenty-seven planes x_r, etc., are themselves triple tangent planes.

The table

	2_r	4_r	6_r
1_r	x_r	y_r	z_r
3_r	ξ_r	η_r	ζ_r
5_r	X_r	Y_r	Z_r

$(r = 1, 2, 3).$

then gives eighteen lines, which form three associated double sixes*, while the remaining nine lines may be written

	U'	V'	W'
U	7	15	11
V	12	8	13
W	14	10	9

.

The equations of these last nine lines are, in the above notation,

7, $x_1 = x_2 = x_3 = 0$; 8, $y_1 = y_2 = y_3 = 0$; 9, $z_1 = z_2 = z_3 = 0$;

10, $\xi_1 = \xi_2 = \xi_3 = 0$; 11, $\eta_1 = \eta_2 = \eta_3 = 0$; 12, $\zeta_1 = \zeta_2 = \zeta_3 = 0$;

13, $X_1 = X_2 = X_3 = 0$; 14, $Y_1 = Y_2 = Y_3 = 0$; 15, $Z_1 = Z_2 = Z_3 = 0$.

2. *The set of three associated Steiner trihedral pairs.*

We shall consider the three associated Steiner trihedral pairs :

$$\text{I} \begin{pmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \\ 13 & 14 & 15 \end{pmatrix}, \qquad \text{II} \begin{pmatrix} 2_1 & 2_2 & 2_3 \\ 4_1 & 4_2 & 4_3 \\ 6_1 & 6_2 & 6_3 \end{pmatrix}, \qquad \text{III} \begin{pmatrix} 1_1 & 3_1 & 5_1 \\ 1_2 & 3_2 & 5_2 \\ 1_3 & 3_3 & 5_3 \end{pmatrix},$$

where each line in each table meets the four lines in its minor group in the same table, but does not meet the four lines in the same row or column as itself.

The three double sixes associated with I are

$1_2\ 3_2\ 5_2\ 2_3\ 4_3\ 6_3$ $\quad 1_3\ 3_3\ 5_3\ 2_1\ 4_1\ 6_1$ $\quad 1_1\ 3_1\ 5_1\ 2_2\ 4_2\ 6_2$,

$1_3\ 3_3\ 5_3\ 2_2\ 4_2\ 6_2$ $\quad 1_1\ 3_1\ 5_1\ 2_3\ 4_3\ 6_3$ $\quad 1_2\ 3_2\ 5_2\ 2_1\ 4_1\ 6_1$.

* Dixon, *loc. cit.* (1), 171.

those associated with II are

$$\begin{matrix} 10\ 11\ 12\ 5_1\ 5_2\ 5_3 \\ 13\ 14\ 15\ 3_1\ 3_2\ 3_3 \end{matrix}' \qquad \begin{matrix} 13\ 14\ 15\ 1_1\ 1_2\ 1_3 \\ 7\ \ 8\ \ 9\ 5_1\ 5_2\ 5_3 \end{matrix}' \qquad \begin{matrix} 7\ \ 8\ \ 9\ 3_1\ 3_2\ 3_3 \\ 10\ 11\ 12\ 1_1\ 1_2\ 1_3 \end{matrix};$$

and those associated with III are

$$\begin{matrix} 4_1\ 4_2\ 4_3\ 9\ 12\ 15 \\ 6_1\ 6_2\ 6_3\ 8\ 11\ 14 \end{matrix}' \qquad \begin{matrix} 6_1\ 6_2\ 6_3\ 7\ 10\ 13 \\ 2_1\ 2_2\ 2_3\ 9\ 12\ 15 \end{matrix}' \qquad \begin{matrix} 2_1\ 2_2\ 2_3\ 8\ 11\ 14 \\ 4_1\ 4_2\ 4_3\ 7\ 10\ 13 \end{matrix}.$$

By the use of the tables giving the twenty-seven lines on the surface it is seen that I is transformed into II, II into III, and III into I, each line by line as they occur in I, II, and III, by the transformation

$$(1) \qquad \left\{ \begin{matrix} \begin{pmatrix} x_1, & y_1, & z_1, & \xi_1, & \eta_1, & \zeta_1, & X_1, & Y_1, & Z_1 \\ x_1, & x_2, & x_3, & y_1, & y_2, & y_3, & z_1, & z_2, & z_3 \end{pmatrix} \\ \begin{pmatrix} x_2, & y_2, & z_2, & \xi_2, & \eta_2, & \zeta_2, & X_2, & Y_2, & Z_2 \\ \xi_1, & \xi_2, & \xi_3, & \eta_1, & \eta_2, & \eta_3, & \zeta_1, & \zeta_2, & \zeta_3 \end{pmatrix} \\ \begin{pmatrix} x_3, & y_3, & z_3, & \xi_3, & \eta_3, & \zeta_3, & X_3, & Y_3, & Z_3 \\ X_1, & X_2, & X_3, & Y_1, & Y_2, & Y_3, & Z_1, & Z_2, & Z_3 \end{pmatrix} \end{matrix} \right.$$

where each symbol in the upper rows is replaced by the corresponding symbol in the lower rows.

The transformation preserves all the necessary identities. The inverse transformation changes III into II, II into I, and I into III.

The double sixes have been so arranged that again this transformation changes the first set of three into the second, the second into the third, and the third into the first, line by line. Having obtained the equation of any quadric associated with the first set of double sixes we shall now be able to write down that of the corresponding quadrics associated with the second and third.

The edges through a point of a trihedral pair.

The nine lines of a trihedral pair meet in eighteen points on the surface. Through each point of each trihedral pair, besides the two lines of the surface through the point, pass sixteen edges*, each of which contains three points of intersection of pairs of lines on the surface. These sixteen edges are the sixteen lines of intersection of two pencils of triple tangent planes, four planes through each line of the surface through the point.

* Henderson, "The twenty-seven lines upon the cubic surface", *Camb. Math. Tracts*, 13 (1911), 26.

Relative to the three trihedral pairs, the sixteen edges through a point are of four types. Consider the typical point 7, 11 in the first trihedral pair. Then it may be verified that the edges and their equations are as follows.

(*a*) The single line containing three points of I:

$$7,\ 11;\quad 12,\ 13;\quad 9,\ 14;\quad x_1/x_2/x_3 = \eta_1/\eta_2/\eta_3 = A_1/A_2/A_3.$$

(*b*) The three lines each of which contains 7, 11, and whose remaining two points are formed each as the intersection of one line from II and one from III:

$$7,\ 11;\quad 2_1,\ 3_1;\quad 4_1,\ 1_1;\quad x_1 = \eta_1 = 0.$$

$$7,\ 11;\quad 2_2,\ 3_2;\quad 4_2,\ 1_2;\quad x_2 = \eta_2 = 0.$$

$$7,\ 11;\quad 2_3,\ 3_3;\quad 4_3,\ 1_3;\quad x_3 = \eta_3 = 0.$$

(c) The six lines each of which contains 7, 11, one point of II, and one point of III:

$$7,\ 11;\quad 1_1,\ 3_2;\quad 2_1,\ 4_2;\quad x_1 = \eta_2 = 0.$$

$$7,\ 11;\quad 1_1,\ 3_3;\quad 2_1,\ 4_3;\quad x_1 = \eta_3 = 0.$$

$$7,\ 11;\quad 1_2,\ 3_3;\quad 2_2,\ 4_3;\quad x_2 = \eta_3 = 0.$$

$$7,\ 11;\quad 1_2,\ 3_1;\quad 2_2,\ 4_1;\quad x_2 = \eta_1 = 0.$$

$$7,\ 11;\quad 1_3,\ 3_1;\quad 2_3,\ 4_1;\quad x_3 = \eta_1 = 0.$$

$$7,\ 11;\quad 1_3,\ 3_2;\quad 2_3,\ 4_2;\quad x_3 = \eta_2 = 0.$$

(*d*) The six lines each of which contains 7, 11, a point of intersection of a line of I and a line of II, and a point of intersection of a line of I and a line of III:

$$7,\ 11;\quad 9,\ 1_1;\quad 13,\ 2_1;\quad x_1 = 0,\ \eta_1/\eta_2/\eta_3 = A_1/A_2/A_3.$$

$$7,\ 11;\quad 9,\ 1_2;\quad 13,\ 2_2;\quad x_2 = 0,\ \eta_1/\eta_2/\eta_3 = A_1/A_2/A_3.$$

$$7,\ 11;\quad 9,\ 1_3;\quad 13,\ 2_3;\quad x_3 = 0,\ \eta_1/\eta_2/\eta_3 = A_1/A_2/A_3.$$

$$7,\ 11;\quad 12,\ 3_1;\quad 14,\ 4_1;\quad \eta_1 = 0,\ x_1/x_2/x_3 = A_1/A_2/A_3.$$

$$7,\ 11;\quad 12,\ 3_2;\quad 14,\ 4_2;\quad \eta_2 = 0,\ x_1/x_2/x_3 = A_1/A_2/A_3.$$

$$7,\ 11;\quad 12,\ 3_3;\quad 14,\ 4_3;\quad \eta_3 = 0,\ x_1/x_2/x_3 = A_1/A_2/A_3.$$

The four triple tangent planes, other than the plane of 7 and 11, through the line 7 are thus

$$x_1 = 0,\quad x_2 = 0,\quad x_3 = 0,\quad x_1/x_2/x_3 = A_1/A_2/A_3,$$

and the four planes through the line 11 are

$$\eta_1 = 0, \quad \eta_2 = 0, \quad \eta_3 = 0, \quad \eta_1/\eta_2/\eta_3 = A_1/A_2/A_3.$$

The two pencils of planes are obviously equi-anharmonic*, and we deduce the result :

The two lines 7 and 11 and the four edges (a) and (b) lie upon a quadric cone†. A similar property holds for every one of the fifty-four points of I, II, and III.

The cone whose vertex is the typical point 7, 11 evidently has the alternative equations $x_1/x_2/x_3 = \eta_1/\eta_2/\eta_3$. We denote the cone by the symbol (7, 11), which symbol will also represent each of the equal expressions

$$x_1\eta_2 - x_2\eta_1, \quad x_2\eta_3 - x_3\eta_2, \quad x_3\eta_1 - x_1\eta_3.$$

3. *The quadrics S through six lines.*

It is easily verified, by an inspection of the tables of lines on the surface, that the quadric $S_r^{(1)}$ ($r = 1, 2, 3$) through the six lines

$$\begin{pmatrix} 1_r & 3_r & 5_r \\ 2_r & 4_r & 6_r \end{pmatrix}$$

is represented by any one of the nine equivalent equations

$$x_r/y_r/z_r = \xi_r/\eta_r/\zeta_r = X_r/Y_r/Z_r.$$

These equations for $r = 1, 2, 3$ are then equations of the three S quadrics associated with the first set of double sixes.

Moreover, if we use $S_r^{(1)}$ to represent any one of the nine equal expressions

$$x_r\eta_r - y_r\xi_r, \quad \xi_r Y_r - \eta_r X_r, \quad X_r y_r - Y_r x_r, \quad y_r\zeta_r - z_r\eta_r, \quad \text{etc.},$$

then it can be shown without much trouble that $2S_r^{(1)}$ is Dixon's expression S_r, namely

$$A_r^2U^2 + B_r^2V^2 + C_r^2W^2 - A_r'^2U'^2 - B_r'^2V'^2 - C_r'^2W'^2.$$

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* The result has been given and proved by W. P. Milne, *Proc. London Math. Soc.* (2), 10 (1911), 447.

† The sixteen edges lie by fours on four quadric cones, each of which passes through the lines 7 and 11. Here, however, we shall deal only with the cones given in the text; the remaining three cones through each point are associated with groupings outside the trihedral pairs chosen.

The equations of the six quadrics $S_1^{(2)}$, $S_2^{(2)}$, $S_3^{(2)}$; $S_1^{(3)}$, $S_2^{(3)}$, $S_3^{(3)}$, associated with II and III respectively through the lines

$$\begin{pmatrix} 7 & 8 & 9 \\ 1_1 & 1_2 & 1_3 \end{pmatrix}, \quad \begin{pmatrix} 10 & 11 & 12 \\ 3_1 & 3_2 & 3_3 \end{pmatrix}, \quad \begin{pmatrix} 13 & 14 & 15 \\ 5_1 & 5_2 & 5_3 \end{pmatrix};$$

$$\begin{pmatrix} 2_1 & 2_2 & 2_3 \\ 7 & 10 & 13 \end{pmatrix}, \quad \begin{pmatrix} 4_1 & 4_2 & 4_3 \\ 8 & 11 & 14 \end{pmatrix}, \quad \begin{pmatrix} 6_1 & 6_2 & 6_3 \\ 9 & 12 & 15 \end{pmatrix},$$

may now be written down by the use of the transformation (1), and we can form the following table, which gives nine alternative equations for each of the nine quadrics :

(2)

I	II	III
$S_1^{(1)}$, $\begin{matrix} \underline{x_1} & \underline{\xi_1} & X_1 \\ \underline{y_1} = \eta_1 = Y_1 \\ z_1 & \zeta_1 & Z_1 \end{matrix}$	$S_1^{(2)}$, $\begin{matrix} \underline{x_1} & y_1 & z_1 \\ \underline{x_2} = y_2 = z_2 \\ x_3 & y_3 & z_3 \end{matrix}$	$S_1^{(3)}$, $\begin{matrix} x_1 & \underline{x_2} & \underline{x_3} \\ \xi_1 = \underline{\xi_2} = \underline{\xi_3} \\ X_1 & X_2 & X_3 \end{matrix}$
$S_2^{(1)}$, $\begin{matrix} \underline{x_2} & \xi_2 & X_2 \\ \underline{y_2} = \eta_2 = Y_2 \\ z_2 & \zeta_2 & Z_2 \end{matrix}$	$S_2^{(2)}$, $\begin{matrix} \xi_1 & \eta_1 & \zeta_1 \\ \underline{\xi_2} = \eta_2 = \zeta_2 \\ \xi_3 & \eta_3 & \zeta_3 \end{matrix}$	$S_2^{(3)}$, $\begin{matrix} \underline{y_1} & \underline{y_2} & \underline{y_3} \\ \eta_1 = \eta_2 = \eta_3 \\ Y_1 & Y_2 & Y_3 \end{matrix}$
$S_3^{(1)}$, $\begin{matrix} \underline{x_3} & \xi_3 & X_3 \\ \underline{y_3} = \eta_3 = Y_3 \\ z_3 & \zeta_3 & Z_3 \end{matrix}$	$S_3^{(2)}$, $\begin{matrix} X_1 & Y_1 & Z_1 \\ X_2 = Y_2 = Z_2 \\ X_3 & Y_3 & Z_3 \end{matrix}$	$S_3^{(3)}$, $\begin{matrix} z_1 & \underline{z_2} & \underline{z_3} \\ \zeta_1 = \underline{\zeta_2} = \underline{\zeta_3} \\ Z_1 & Z_2 & Z_3 \end{matrix}$

4. *The fifty-four associated cones.*

It is readily seen that the equations of each S in any table column of (2) have a row or a column identical with a column or a row in the equations of any S in each of the other two table columns.

This common row or column represents the generator common to the two S quadrics.

By equating the remaining ratios in the equations of these two S quadrics we obtain just four quadrics which pass through the residual cubic curve of intersection of the S quadrics.

We find that in all cases the four quadrics so obtained are, in fact, cones, for the four variables which occur in each such ratio equation reduce in effect to three by virtue of the identical relations between the variables.

For example, one cone through the cubic curve $S_1^{(2)}$, $S_2^{(3)}$ is

$$x_1/x_2/x_3 = \eta_1/\eta_2/\eta_3.$$

Considering only the cones derived from the second and third columns, since the remaining two sets may be derived by the transformation (1), we find that each cone occurs twice. For example, the cone $x_1/x_2/x_3 = \eta_1/\eta_2/\eta_3$ is derived from $S_1^{(2)}$, $S_2^{(3)}$ and also from $S_2^{(2)}$, $S_1^{(3)}$. There are thus, in all, eighteen different cones arising from the second and third table columns of (2).

The eighteen cones derived from the S quadrics II and III are just the eighteen cones of the type (7, 11) *whose vertices are the eighteen points of the trihedral pair I.*

For the equations of the cones derived from these S quadrics are easily seen to be of the form $u_1/u_2/u_3 = v_1/v_2/v_3$, where u and v are two of the variables $x, y, z, \xi, \eta, \zeta, X, Y, Z$, and also the nine lines $u_1 = u_2 = u_3 = 0$ are just the nine lines of the trihedral pair I.

We thus obtain the geometrical result :

The two cubic curves $S_p^{(2)}$, $S_q^{(3)}$; $S_{p'}^{(2)}$, $S_{q'}^{(3)}$ $(p' \neq p,\ q' \neq q)$ *lie on the same quadric cone of the trihedral pair I.*

It may be verified, in fact, by the use of the equations of the quadrics and cones, that the quartic surface

$$S_p^{(i)} S_{p'}^{(i)} - S_q^{(j)} S_{q'}^{(j)}$$

breaks up into two cones of the fifty-four for all values of i, j, p, p', q, q', chosen from 1, 2, 3, with the restrictions $j \neq i$, $p' \neq p$, $q' \neq q$. The two vertices in each case are the two points of intersection of the four lines in a two by two block of that trihedral pair which corresponds neither to i nor j.

For example*,

$$(7,\ 11)(8,\ 10) = S_1^{(2)} S_2^{(2)} - S_1^{(3)} S_2^{(3)},$$

$$(4_3,\ 6_1)(6_3,\ 4_1) = S_2^{(3)} S_3^{(3)} - S_1^{(1)} S_3^{(1)}.$$

There are, in all, twenty-seven such cubic curves of intersection of the S quadrics, since any quadric from one column of (2) can be taken with any quadric from each of the other two columns. Four cones pass through each curve, and each cone contains two curves. The fifty-four cones (l, m) whose vertices are the fifty-four points of the three trihedral pairs

* For the sake of simplicity we give no convention for the sign to be associated with (l, m) in the equations of the cones. It will be understood that in these and later equations suitable signs are taken. No ambiguity will be so caused. The sign of $S_r^{(i)}$ is, however, determinate by the convention adopted at the beginning of § 3.

are thus accounted for, and fall into three groups of eighteen, corresponding to the trihedral pairs I, II, III.

By an inspection of the tables (2) it can be verified that the four vertices of the four cones of I through each of the nine curves derived from II and III form a quadrilateral on the cubic surface, each side of which is a line of I.

It can also be verified that the four cones of I whose vertices lie on a line of I have this line as a common generator, and pass through four cubic curves in pairs, so that each cone passes through two of the four curves and each curve lies on two of the four cones.

For example, the four cones (7, 11); (7, 15); (7, 12); (7, 14) pass respectively through the pairs of four cubic curves

$$S_1^{(2)}, S_2^{(3)}; \quad S_2^{(2)}, S_1^{(3)}: \quad S_1^{(2)}, S_3^{(3)}; \quad S_3^{(2)}, S_1^{(3)}:$$

$$S_1^{(2)}, S_3^{(3)}; \quad S_2^{(2)}, S_1^{(3)}: \quad S_1^{(2)}, S_2^{(3)}; \quad S_3^{(2)}, S_1^{(3)}.$$

These four cones evidently pass two at a time through each of two cubic curves on each of the two quadrics which contain the line of vertices, and through one cubic curve on each of the remaining four quadrics of the six.

Moreover, the cones are connected by the following property independently of the S quadrics.

The eighteen cones I fall into six groups of three, such that the lines which join the vertices of the three cones in each group form a triangle of the trihedral pair I; each side of the triangle is a common generator of two cones of the three, and the three cones meet in a common cubic curve which is not a curve of intersection of the S quadrics.

For consider a typical triangle 7, 11; 11, 15; 15, 7, of the trihedral pair I.

The equations of the three cones with vertices at these points are respectively

$$x_1/x_2/x_3 = \eta_1/\eta_2/\eta_3; \quad \eta_1/\eta_2/\eta_3 = Z_1/Z_2/Z_3; \quad Z_1/Z_2/Z_3 = x_1/x_2/x_3;$$

and the results follow at once.

Analytically we thus have the typical identities

$$x_r(11, 15)+\eta_r(15, 7)+Z_r(7, 11) = 0 \quad (r = 1, 2, 3).$$

If we consider the whole group of fifty-four cones we find that their equations satisfy linear relations in sets of three, one cone in each trio from each trihedral pair. The general relation of the set is as follows, and can be verified from the equations of the cones.

If $(S_p^{(i)}, S_q^{(j)}; S_{p'}^{(i)}, S_{q'}^{(j)}) = 0$ *represents the equation of the cone through the residual intersection of* $S_p^{(i)}, S_q^{(j)}$, *and* $S_{p'}^{(i)}, S_{q'}^{(j)}$, *then*

$$(S_p^{(2)}, S_q^{(3)}; S_{p'}^{(2)}, S_{q'}^{(3)}) + (S_q^{(3)}, S_r^{(1)}; S_{q'}^{(3)}, S_{r'}^{(1)}) + (S_r^{(1)}, S_p^{(2)}; S_{r'}^{(1)}, S_{p'}^{(2)}) = 0,$$

for all p, q, r, p', q', r' *chosen from* 1, 2, 3 *with the restrictions* $p' \neq p$, $q' \neq q$, $r' \neq r$.

There are one hundred and eight relations of this form; any one cone enters into six of the relations. Thus we have the result:

Each of the fifty-four cones passes through the six quartic curves of intersection of six pairs of cones, one cone in each pair belonging to each of the two sets I, II, III, to which the given cone does not belong.

The cones thus pass in threes through one hundred and eight quartic curves, and six curves lie on each cone.

The linear relations, however, are best considered in relation to the Schur quadrics, as will be seen in the next paragraph.

5. *The nine Schur quadrics.*

The three Schur quadrics $2T_1^{(1)}$, $2T_2^{(1)}$, $2T_3^{(1)}$ have been expressed by A. L. Dixon* in the forms

$$S_1^{(1)} - S_2^{(1)} - S_3^{(1)}, \quad S_2^{(1)} - S_3^{(1)} - S_1^{(1)}, \quad S_3^{(1)} - S_1^{(1)} - S_2^{(1)}.$$

It is apparent, by the use of the transformation (1), that the Schur quadrics $2T_p^{(2)}$, $2T_p^{(3)}$ associated with II and III of table (2) are formed by the same process upon the expressions S of II and III.

Each of the nine expressions $2T$ can be written down in various equivalent ways from the various ratio equations of the S quadrics, but by finding the various forms of a typical T, say $2T_3^{(1)} \equiv S_3^{(1)} - S_1^{(1)} - S_2^{(1)}$, it can be shown that there are just six different ways in which each $2T$ of the nine can be represented as a sum or difference of a different pair of the fifty-four cones; all the remaining forms merely repeat the same cones in their various forms. The twelve cones which occur in this sixfold representation of each $2T$ are all different.

It is then found that each of the Schur quadrics in the remaining two table columns of (2) can be expressed as the difference or sum of one of the above six pairs of cones.

* *Loc. cit.* (1), 171.

For example, $2T_3^{(1)}$ can be expressed in the six forms :-

(i) $(\xi_1 Y_2 - \eta_1 X_2) - (X_1 \eta_2 - Y_1 \xi_2)$, (iv) $(y_1 \xi_2 - \eta_1 z_2) + (y_2 \xi_1 - \eta_2 z_1)$,

(ii) $(X_1 y_2 - Y_1 x_2) - (x_1 Y_2 - y_1 X_2)$, (v) $(z_1 \xi_2 - \xi_1 x_2) + (z_2 \xi_1 - \xi_2 x_1)$,

(iii) $(x_1 \eta_2 - y_1 \xi_2) - (\xi_1 y_2 - \eta_1 x_2)$, (vi) $(x_1 \eta_2 - \xi_1 y_2) + (x_2 \eta_1 - \xi_2 y_1)$,

while we find

$$\text{(i)} \quad 2T_1^{(2)} = (\xi_1 Y_2 - \eta_1 X_2) + (X_1 \eta_2 - Y_1 \xi_2),$$

$$\text{(ii)} \quad 2T_2^{(2)} = (X_1 y_2 - Y_1 x_2) + (x_1 Y_2 - y_1 X_2),$$

$$\text{(iii)} \quad 2T_3^{(2)} = (x_1 \eta_2 - y_1 \xi_2) + (\xi_1 y_2 - \eta_1 x_2),$$

$$\text{(iv)} \quad 2T_1^{(3)} = (y_1 \xi_2 - \eta_1 z_2) - (y_2 \xi_1 - \eta_2 z_1),$$

$$\text{(v)} \quad 2T_2^{(3)} = (z_1 \xi_2 - \xi_1 x_2) - (z_2 \xi_1 - \xi_2 x_1),$$

$$\text{(vi)} \quad 2T_3^{(3)} = (x_1 \eta_2 - \xi_1 y_2) - (x_2 \eta_1 - \xi_2 y_1).$$

We thus have the result :

Any two given Schur quadrics $2T_p^{(i)}$, $2T_q^{(j)}$ $(j \neq i)$ *can be expressed as the sum and difference of two cones chosen from the fifty-four.*

The particular cones to be chosen can easily be obtained from a consideration of the double sixes, for in each pair of double sixes, corresponding to each pair of quadrics considered, occur a common pair of lines in the upper rows and a common pair in the lower rows. These four lines meet in two points which are (from the polar property of the quadrics) the vertices of the cones required.

There are twenty-seven ways of choosing such a pair of the nine Schur quadrics, so that all the fifty-four cones are accounted for in the representation.

It follows that every cone of the set can be expressed in the form $T_p^{(i)} \pm T_q^{(j)}$ $(j \neq i)$.

We can, moreover, give the geometrical property :

Two cones of the fifty-four pass through each of the twenty-seven curves of intersection of such pairs of Schur quadrics.

Thus, finally, using a result in § 4, it can be shown that the quadrics through six lines and the Schur quadrics are related by the cones as follows :

Through the curve of intersection of two Schur quadrics $T_p^{(i)}$, $T_q^{(j)}$ *pass two quadric cones of the set ; one of the two passes also through the*

two cubic curves $S_p^{(i)}$, $S_{q'}^{(j)}$; $S_{p'}^{(i)}$, $S_{q''}^{(j)}$, *while the other passes through the two cubic curves* $S_{p'}^{(i)}$, $S_{q''}^{(j)}$; $S_{p''}^{(i)}$, $S_q^{(j)}$, *where each trio* p, p', p''; q, q', q'' *is* 1, 2, 3 *in some order and* $j \neq i$.

The one hundred and eight linear relations noted at the end of § 4 are all contained in the expressions for the cones in terms of the nine Schur quadrics.

6. *Mutual apolarity of certain of the Schur quadrics.*

It is known* that the invariant relation $\Theta\Theta' = 4\Delta\Delta'$ holds for any pair of the three Schur quadrics derived from each table column of (2). The relations between Schur quadrics from different table columns may be investigated as follows.

We have shown that two of the values of λ, which make the quadric $T_p^{(i)} + \lambda T_q^{(j)} = 0$ a cone, are 1 and -1. In order to find the remaining two cones through the curve of intersection, we express the conditions that the polar plane of some point $(U_1, V_1, W_1, U_1', V_1', W_1')$ with respect to $T_p^{(i)} + \lambda T_q^{(j)} = 0$ should vanish identically, that is, that the coefficients of U, etc., should be of the form $mA_1 + nA_2$, etc. We are led, after some simple algebra, to the result that these two cones also are of the forms† $T_p^{(i)} \pm kT_q^{(j)} = 0$.

Thus the equation $\Delta\lambda^4 + \Theta\lambda^3 + \Phi\lambda^2 + \Theta'\lambda + \Delta' = 0$ has the four roots $1, -1, k, -k$, and the two quadrics are related by the invariant equations $\Theta = 0, \Theta' = 0$. We have then the property‡ :

Any two Schur quadrics $T_p^{(i)}$, $T_q^{(j)}$ $(p = 1, 2, 3;\ q = 1, 2, 3;\ j \neq i)$ *are mutually apolar.*

Since, for example, $T_1^{(i)} + T_2^{(i)} + S_3^{(i)} = 0$, it is apparent that the quadrics $T_p^{(i)}$, $S_q^{(j)}$ are related by $\Theta = 0$.

If we consider the whole set of thirty-six Schur quadrics we thus obtain the result :

Any one Schur quadric is related to twenty others of the set by the equation $\Theta\Theta' = 4\Delta\Delta'$, *and to the remaining fifteen by the equations* $\Theta = 0,\ \Theta' = 0$.

* A. L. Dixon, *loc. cit.* (1), 172.

† The value of k^2 for $i = 2$, $j = 3$, $p = q = 3$, for example, is

$$\frac{(B_1A_2' - B_2A_1')(C_1C_2' - C_2C_1')}{(B_1C_2' - B_2C_1')(C_1A_2' - C_2A_1')}.$$

‡ Shortly after this paper was written, a paper by S. Bishara was published in which this and the next result and results in § 8 were obtained by a different method. See S. Bishara, *Journal London Math. Soc.*, 6 (1931), 12–15.

7. *The quartic surfaces* $Q \equiv S_1^{(i)} S_2^{(i)} - T_3^{(i)2}$.

The quartic surface $S_1^{(1)} S_2^{(1)} - T_3^{(1)2} \equiv T_2^{(1)} T_3^{(1)} + T_3^{(1)} T_1^{(1)} + T_1^{(1)} T_2^{(2)} = 0$, associated with the three associated double sixes I, has been investigated*.

We recall the result of § 4,

$$(7,\ 11)(8,\ 10) = S_1^{(2)} S_2^{(2)} - S_1^{(3)} S_2^{(3)},$$

which can be written

$$(T_3^{(2)} + T_3^{(3)})(T_3^{(2)} - T_3^{(3)}) = S_1^{(2)} S_2^{(2)} - S_1^{(3)} S_2^{(3)},$$

or

$$S_1^{(2)} S_2^{(2)} - T_3^{(2)2} = S_1^{(3)} S_2^{(3)} - T_3^{(3)2}.$$

This identity is typical and leads to the result :

The three quartic surfaces derived from the three sets of three associated double sixes are identical.

8. *The common self-conjugate tetrahedron of two Schur quadrics* $T_p^{(i)}$, $T_q^{(j)}$.

It was shown in § 5 that two of the vertices of the common self-conjugate tetrahedron of the typical pair of Schur quadrics $T_3^{(2)}$, $T_3^{(3)}$ are the points 7, 11 ; 8, 10. The remaining two vertices are found, by the process described in § 6, to be the two intersections of the line 15 with the quadric

$$(B_1 V - C_1 W)(A_2' U' - C_2' W') - (B_2 V - C_2 W)(A_1' U' - C_1' W') = 0,$$

which may be written

$$(B_1 C_2' - B_2 C_1') VW' + (C_1 A_2' - C_2 A_1') WU'$$
$$= (C_1 C_2' - C_2 C_1') WW' + (B_1 A_2' - B_2 A_1') VU'.$$

The left- and right-hand sides of this equation can easily be shown to be the values of the expressions $T_2^{(2)}$ and $T_2^{(3)}$ respectively for $U = V' = 0$.

The two points required are thus the two points of intersection of the line 15 with the cone $T_2^{(2)} - T_2^{(3)} = 0$. Also the cone $T_2^{(2)} + T_2^{(3)} = 0$ is the cone (7, 15), and so contains the line 15.

It follows that the two quadrics $T_2^{(2)}$, $T_2^{(3)}$ meet the line 15 in the two points sought.

Similarly it can be shown that the quadrics $T_1^{(2)}$, $T_1^{(3)}$ also meet the line 15 in the same two points.

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* A. L. Dixon, *loc. cit.* (2), 352.

We now make use of the identity* $H = CE - 4Q$, where H is the Hessian of the surface and E is a certain plane. This identity may be written

$$H = CE - 4(S_3^{(2)} S_1^{(2)} - T_2^{(2)^2}).$$

Since the line 15 lies on C and also on $S_3^{(2)}$ it follows that the line touches the Hessian at the two points sought. These points are thus the two parabolic points on the line 15.

We can thus state the following results :

(*a*) *The typical set of four Schur quadrics* $T_1^{(2)}$, $T_2^{(2)}$, $T_1^{(3)}$, $T_2^{(3)}$ *meet at the two parabolic points of the line* 15, *which is the common generator of the quadrics* $S_3^{(2)}$, $S_3^{(3)}$.

(*b*) *If two cones, of the set of fifty-four, whose sum and difference are two Schur quadrics are said to be related, then each of the typical*† *related cones* (7, 11), (8, 10) *passes through the four parabolic points which lie on the two lines indicated in the symbol for the other.*

(*c*) *The common self-conjugate tetrahedron of any two Schur quadrics which are mutually apolar has for its vertices four points on the cubic surface; two of these points are points of intersection of two pairs of lines on the surface; the remaining two are the parabolic points on the line of the surface which meets both pairs.*

In the case of the three trihedral pairs considered, we have seen, given the two double sixes which determine the quadrics, how to find the first pair of points as two points of a definite trihedral pair of the three; the second pair of points are the two parabolic points on that one line (of the same trihedral pair) which does not occur in either double six.

* A. L. Dixon, *loc. cit.* (2), 359.

† We replace the cone (7, 15) used above by (7, 11).

ON THE CONDITION THAT A CERTAIN INTEGRAL MAY BE RATIONAL

By M. HANNA.

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It has been shown by Hardy* that the integral

$$I_1 \equiv \int \frac{ax^2+2\beta x+\gamma}{(Ax^2+2Bx+C)^2}\,dx$$

is rational when the two quadratics (a, β, γ), (A, B, C) are related harmonically, and more generally that the integral

$$I_2 \equiv \int \frac{(ax^2+2\beta x+\gamma)(a_1x^2+2\beta_1 x+\gamma_1)\ldots(a_n x^2+2\beta_n x+\gamma_n)}{(Ax^2+2Bx+C)^{n+2}}\,dx$$

is rational if all the quadratics are related harmonically to any one of those in the numerator. The condition in the latter case is sufficient but not necessary.

The aim of this paper is to show that the condition, both necessary and sufficient, that the integral

$$I \equiv \int \frac{b_0x^{2n}+2nb_1x^{2n-1}+\ldots+{}_{2n}C_r\,b_r x^{2n-r}+\ldots+b_{2n}}{(a_0x^2+2a_1x+a_2)^{n+1}}\,dx$$

shall be algebraic is the vanishing of the invariant, of the two forms (a_0, a_1, a_2), $(b_0, b_1, \ldots, b_{2n})$, whose leading term is $a_2^n b_0$.

1. If rational, I takes the form

$$f \equiv \frac{c_0x^{2n-1}+\ldots+{}_{2n-1}C_r c_r x^{2n-r-1}+\ldots+c_{2n-1}}{(a_0x^2+2a_1x+a_2)^n}.$$

* G. H. Hardy, *Integration of functions of a single variable* (1905), 15–16.

Differentiating and identifying, the necessary and sufficient condition is

$$2nb_r = r(r-1)\,a_2 c_{r-2} + 2r(n-r)\,a_1 c_{r-1} - (r+1)(2n-r)\,a_0 c_r$$

$$(r = 0,\ 1,\ 2,\ \ldots,\ 2n),$$

i.e. the vanishing of the eliminant*

$$\Delta \equiv \sum_{r=0}^{2n} A_r b_r$$

$$\equiv \begin{vmatrix}
b_0 & -2na_0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
b_1 & (2n-2)a_1 & -(2n-1)a_0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
b_2 & 2a_2 & (2n-4)a_1 & -(2n-2)a_0 & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
b_{2n-2} & & & & \ldots & \ldots & -2a_0 & & \\
b_{2n-1} & & & & & \ldots & -(2n-2)a_1 & -a_0 & \\
b_{2n} & & & & & & 2na_2 & & -2na_1
\end{vmatrix}.$$

On putting $A_r = \dfrac{(2n)!}{(2n-r)!}\,a_0^r D_r$, where

$$D_r \equiv \begin{vmatrix}
(2n-2r-2)a_1 & -(2n-r-1)a_0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
(r+2)a_2 & (2n-2r-4)a_1 & -(2n-r-2)a_0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
 & (r+3)a_2 & (2n-2r-6)a_1 & -(2n-r-3)a_0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
 & & & & & & \ldots & \ldots & -2a_0 & \\
 & & & & & & & \ldots & -(2n-2)a_1 & -a_0 \\
 & & & & & & & & 2na_2 & -2na_1
\end{vmatrix},$$

we find that

$$D_r = (2n-2r-2)a_1 D_{r+1} + (r+2)(2n-r-1)a_0 a_2 D_{r+2}.$$

Hence A_r satisfies the difference equation

$$(2n-r)a_0 A_r = 2(n-r-1)a_1 A_{r+1} + (r+2)a_2 A_{r+2} \qquad (1)$$

$$(r = 0,\ 1,\ 2,\ \ldots,\ 2n-2).$$

* The second, third, ... columns, from left to right, correspond to $c_0, c_1, \ldots, c_{2n-1}$. From these columns, respectively, the factors 1, 2, ..., $2n$ have been removed.

2. We now proceed to show that Δ is a seminvariant, *i.e.* that it is annihilated by the operator

$$\Omega \equiv a_0 \frac{\partial}{\partial a_1} + 2a_1 \frac{\partial}{\partial a_2} + b_0 \frac{\partial}{\partial b_1} + 2b_1 \frac{\partial}{\partial b_2} + \ldots + rb_{r-1} \frac{\partial}{\partial b_r} + \ldots + 2nb_{2n-1} \frac{\partial}{\partial b_{2n}}.$$

Assume that the equation

$$\Omega A_r + (r+1) A_{r+1} = 0 \tag{2}$$

holds for two values $r+2$, $r+1$, operate with Ω on equation (1), and then substitute for ΩA_{r+1}, ΩA_{r+2}. The result is

$$(2n-r) a_0 \Omega A_r$$
$$= 2(n-r-1) a_0 A_{r+1} - (r+2)\{2(n-r-2) a_1 A_{r+2} + (r+3) a_2 A_{r+3}\}.$$

Now, using equation (1) for $r+1$ and simplifying, we get

$$\Omega A_r + (r+1) A_{r+1} = 0,$$

i.e., assuming that equation (2) holds good for $r+2$, $r+1$, we find it also to be true for r. Since

$$A_{2n} = (2n)!\, a_0^{2n},$$
$$A_{2n-1} = -(2n)!\, 2n a_0^{2n-1} a_1,$$
$$A_{2n-2} = (2n)!\, \{2n(n-1) a_1^2 + n a_0 a_2\} a_0^{2n-2},$$

the assumption is valid for $2n-1$, $2n-2$, and hence, by induction, equation (2) is true for $r = 2n-1, 2n-2, \ldots, 2, 1, 0$. Also $\Omega A_{2n} = 0$.

Since

$$\Omega\Delta \equiv \Omega \sum_{r=0}^{2n} b_r A_r = b_{2n}\, \Omega A_{2n} + \sum_{r=0}^{2n-1} b_r \{\Omega A_r + (r+1) A_{r+1}\},$$

it follows that $\Omega\Delta = 0$, *i.e.* Δ is a seminvariant. It also follows that

$$A_r = \frac{(-)^r}{r!} \Omega^r A_0. \tag{3}$$

3. We shall now prove that A_0, and consequently Δ, contains a_0^n as a factor. Evidently A_0 is homogeneous, of degree $2n$, and isobaric, of weight $2n$, in a_0, a_1, a_2, and hence

$$A_0 = \sum_{m=0}^{n} l_m a_0^{n-m} a_1^{2m} a_2^{n-m}.$$

Hence, from equation (3), we have

$$-A_1 = \sum_{m=0}^{n-1} \{2(n-m)l_m + 2(m+1)l_{m+1}\} a_0^{n-m} a_1^{2m+1} a_2^{n-m-1},$$

$$2A_2 = \sum_{m=0}^{n-2} \{4(n-m)(n-m-1)l_m + 2(n-m-1)(4m+5)l_{m+1} + 2(m+2)(2m+3)l_{m+2}\} a_0^{n-m} a_1^{2m+2} a_2^{n-m-2} + 2(nl_0+l_1)a_0^{n+1} a_2^{n-1}.$$

But, putting $r = 0$ in equation (1), we have

$$nA_0 a_0 = (n-1)A_1 a_1 + A_2 a_2. \tag{4}$$

Substituting for A_0, A_1, A_2 in the last equation and collecting terms, we get

$$-l_1 a_0^{n+1} a_2^n + \sum_{m=0}^{n-2} \{2m(n-m)l_m - (2n-4m-3)(m+1)l_{m+1} - (m+2)(2m+3)l_{m+2}\} a_0^{n-m} a_1^{2m+2} a_2^{n-m-1} + \{2(n-1)l_{n-1} + n(2n-1)l_n\} a_0 a_1^{2n} \equiv 0,$$

i.e. l_m satisfies the equations

$$l_1 = 0, \tag{5}$$

$$2m(n-m)l_m = (2n-4m-3)(m+1)l_{m+1} + (m+2)(2m+3)l_{m+2}, \tag{6}$$

$$2(n-1)l_{n-1} + n(2n-1)l_n = 0, \tag{7}$$

with $m = 0, 1, 2, \ldots, n-2$.

Now, from equation (5), $l_1 = 0$, and then, putting $m = 0, 1, \ldots, n-2$, in succession, in equation (6), we get, one by one,

$$l_2 = l_3 = l_4 = \ldots = l_n = 0.$$

Equation (7) serves only as a check on the evaluation of the l's.

It therefore follows that

$$A_0 = l_0 a_0^n a_2^n.$$

4. We have thus seen that $C \equiv k a_0^{-n} \Delta$, where k is a numerical constant, is homogeneous, linear in $b_0, b_1, \ldots, b_{2n}$ and of degree n in a_0, a_1, a_2, and isobaric of weight $2n$. Hence C satisfies the relation $\Sigma ip = 2w$, and, since it is annihilated by Ω, it is therefore an invariant.

Furthermore, since $A_{2n} = (2n)!\, a_0^{2n}$, the end-terms of C are $a_0^n b_{2n}$, $a_2^n b_0$ $[k = (2n!)^{-1}]$. Hence we have proved the theorem previously stated, namely:

The necessary and sufficient condition that I shall be algebraic is the vanishing of the invariant whose leading term is $a_2^n b_0$, *e.g.**

for $n = 1$,

$$C_1 = a_2 b_0 - 2a_1 b_1 + a_0 b_2;$$

for $n = 2$,

$$C_2 = a_2^2 b_0 - 4a_1 a_2 b_1 + 2(a_0 a_2 + 2a_1^2) b_2 - 4a_0 a_1 b_3 + a_0^2 b_4;$$

for $n = 3$,

$$C_3 = a_2^3 b_0 - 6a_1 a_2^2 b_1 + 3(a_0 a_2 + 4a_1^2) a_2 b_2 - 4(3a_0 a_2 + 2a_1^2) a_1 b_3 + 3a_0(a_0 a_2 + 4a_1^2) b_4 - 6a_0^2 a_1 b_5 + a_0^3 b_6.$$

In presenting this paper, I am greatly indebted to Prof. W. E. H. Berwick for suggestion and criticism.

* It is not difficult to see that C_1, C_2, C_3 vanish when Hardy's conditions are satisfied.

THE DERIVATION OF THE PATTERN FORMULAE OF TWO-WAY PARTITIONS FROM THOSE OF SIMPLER PATTERNS

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1. *Introductory.*

If the moment generating function of a variable be defined in terms of the frequency element $df(x)$ of a variate x in the form

$$M = \int e^{tx} df(x),$$

then, when M can be expanded in a series of powers of t, the cumulative function K, defined by

$$K = \log M,$$

can also be expanded in the form

$$K = \sum_r \frac{\kappa_r t^r}{r!}.$$

The coefficients κ_r of this expansion have by different writers been termed semi-invariants and cumulative moment functions. Since both terms are lengthy, and the first somewhat misleading, we propose in what follows to refer to them as the *cumulants* of the distribution.

In a recent paper* it was shown that, if symmetric functions k_r, of the r-th degree, are calculated from the observations of a sample, in such a manner that the mean value of k_r in all possible samples is equal to the r-th cumulant (κ_r) of the infinite population from which the sample has been taken, then the cumulants of the distribution of any k, and of the simultaneous distribution of all such statistics, which are necessarily expressible in terms of the cumulants of the population, may be readily and simply determined by combinatorial methods. The essence of the method is that the coefficient of any single term in the required

* R. A. Fisher, *Proc. London Math. Soc.* (2), 30 (1929), 199–238.

formulae is a composite of contributions from one or more two-way partitions, of which one marginal partition represents the formula, while the other specifies the particular term; and for each such partition the numerical coefficient is derived as the number of ways of setting up the partition, while the coefficient in n, the size of the sample, depends on the pattern of the two-way partition, *i.e.* on the number of rows and columns, and the number and distribution of zero entries. A useful list of such pattern formulae has already been given (*loc. cit.*).

The listing of the great number of patterns possible for two-way partitions of larger numbers would, however, be impracticable; the present paper will show how the pattern formulae for such cases may be readily derived from those already listed. Patterns with a large number of rows have necessarily, for formulae of given degree, relatively few entries in each row. The extreme case is that for the terms involving only the variance of the sampled population, in which each row has only two entries; these patterns are of particular interest, since only these occur in the distribution of moment statistics derived from the normal distribution. It will be seen that the method of deriving the pattern function by the addition of a new column is particularly simple in these cases. It should be noted that the pattern functions are the same for multivariate as for univariate problems.

2. *The general method of evaluating a pattern.*

The general procedure for determining the function of n associated with any specified pattern is to consider all the possible ways in which the rows can be separated into separate groups, or separates. Thus with two rows we have only two possible separations, the rows being either amalgamated as in the marginal total, or kept separate; these two separations correspond to the partitions (2) and (1^2) respectively. With three rows we have one separation corresponding to the partition (3), three corresponding to the partition (21), and one corresponding to the partition (1^3). In general the number of separations of r rows corresponding to the partition $(p_1^{\pi_1} p_2^{\pi_2} \ldots)$ is

$$\frac{r!}{\pi_1!\,(p_1!)^{\pi_1} \cdot \pi_2!\,(p_2!)^{\pi_2} \ldots},$$

and the total number of separations of r rows into s separates is

$$\frac{1}{(s-1)!}\,\Delta^{s-1}(1^{r-1}),$$

the leading $(s-1)$-th divided difference of the series of the $(r-1)$-th

powers of the positive integers. Representative numbers are given in Table I.

TABLE I.

Number of separations of r rows into s separates.

Number of rows	Number of separates									Total
	1	2	3	4	5	6	7	8	9	
2	1	1								2
3	1	3	1							5
4	1	7	6	1						15
5	1	15	25	10	1					52
6	1	31	90	65	15	1				203
7	1	63	301	350	140	21	1			877
8	1	127	966	1701	1050	266	28	1		4140
9	1	255	3025	7770	6951	2646	462	36	1	21147

The total number of separations, being [Whitworth, *Choice and Chance* (1886), 95] the coefficient of $x^r/r!$ in the expansion of

$$e^{e^x-1},$$

increases rapidly, and although, usually, large groups of separations making similar contributions may be treated together, it is evidently desirable to shorten the method for cases of more than six or eight rows.

The contribution of each separation to the function required is the product of the factor $n(n-1)\ldots(n-s+1)$ with factors for the several columns; these factors are

$$\frac{1}{n}, \quad \frac{-1}{n(n-1)}, \quad \frac{2!}{n(n-1)(n-2)}, \quad \ldots,$$

according as the column is represented in 1, 2, 3, ... separates. This process has been already sufficiently exemplified (R. A. Fisher, *loc. cit.*). We shall now show how the pattern-function for a pattern containing a column with only two entries may be obtained from the pattern-functions of simpler patterns, and then consider the parallel procedure to be used when no column contains less than three or more entries.

3. *Expansion of a pattern function in terms of the functions of simpler patterns.*

Consider the pattern

. × × ×
× . × ×
× × . .
× × . .

of which the right-hand column contains only two entries. The fifteen separations which are possible with four rows may be divided into two classes, in one of which (α) the two rows represented in the fourth column lie in the same separate, while in the second class (β) they lie in different separates. In the first class the fourth column contributes the factor n^{-1}, in the second class the factor is $-1/n(n-1)$.

Now in separations of the first class the cofactors of n^{-1} will be the contributions of all its possible separations to the pattern-function of the pattern

$$\begin{matrix} \times\times\times \\ \times\times\,. \\ \times\times\,. \end{matrix}$$

in which the fourth column is omitted, and the first two rows of the original pattern are amalgamated. In general we may designate the function of this reduced pattern by A. If now we represent the function of the pattern

$$\begin{matrix} .\times\times \\ \times\,.\times \\ \times\times. \\ \times\times\,. \end{matrix}$$

in which the rows have not been amalgamated, by B, it appears that the cofactor of $-1/n(n-1)$ in the pattern-function to be evaluated will consist of all the contributions to B which do not occur in A, and the required function is reduced to the form

$$\frac{A}{n}-\frac{B-A}{n(n-1)} \equiv \frac{A}{n-1}-\frac{B}{n(n-1)},$$

which is a general formula for the function of any pattern having a column with only two entries.

In the particular case considered we may at once substitute

$$A=\frac{1}{(n-1)(n-2)}, \qquad B=\frac{n}{(n-1)^2(n-2)},$$

and obtain

$$\frac{1}{(n-1)^2(n-2)}\left(1-\frac{1}{n-1}\right) \equiv \frac{1}{(n-1)^3},$$

in accordance with the value given in Fisher's list of useful patterns.

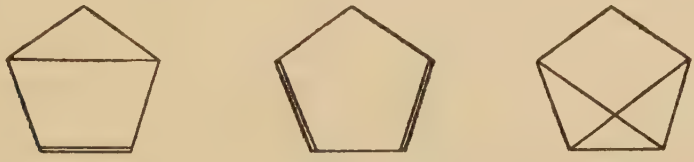
Alternatively, of course, B could have been reduced in turn to functions of two column patterns.

Whenever one or both rows represented in the column to be removed contain only two entries, the pattern-function B vanishes, and we are left simply with

$$A/(n-1);$$

this will always be the case with the pattern of terms involving only variances and covariances, which are the only terms which appear in sampling from normal populations.

For these normal cases use may be made of the symbolical diagram in which each column is represented by a point and each row by a line joining the two points corresponding to the two columns in which it is represented. Thus for the evaluation of $\kappa(3^4 2)$, we have partitions represented by the figures



in each of which one angle at which two lines meet represents the column with only two entries. The reduction formula thus shows that the functions associated with these patterns will be each $1/(n-1)$ of the functions associated with the simpler patterns represented by the three figures below (two of which are equivalent),

that is of those which occur in the evaluation of $\kappa(3^4)$. In respect of the numerical factor, too, it should be noticed that every way of setting up a partition for $\kappa(3^4)$ corresponds to twelve ways of setting up one of the partitions for $\kappa(3^4 2)$, since any one of the six rows may be broken and connected in two ways with the elements of the new column. Thus a correspondence is established for the whole coefficient, and we have for the normal case

$$\kappa(3^4 2) = \frac{12}{n-1} \kappa(3^4)\kappa_2,$$

or, in general, since the number of rows in the partition is equal to the power of κ_2 by which the coefficient is multiplied, the addition of a new part 2 is equivalent to the action of the operator

$$\frac{2\kappa_2^2}{n-1} \frac{d}{d\kappa_2},$$

an operator by means of which the higher cumulants of simultaneous distributions involving the estimated variance k_2 may be very readily

obtained. In the multivariate case the operator for adding a variance or covariance k_{pq} is

$$\sum_{rs} \frac{1}{n-1} (\kappa_{pr}\kappa_{qs}+\kappa_{ps}\kappa_{qr}) \frac{d}{d\kappa_{rs}},$$

where κ_{pq} stands for the covariance of the variates p and q.

4. *Removal of a column of three entries.*

The direct generalization of the method of the last section to partitions containing a column of three entries will evidently express the new function in terms of the functions of five simpler partitions, each obtained by suppressing the column of three entries, namely A, formed by the amalgamation of all three of the rows used in the column suppressed; B_1, B_2, B_3, formed by the amalgamation of two only of these three rows; and C by leaving all three rows distinct. In the new function the cofactor of $1/n$ will evidently be A, that of $-1/n(n-1)$ will be the sum of three quantities $B-A$, and that of $2/n(n-1)(n-2)$ will be

$$C-(B_1-A)-(B_2-A)-(B_3-A)-A = C-B_1-B_2-B_3+2A.$$

Hence the general formula is

$$A\left(\frac{1}{n}+\frac{3}{n(n-1)}+\frac{4}{n(n-1)(n-2)}\right)$$

$$-(B_1+B_2+B_2)\left(\frac{1}{n(n-1)}+\frac{2}{n(n-1)(n-2)}\right)+\frac{2C}{n(n-1)(n-2)}$$

$$\equiv \frac{n}{(n-1)(n-2)} A-\frac{1}{(n-1)(n-2)} (B_1+B_2+B_3)+\frac{2}{n(n-1)(n-2)} C.$$

As an example we may derive the function for the pattern

$$\begin{array}{l} \times\times\times \\ \times\times\times \\ \times\times\times \\ \times\times\ . \end{array}$$

from those for the simpler patterns

$\times\times$	$\times\times$	$\times\times$
$\times\times$	$\times\times$	$\times\times$
	$\times\times$	$\times\times$
		$\times\times$
A	$B_1=B_2=B_3$	C
$\frac{1}{n-1}$	$\frac{n}{(n-1)(n-2)}$	$\frac{n(n+1)}{(n-1)(n-2)(n-3)}$.

We have then

$$\frac{n}{(n-1)^2(n-2)} - \frac{3n}{(n-1)^2(n-2)^2} + \frac{2(n+1)}{(n-1)^2(n-2)^2(n-3)}$$

$$= \frac{n^3-8n^2+17n+2}{(n-1)^2(n-2)^2(n-3)}.$$

In its application to normal patterns the formula for removing a column of three entries is reduced to the simple formula

$$\frac{n}{(n-1)(n-2)} A,$$

but it should be noticed that A here is not necessarily the function of a normal pattern, for the other entries of the rows represented in the suppressed column may be in three different columns, in which case A will be the function of a pattern having a row with three entries. When, however, two of the amalgamated rows are duplicates joining the same pair of columns, the function will be expressed in terms of that of a normal pattern. In the representative diagram the effect of this will be to suppress a point, two of the lines from which are duplicates, and to replace the set of three lines meeting there by a single line joining their extremities

as does the broken line in the figure. By two such operations such a diagram as

is reduced to , representing the pattern

$$\begin{matrix} \times & \times \\ \times & \times \end{matrix}$$

and having the function $1/(n-1)$, whence it easily follows that the diagram in question is associated with the function

$$n^2/(n-1)^3(n-2)^2.$$

In the case where three entries appear in the amalgamated row the partition of which A is the function, though not a normal one, is of the kind which must occur in the term in $\kappa_3\kappa_2^{r-1}$, that is in the final term of the corresponding formula. Since any arrangement of a two-way partition having a marginal column represented by the partition $(3\ 2^{r-1})$ corresponds to six arrangements of the normal partition of the formula in which a new row of three entries has been added, these coefficients must be in the simple ratio

$$\frac{6n}{(n-1)(n-2)}.$$

For example, the coefficient of the term in $\kappa_3\kappa_2^2$ of the formula for $\kappa(4\ 3)$ is

$$\frac{36n}{(n-1)(n-2)},$$

whence it follows that the coefficient of the term in κ_2^5 of the formula for $\kappa(4\ 3^2)$ must be

$$\frac{216n^2}{(n-1)^2(n-2)^2},$$

as may be verified from the formula given previously. Equally the corresponding coefficients in the formulae for $\kappa(3\,2^2)$ and $\kappa(3^2 2^2)$ are

$$\frac{48}{(n-1)^2} \quad \text{and} \quad \frac{288n}{(n-1)^3(n-2)}.$$

All the normal terms occurring in the distribution of k_3 and of its simultaneous distribution with other such statistics may then be obtained from the coefficients of simpler formulae.

5. *The removal of a column containing any number of entries.*

In expressing the function of a pattern as a linear function of the patterns formed by deleting a column of r entries, and amalgamating the r corresponding rows into ρ' rows in accordance with the partition $(p_1^{\pi_1}p_2^{\pi_2}\ldots)$, where

$$\Sigma p\pi = r,$$

$$\Sigma\pi = \rho',$$

let $a(p_1^{\pi_1}p_2^{\pi_2}\ldots)$ be the coefficient of the function of any one of the patterns so formed.

The functions consist of the sums of contributions from all possible separations of all the rows; we shall designate by $Q(q_1^{\chi_1} q_2^{\chi_2} \ldots)$ the total contribution of all such separations in which the r rows of the deleted column are separated in any particular way into χ_1 separates containing q_1 of these rows each, χ_2 separates containing q_2 of these rows each, and so on; then, if the partition $(p_1^{\pi_1} p_2^{\pi_2} \ldots)$ can be found by subdividing the parts of the partition $(q_1^{\chi_1} q_2^{\chi_2} \ldots)$ in λ ways, any Q will appear in the expansion with coefficient

$$\Sigma \lambda a (p_1^{\pi_1} p_2^{\pi_2} \ldots),$$

the summation being over all partitions $(p_1^{\pi_1} p_2^{\pi_2} \ldots)$, and this must be equated to its coefficient in the function of the whole pattern, namely

$$(-)^{\rho-1} (\rho-1)! \frac{(n-\rho)!}{n!},$$

where $\Sigma(\chi) = \rho$. We have in this way one such equation for every kind of partition $(q_1^{\chi_1} q_2^{\chi_2} \ldots)$, and these are sufficient to determine the unknown coefficients a.

But if the statistic k_r, defined as having its mean sampling value equal to κ_r, be expanded in the form

$$k_r = \Sigma A (p_1^{\pi_1} p_2^{\pi_2} \ldots) s_{p_1}^{\pi_1} s_{p_2}^{\pi_2} \ldots,$$

where s_p is the sum of the p-th powers of the observations, then the coefficient of $\mu_{q_1}^{\chi_1} \mu_{q_2}^{\chi_2} \ldots$ in the mean value of the expansion, when the mean values of the s-products are expanded in terms of the moments μ, by the general formula previously given (Fisher, *loc. cit.*, p. 207), will be

$$\Sigma \frac{n!}{(n-\rho)!} \mu A (p_1^{\pi_1} p_2^{\pi_2} \ldots),$$

where μ is the number of ways in which the parts of $(p_1^{\pi_1} p_2^{\pi_2} \ldots)$ may be amalgamated to form those of $(q_1^{\chi_1} q_2^{\chi_2} \ldots)$, and this is to be equated to its known coefficient in the expansion of κ_r, namely

$$\frac{(-)^{\rho-1} (\rho-1)!}{\chi_1! \chi_2! \ldots} \frac{r!}{(q_1!)^{\chi_1} (q_2!)^{\chi_2} \ldots}.$$

The relation between λ and μ may be found from the following consideration; the number of ways of dividing r objects into ρ' parts, $p_1^{\pi_1} p_2^{\pi_1} \ldots$, these being grouped into ρ divisions $q_1^{\chi_1} q_2^{\chi_2} \ldots$, may be obtained

either by multiplying by λ the number of ways of dividing r objects into parts $q_1^{\chi_1} q_2^{\chi_2} \ldots$, *i.e.*

$$\frac{\lambda}{\chi_1 ! \chi_2 ! \ldots} \frac{r !}{(q_1 !)^{\chi_1} (q_2 !)^{\chi_2} \ldots},$$

or equally by multiplying by μ the number of ways of dividing r objects into parts $p_1^{\pi_1} p_2^{\pi_2} \ldots$, *i.e.*

$$\frac{\mu}{\pi_1 ! \pi_2 ! \ldots} \frac{r !}{(p_1 !)^{\pi_1} (p_2 !)^{\pi_2} \ldots}.$$

This relationship must hold for each particular separation of the partition $(p_1^{\pi_1} p_2^{\pi_2} \ldots)$ having *specification* $(q_1^{\chi_1} q_2^{\chi_2} \ldots)$, as defined by MacMahon*, or, as is here required, for all separations taken together. Consequently our second set of equations may be written

$$\Sigma \left\{ \lambda A (p_1^{\pi_1} p_2^{\pi_2} \ldots) \div \frac{r !}{\pi_1 ! \pi_2 ! \ldots (p_1 !)^{\pi_1} (p_2 !)^{\pi_2} \ldots} \right\} = (-)^{\rho - 1} (\rho - 1) ! \frac{(n - \rho) !}{n !},$$

showing that we can satisfy the equations for a by putting

$$a (p_1^{\pi_1} p_2^{\pi_2} \ldots) = A (p_1^{\pi_1} p_2^{\pi_2} \ldots) \div \frac{r !}{\pi_1 ! \pi_2 ! \ldots (p_1 !)^{\pi_1} (p_2 !)^{\pi_2} \ldots},$$

where the partition functions A have been already given up to partitions of six, that is, as far as is needed for the deletion of a column of six entries, in Fisher's expressions for k_1 to k_6 (*loc. cit.*, 203–4).

Let us illustrate this general method of proof by the case when $r = 4$. The function of a pattern containing a column of four entries is to be expressed in terms of the functions of the fifteen simpler patterns formed by deleting this column, and amalgamating the entries of the four rows in which it is represented in every possible way. We represent by $F(4)$ the function of the patterns found by amalgamating all four rows into one, by $F_1(3\,1)$, $F_2(3\,1)$, $F_3(3\,1)$, $F_4(3\,1)$ the functions of the four patterns which can be found by leaving one row untouched and amalgamating into one the remaining three. Similarly there will be three functions $F(2^2)$, six functions $F(2\,1^2)$, and one function $F(1^4)$. We have to find five coefficients $a(4)$, $a(3\,1)$, $a(2^2)$, $a(2\,1^2)$, and $a(1^4)$, such that the new function shall be

$$a(4) F(4) + \sum_1^4 a(3\,1) F(3\,1) + \sum_1^3 a(2^2) F(2^2) + \sum_1^6 a(2\,1^2) F(2\,1^2) + a(1^4) F(1^4).$$

* P. A. MacMahon, *Combinatory analysis*, 1 (1915), 46.

Now the separations in which all four rows lie in different separates appear only in $F(1^4)$, and the sum of their contributions, $Q(1^4)$, comes into the new function with an additional factor

$$\frac{-6}{n(n-1)(n-2)(n-3)},$$

hence the first of the five equations required is

$$a(1^4) = \frac{-6}{n(n-1)(n-2)(n-3)}.$$

Next any separation in which the four rows lie in three different separates will appear in one of the functions $F(2\,1^2)$ and also in $F(1^4)$, and the sum of the contributions from such separations, $Q(2\,1^2)$, comes into the new function with the additional factor $2/n(n-1)(n-2)$, so that we have the second equation

$$a(2\,1^2)+a(1^4) = \frac{2}{n(n-1)(n-2)}.$$

Thirdly, any separation in which the rows fall two each into two different separates will appear in one of the functions $F(2^2)$, in two of the functions $F(2\,1^2)$, and in the function $F(1^4)$, so that

$$a(2^2)+2a(2\,1^2)+a(1^4) = \frac{-1}{n(n-1)}.$$

It should be noted that the coefficient 2 is the number of ways in which a partition (2^2) can be subdivided into a partition $(2\,1^2)$.

Fourthly, any separation in which three of the rows fall into one separate, and the remaining row into a different one, appears in one function $F(3\,1)$, in three functions $F(2\,1^2)$, and in the function $F(1^4)$, giving the equation

$$a(3\,1)+3a(2\,1^2)+a(1^4) = \frac{-1}{n(n-1)},$$

in which the coefficient 3 is the number of ways in which a partition $(3\,1)$ may be subdivided into partitions $(2\,1^2)$.

Finally, any separation in which all four rows fall in the same separate appears in all the functions, so we have

$$a(4)+4a(3\,1)+3a(2^2)+6a(2\,1^2)+a(1^4) = \frac{1}{n},$$

completing the set of five equations necessary to determine the coefficients a.

The equations may evidently be solved by direct substitution, yielding the solution

$$a(1^4) = \frac{-6}{n(n-1)(n-2)(n-3)},$$

$$a(2\,1^2) = \frac{2}{(n-1)(n-2)(n-3)},$$

$$a(2^2) = \frac{-1}{(n-2)(n-3)},$$

$$a(3\,1) = \frac{-(n+1)}{(n-1)(n-2)(n-3)},$$

$$a(4) = \frac{n(n+1)}{(n-1)(n-2)(n-3)}.$$

The proof of the relationship under discussion depends, however, upon the correspondence of the equations for a with those obtained by putting

$$k_4 = A(4)s_4 + A(3\,1)s_3s_1 + A(2^2)s_2^2 + A(2\,1^2)s_2s_1^2 + A(1^4)s_1^4,$$

and expressing the conditions that the mean value of the statistic so constructed shall be equal to the population parameter κ_4 with its identical expression in terms of the moments

$$\kappa_4 = \mu_4 - 4\mu_3\mu_1 - 3\mu_2^2 + 12\mu_2\mu_1^2 - 6\mu_1^4.$$

Here μ_1^4 will appear in the expansion of the mean value of s_1^4 only, giving

$$n(n-1)(n-2)(n-3)\,A(1^4) = -6\,;$$

$\mu_2\mu_1^2$ will appear in the means of $s_2s_1^2$ and of s_1^4, giving

$$n(n-1)(n-2)\{A(2\,1^2) + 6A(1^4)\} = 12\,;$$

μ_2^2 will appear in the means of s_2^2, $s_2s_1^2$, and s_1^4, giving

$$n(n-1)\{A(2^2) + A(2\,1^2) + 3A(1^4)\} = -3,$$

while the two remaining equations are

$$n(n-1)\{A(3\,1) + 2A(2\,1^2) + 4A(1^4)\} = -4$$

and

$$n\{A(4) + A(3\,1) + A(2^2) + A(2\,1^2) + A(1^4)\} = 1,$$

the numerical factors in these equations being the number of ways in which the parts of any particular partition may be amalgamated to form those of the partition in the equation of which it occurs.

These equations also may be solved by direct substitution, giving the familiar relations

$$A(1^4) = \frac{-6}{n(n-1)(n-2)(n-3)},$$

$$A(2\,1^2) = \frac{12}{(n-1)(n-2)(n-3)},$$

$$A(2^2) = \frac{-3}{(n-2)(n-3)},$$

$$A(3\,1) = \frac{-4(n+1)}{(n-1)(n-2)(n-3)},$$

$$A(4) = \frac{n(n+1)}{(n-1)(n-2)(n-3)},$$

from which it is evident that we can obtain the corresponding solutions for a by dividing by the number of ways in which four objects may be distributed in the required partition, or in general by

$$\frac{1}{\pi_1!\ \pi_2!\ \ldots}\ \frac{r!}{(p_1!)^{\pi_1}(p_2!)^{\pi_2}\ldots}.$$

6. *Proof of the vanishing of a class of zero functions.*

The fact that the function of a two-way partition can be found by an expression linear in the functions of partitions with one less column, and with the rows represented in the deleted column more or less amalgamated, may be used to prove that the partition function is necessarily zero in a class of cases in which its vanishing has hitherto only been noted empirically. The class in question is that of partitions the rows of which can be divided into two groups, which, whatever may be their internal connections, are only connected with each other by entries in a single column. For example, in the pattern

$$\begin{array}{cccc} \times & \times & \times & . \\ \times & \times & \times & . \\ . & . & \times & \times \\ . & . & \times & \times \end{array}$$

the group of columns represented in the two upper rows and the group represented in the two lower rows have in common only a single column, namely the third from the left. Evidently all patterns in which any row has only one entry belong to this class, but, whereas a statistical reason

has already been given for patterns with such rows having zero functions, no general demonstration applicable to the whole class has been given.

The demonstration which suggests itself from the methods of this paper is as follows. If any of the columns which are not represented in both groups of rows is removed, all the patterns in terms of which the entire pattern-function is expanded still possess two groups of rows having only one column in common. If all such columns are removed in succession, the expansion contains only the functions of patterns having a single column, all of which have, however, two or more rows.

It is obvious statistically that the function of such patterns must be zero, for the statistics k_r have been defined so that their mean sampling values shall be κ_r and shall contain no product terms in κ's of lower degree; we may, however, give an algebraical proof for a pattern of r rows and one column.

The contribution to the function of any separation into s separates will be

$$n(n-1)\dots(n-s+1)\,\frac{(-)^{s-1}(s-1)!}{n(n-1)\dots(n-s+1)} = (-)^{s-1}(s-1)!,$$

and, the number of separations being

$$\frac{1}{(s-1)!}\,\Delta^{s-1}(1^{r-1}),$$

the complete function must be

$$\sum_{s=1}^{r}(-)^{s-1}\Delta^{s-1}(1^{r-1}),$$

which is the finite difference expansion of 0^{r-1}, and is consequently zero when r exceeds unity, as in the class of patterns under consideration.

Summary.

A method is developed of calculating the function of n to be associated with any two-way partition in the evaluation of the cumulants of the sampling distribution of the appropriate moment statistics k, by expanding it in terms of the functions of partitions having simpler patterns. When columns of two or three entries occur the simplification is extremely rapid. The method is, however, generalized for all cases.

A proof is given of the vanishing of the functions corresponding to all patterns in which the rows may be divided into two groups having only a single column in common.

ON INTEGRAL FUNCTIONS OF INTEGRAL ORDER

By M. L. CARTWRIGHT.

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1.1. Suppose* that $f(z) = f(re^{i\theta})$ is an integral function of order ρ, where ρ is an integer. We may suppose, without loss of generality in what follows, that $f(0) = 1$. Let $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$, ..., where $0 < r_1 \leqslant r_2 \leqslant \ldots$, be the zeros of $f(z)$; let $n(r)$ be the number of zeros for which $r_n \leqslant r$, and let $M(r) = \max |f(re^{i\theta})|$. Then we may write

$$(1.11) \quad f(z) = e^{c_1 z + c_2 z^2 + \ldots + c_\rho z^\rho} \prod_1^\infty \left(1 - \frac{z}{z_n}\right) e^{z/z_n + z^2/2z_n^2 + \ldots + z^{\rho-1}/[(\rho-1)z_n^{\rho-1}]}$$

$$= e^{g(z)} \prod_1^\infty \left(1 - \frac{z}{z_n}\right) E_n(z, \rho-1),$$

or

$$(1.12) \quad f(z) = e^{g(z)} \prod_1^\infty \left(1 - \frac{z}{z_n}\right) E_n(z, \rho),$$

according as the series

$$(1.13) \quad \sum_1^\infty \frac{1}{r_n^\rho}$$

is convergent or divergent, the products being absolutely convergent. It is known that the series (1.13) is convergent if

$$(1.14) \quad \int_1^\infty \frac{\log M(r)}{r^{\rho+1}}\, dr$$

is convergent†. Also, if (1.14) is divergent, then

$$\sum_1' \frac{1}{|r_n(a)|^\rho},$$

* I should like to thank the Referee for some important suggestions, particularly in connection with Theorem IV. In consequence of his suggestions I have rewritten the earlier part of the paper and added Theorems V, VIII, and IX. Theorems VI and VII remain unchanged.

† See G. Valiron, *Lectures on the general theory of integral functions* (Toulouse, 1923), 185.

where $z_n(a)$ is the n-th zero of $f(z)-a$, is divergent, except possibly for a single exceptional value of a.

1.2. The object of this paper is to consider the behaviour of the series

$$\sum_1^\infty \frac{1}{z_n^\rho}. \tag{1.21}$$

The general conclusion is that, if $f(z)$ is of minimum type or even a very smooth function of mean type, then (1.21) is convergent, and we may write $f(z)$ in the form (1.12) although the product may be only conditionally convergent. Even when (1.21) is not convergent we can often find quite narrow limits for the oscillation of the partial sums; and for certain functions of mean type the behaviour of the function is dominated by the behaviour of the partial sums of (1.21).

We say that $f(z)$ is of order ρ and *mean type** or *type A* if

$$\overline{\lim_{r\to\infty}} \frac{\log|f(re^{i\theta})|}{r^\rho} = A, \tag{1.22}$$

where $A > 0$. If (1.22) is satisfied with $A = 0$, we say that $f(z)$ is of *minimum type*, and if $e^{cz^\rho}f(z)$ is of minimum type for some $c \neq 0$, $f(z)$ is of *reducible mean type*. Lastly, if (1.22) is not satisfied for any finite A, $f(z)$ is said to be of *maximum type*. I suppose, of course, in all cases that $f(z)$ is of order ρ.

1.3. Lindelöf† proved the three following theorems.

THEOREM I. *If $f(z)$ is of order ρ and minimum type, then*

$$n(r) = o(r^\rho) \quad \text{and} \quad c_\rho + \frac{1}{\rho}\sum_1^\infty \frac{1}{z_n^\rho} = 0. \tag{1.31}$$

Conversely, if both the conditions (1.31) *are satisfied, and if $\Sigma(1/r_n^{\rho-\epsilon})$ is divergent for every $\epsilon > 0$, then $f(z)$ is of order ρ and minimum type.*

COROLLARY. *If $f(z)$ is of order ρ and reducible mean type, then*

$$n(r) = o(r^\rho) \quad \text{and} \quad c_\rho + \frac{1}{\rho}\sum_1^\infty \frac{1}{z_n^\rho} = c \neq 0. \tag{1.32}$$

Conversely, if both conditions (1.32) *are satisfied, then $f(z)$ is of order ρ and reducible mean type.*

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* See E. Lindelöf, *Annales Sci. de l'Ecole Normale* (3), 22 (1905), 369–395.

† *Loc. cit.*

THEOREM II. *If $f(z)$ is of order ρ and mean type, then there is a K such that*

$$n(r) < Kr^{\rho} \quad \text{and} \quad \left| c_{\rho} + \frac{1}{\rho} \sum_{1}^{N} \frac{1}{z_n^{\rho}} \right| < K. \tag{1.33}$$

Conversely, if conditions (1.33) *are both satisfied for some $K > 0$, but conditions* (1.31) *are not both satisfied, then $f(z)$ is of order ρ and mean type.*

THEOREM III. *If $f(z)$ is of order ρ and maximum type, then one at least of the conditions* (1.33) *is not satisfied for any finite K. Conversely, if* (1.33) *is not satisfied for any finite K, and if $\Sigma(1/r_n^{\rho+\epsilon})$ is convergent for every $\epsilon > 0$, then $f(z)$ is of order ρ and maximum type.*

Throughout the paper the results for $n(r)$ are trivial compared with those for the series (1.21). I state them because the series results depend to a very large extent on $n(r)$ and correspond closely to the results for $n(r)$.

1.4. We can be more precise with regard to functions of mean type. Let

$$S(r) = c_{\rho} + \frac{1}{\rho} \sum_{r_n \leqslant r} \frac{1}{z_n^{\rho}},$$

and let

$$S(r) = S'(r) + iS''(r),$$

where $S'(r)$ and $S''(r)$ are real. Further, let

$$\overline{\lim_{r \to \infty}} \frac{\log |f(re^{i\theta})|}{r^{\rho}} = h(\theta).$$

Then, if $f(z)$ is of type A, it follows from the definition that $h(\theta) \leqslant A$ uniformly for all θ, and $h(\theta) = A$ for some θ. Also Phragmén and Lindelöf* have shown that under these conditions $h(\theta)$ is continuous.

THEOREM IV. *Suppose that $f(z)$ is of order ρ and type A, and that*

$$\overline{\lim_{r \to \infty}} \frac{n(r)}{\rho r^{\rho}} = B, \quad \overline{\lim_{r \to \infty}} \{S'(r) \cos \rho\theta - S''(r) \sin \rho\theta\} = C(\theta).$$

Then $B \leqslant Ae$ and

$$h(\theta) - K_1 B \leqslant C(\theta) \leqslant h(\theta) + K_2 B, \tag{1.41}$$

where K_1 and K_2 depend only on ρ.

COROLLARY 1. *If $B = 0$, then $h(\theta) = C(\theta)$.*

* E. Phragmén and E. Lindelöf, *Acta Math.*, 31 (1908), 381–406.

COROLLARY 2. *There are functions of order ρ and type A for which $B = 0$ and $S(r)$ oscillates between $\pm A$.*

The inequalities still hold when we remove the restriction on A; if $A = 0$ the theorem reduces to Theorem I, and if $A = \infty$ we have Theorem III. The left-hand inequality in (1.41) is substantially due to Pringsheim*, and the proof is similar to his. The method which I use for the right-hand side is a little more direct than that used by Lindelöf in Theorems I, II, and III. The value of K_2 is large, so that the right-hand inequality is poor except when B is very small. In Theorem IX I shall obtain a different inequality which is much better for large B, but not so good for very small B.

If $B = 0$, the method yields more than Corollary 1; in fact, with a few minor refinements, we have

THEOREM V. *Suppose that $f(z)$ is of order ρ and mean type and that $n(r) = o(r^\rho)$. Then (1) given any $\epsilon > 0$, we can choose $r_0 = r_0(\epsilon)$ so that*

$$\log|f(re^{i\theta})| < \{S'(r)\cos\rho\theta - S''(r)\sin\rho\theta + \epsilon\}r^\rho$$

uniformly for $r \geqslant r_0$, (2) given any $\epsilon > 0$, $\eta > 0$, we can choose $R_0 = R_0(\epsilon, \eta)$ so that

$$\log|f(re^{i\theta})| > \{S'(r)\cos\rho\theta - S''(r)\sin\rho\theta - \epsilon\}r^\rho \tag{1.42}$$

in $0 < r < R$, $R > R_0$, except in a set of measure less than ηR†, (3) if $f(z)$ has no zeros for which $\alpha < \theta_n < \beta$, then, given any $\epsilon > 0$, $\delta > 0$, we can choose $r_0 = r_0(\epsilon, \delta)$ so that (1.42) holds uniformly for

$$\alpha + \delta \leqslant \theta \leqslant \beta - \delta, \quad r \geqslant r_0.$$

Parts (1) and (2) can be expressed roughly by saying that

$$\log|f(re^{i\theta})| \sim \{S'(r)\cos\rho\theta - S''(r)\sin\rho\theta\}r^\rho$$

in a set of density 1. Using this theorem we can construct functions with very peculiar properties by making $S(r)$ oscillate appropriately.

1.5. We now pass to theorems depending on a different order of ideas. All the proofs of the previous theorems depend on splitting up the canonical product for $f(z)$ so as to separate the factor $\exp\{z^\rho S(r)\}$ from the rest. The behaviour of $S'(r)$ and $S''(r)$ is then deduced from the behaviour of

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* A. Pringsheim, *Math. Annalen*, 58 (1904), 311

† Compare E. C. Titchmarsh, *Proc. London Math. Soc.* (2), 26 (1927), 191–194.

$\exp\{z^\rho S(r)\}$. The next theorem depends on quite a different set of ideas, and shows that $\lim S(r)$ exists for a certain class of functions of mean type. We use a species of mean value and a formula, similar in some respects to Jensen's. The mean values are practically Rieszian* means of type n^2 and order 1, and we complete the theorem by a Tauberian argument. This method is due to Titchmarsh†, who used it for functions of the form

$$f(z) = \int_{-1}^{1} e^{zt}\,\phi(t)\,dt.$$

THEOREM VI. *Suppose that $f(z)$ is of order ρ and mean type and that*

$$\lim_{r\to\infty} \frac{n(r)}{\rho r^\rho} = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta)\,d\theta = H. \tag{1.51}$$

Then (1.21) *is convergent and*

$$\lim_{r\to\infty} S(r) = \frac{1}{\pi}\int_{-\pi}^{\pi} h(\theta)\,e^{-i\theta\rho}\,d\theta.$$

COROLLARY. *If the conditions of the theorem are satisfied, and also $h(\theta) = h(\theta+\pi/\rho)$ for all θ [in particular if $h(\theta) = A$ for all θ], then* $\lim S(r) = 0$.

From one point of view this is a best possible result. For, by the methods of Theorem IV, we can prove

THEOREM VII. *There are functions of order ρ and mean type for which*

$$\overline{\lim_{r\to\infty}} \frac{n(r)}{\rho r^\rho} = B < H,$$

and for which $S(r)$ oscillates between $\pm\frac{1}{2}\pi(H-B)$.

We may replace (1.51) by

$$\lim_{r\to\infty} \frac{1}{2\pi r^\rho}\int_{-\pi}^{\pi} \log|f(re^{i\theta})|\,d\theta = H, \tag{1.511}$$

which is equivalent to it. For, by Jensen's formula,

$$\int_0^r \frac{n(u)}{u}\,du = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\theta})|\,d\theta, \tag{1.52}$$

* See G. H. Hardy and M. Riesz, "General theory of Dirichlet's series", *Cambridge Math. Tracts*, 18 (1915), 21, 22.

† E. C. Titchmarsh, *Proc. London Math. Soc.* (2), 25 (1926), 299–301.

and it follows by a well-known argument* that if

$$\lim_{r\to\infty} \frac{1}{r^\rho} \int_0^r \frac{n(u)}{u}\, du = H,$$

then $\lim\{n(r)/r^\rho\} = H\rho$. The converse is obviously true. If $h(\theta) = 0$, Theorem VI reduces to the first part of Theorem I. If $\lim\{n(r)/r^\rho\}$ exists, it must be less than or equal to $H\rho$ in virtue of (1 . 52). Hence we may express Theorem VI in the form:

If $f(z)$ is of mean type and has its full complement of zeros evenly distributed, then (1 . 21) *is convergent.*

1 . 6. Let

$$T(r, \rho) = c\rho + \frac{1}{\rho} \sum_{r_n < r} \frac{1}{z_n^\rho} \left(1 - \frac{r_n^{2\rho}}{r^{2\rho}}\right) \quad \text{and} \quad n_1(r) = \frac{1}{r^\rho} \int_0^r \frac{n(u)}{u}\, du.$$

For these mean values we have

THEOREM VIII. *Suppose that $f(z)$ is of order ρ and type A and that*

$$\overline{\lim_{r\to\infty}}\, n_1(r) = B_1, \qquad \underline{\lim_{r\to\infty}}\, n_1(r) = B_1'.$$

Then

$$B_1 \leqslant H \leqslant A,$$

where H is defined as in (1 . 51) *and*

$$\overline{\lim_{r\to\infty}} \{ |T(r, \rho)| + 2n_1(r)/r^\rho \} \leqslant 2A,$$

i.e.

$$\overline{\lim_{r\to\infty}} |T(r, \rho)| \leqslant 2(A - B_1') \leqslant 2A. \tag{1 . 61}$$

From (1 . 61) we deduce

THEOREM IX. *Suppose that $f(z)$ is of order ρ and type A and that H, B, B_1', and $S(r)$ are defined as before. Then*

$$B \leqslant (He - B_1') \leqslant He \leqslant Ae$$

and

$$\overline{\lim_{r\to\infty}} |S(r)| \leqslant 2(A - B_1') + B \leqslant (2+e)A - 2B_1' \leqslant (2+e)A.$$

The last theorem emphasizes the importance of even distribution of the zeros; for if B_1' is nearly equal to B_1 and therefore to B, we find that $|S(r)|$ is small. If $B_1' = H$, the results are superseded by those of Theorem VI.

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* See E. C. Titchmarsh, *Proc. London Math. Soc.* (2), 25 (1926), 298.

The mean value method fails to separate the real and imaginary parts of $S(r)$, and consequently, when $B = 0$, the results are not so good as those of Theorem IV. I shall conclude by showing how the mean values $T(r, \rho)$ are related to the canonical product for $f(z)$.

1.7. Lindelöf has defined functions of order $(\rho, \rho_1, \rho_2, \ldots, \rho_k)$ by introducing logarithms, and has proved theorems corresponding to I, II, and III for functions of order $(\rho, \rho_1, \ldots, \rho_k)$ and the various types. The other theorems of this paper have similar extensions.

Proof of Theorems IV *and* V.

2.1. We first need the following lemma due to Boutroux*.

LEMMA A. *Suppose that*

$$P(z) = \prod_1^N \left(1 - \frac{z}{z_n}\right),$$

where $N = n(R)$. *Then, in the annular regions*

$$Rk^{-1} \leqslant |z| \leqslant R \quad \text{and} \quad Rk^{-3} \leqslant |z| \leqslant Rk^{-2} \quad (k > 1),$$

there are circles on which

$$\log |P(z)| > -K_3 N,$$

where $K_3 = K_3(k)$.

The fundamental difference between this proof and Lindelöf's proof of II lies in the use of Lemma A. We also need a result in the opposite direction†.

LEMMA B. *If*

$$n(r) \leqslant B\rho r^\rho, \tag{2.11}$$

then $\log |P(z)| < (1 + \rho \log 2) BR^\rho$, *provided that* $|z| \leqslant R$.

It follows from (2.11) that

$$\log \frac{R^N}{r_1 r_2 \ldots r_n} = \int_0^R \frac{n(u)}{u}\, du \leqslant BR^\rho.$$

Hence

$$\log |P(Re^{i\theta})| \leqslant \log \frac{R^N}{r_1 r_2 \ldots r_n} + \sum_1^N \log\left(\frac{r_n}{R} + 1\right)$$

$$\leqslant BR^\rho + N \log 2 \leqslant B(1 + \rho \log 2) R^\rho.$$

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* See G. Valiron, *loc. cit.*, 80.

† Compare E. Lindelöf, *loc. cit.*

This proves the lemma for $|z| = R$, and it follows by the maximum modulus principle for $|z| < r$.

2.2. *Proof of the right-hand inequality in* (1.41). We may express $f(z)$ in the form (1.12) and we write

$$f(z) = P(z)\,E(z)\,\Phi(z), \tag{2.21}$$

where

$$E(z) = e^{g(z)} \prod_{1}^{N} E_n(z, \rho),$$

$$\Phi(z) = \prod_{N+1}^{\infty} \left(1 - \frac{z}{z_n}\right) E_n(z, \rho),$$

and $P(z)$ is defined as before. Now*

$$\left(1 - \frac{z}{z_n}\right) E_n(z, \rho) = \exp\left\{\frac{\gamma_n}{\rho+1}\,\frac{k}{k-1}\left(\frac{r}{r_n}\right)^{\rho+1}\right\},$$

where $|\gamma_n| < 1$, provided that $r/r_n \leqslant k^{-1}$. Hence

$$\Phi(re^{i\theta}) = \exp\left\{\frac{k}{k-1}\,\frac{1}{\rho-1}\sum_{N+1}^{\infty} \gamma_n \left(\frac{r}{r_n}\right)^{\rho+1}\right\} \tag{2.22}$$

for $r \leqslant Rk^{-1}$. We know that

$$\frac{1}{(\rho+1)}\sum_{N+1}^{M}\frac{1}{r_n^{\rho+1}} = \int_{r_{N+1}}^{r_{M+1}} \frac{n(x)}{x^{\rho+2}}\,dx + \frac{M}{(\rho+1)r_{M+1}^{\rho+1}} - \frac{N}{(\rho+1)r_{N+1}^{\rho+1}}. \tag{2.221}$$

Since $M < (B+\epsilon)\rho r_{M+1}^{\rho}$, we have

$$\left|\frac{1}{(\rho+1)}\sum_{N+1}^{\infty}\gamma_n\left(\frac{r}{r_n}\right)^{\rho+1}\right| \leqslant r^{\rho+1}\int_{r_{N+1}}^{\infty}\frac{n(x)}{x^{\rho+2}}\,dx \leqslant (B+\epsilon)\rho r^{\rho}. \tag{2.222}$$

Hence

$$\log|\Phi(re^{i\theta})| > -K_4(B+\epsilon)r^{\rho}, \tag{2.23}$$

where $K_4 = K_4(k, \rho)$, provided that $r \leqslant Rk^{-1}$.

Given any $k > 1$ it follows from Lemma A that there is a circle $|z| = r$, such that $Rk^{-2} \leqslant r \leqslant Rk^{-1}$, on which

$$\log|P(z)| > -K_3 N > -K_5(B+\epsilon)r^{\rho}, \tag{2.24}$$

* See G. Valiron, *loc. cit.*, 10.

where $K_5 = K_5(k, \rho)$. Now combining (2.23) and (2.24) with (2.21) we have

$$(2.25)\quad \log|E(z)| = \log\frac{|f(z)|}{|P(z)|\,|\Phi(z)|} < \{h(\theta)+\epsilon+(K_4+K_5)(B+\epsilon)\}r^\rho$$

on $|z| = r$, provided that r and therefore R is greater than $r_0(\epsilon)$. Hence, taking logarithms, we have

$$(2.26)\quad \Re\{c_1 re^{i\theta}+\ldots+c_\rho r^\rho e^{i\theta\rho}\}+\sum_1^N \Re\left\{\frac{re^{i(\theta-\theta_n)}}{r_n}+\ldots+\frac{r^\rho e^{i(\theta-\theta_n)\rho}}{\rho r_n^\rho}\right\}$$
$$< \{h(\theta)+\epsilon+(K_4+K_5)(B+\epsilon)\}r^\rho$$

for all values of θ.

Put $c_p = a_p+ib_p$, where a_p and b_p are real, $p = 1, 2, \ldots, \rho$. Then (2.26) may be written

$$(2.27)\quad \sum_{p=1}^{\rho} r^p(a_p\cos p\theta - b_p\sin p\theta) + \sum_{p=1}^{\rho}\sum_{n=1}^{N}\frac{r^p}{pr_n^p}(\cos p\theta\cos p\theta_n+\sin p\theta\sin p\theta_n)$$
$$< \{h(\theta)+\epsilon+(K_4+K_5)(B+\epsilon)\}r^\rho,$$

where $Rk^{-2} \leqslant r \leqslant Rk^{-1}$, K_4 and K_5 depend on k and ρ, $N = n(R)$, provided that $r > r_0(\epsilon)$.

Since $r_n^\rho > n/(B+\epsilon)\rho$ for $n > n_0(\epsilon)$, we have, for $Rk^{-2} \leqslant r \leqslant Rk^{-1}$,

$$(2.28)\quad \frac{1}{p}\sum_{n_0}^{N}\left(\frac{r}{r_n}\right)^p < \frac{1}{p}(B+\epsilon)^{p/\rho}\rho^{p/\rho}r^p\sum_1^N\frac{1}{n^{p/\rho}} \leqslant \frac{\rho^{p/\rho}}{p}(B+\epsilon)^{p/\rho}r^p\int_1^N\frac{dt}{t^{p/\rho}}$$
$$\leqslant \frac{\rho^{p/\rho+1}}{p(\rho-p)}(B+\epsilon)^{p/\rho}r^pN^{1-p/\rho} < \frac{(B+\epsilon)\rho^2r^\rho k^{(1-p/\rho)2}}{p(\rho-p)},$$

provided that $p < \rho$. Hence, dividing (2.27) by r^ρ, we have

$$(2.29)\quad a_\rho\cos\rho\theta - b_\rho\sin\rho\theta+\frac{1}{\rho}\sum_1^N\frac{1}{r_n^\rho}(\cos\rho\theta_n\cos\rho\theta+\sin\rho\theta_n\sin\rho\theta)$$
$$= S'(R)\cos\rho\theta - S''(R)\sin\rho\theta < h(\theta)+\epsilon+(K_4+K_5+K_6)(B+\epsilon),$$

where K_4, K_5, K_6 depend on k, for every R such that $Rk^{-2} > r_0(\epsilon)$, *i.e.* $R > R_0(\epsilon, k)$. Keep k fixed and make $R\to\infty$. Then we have

$$\overline{\lim_{R\to\infty}}\{S'(R)\cos\rho\theta - S''(R)\sin\rho\theta\} < h(\theta)+K_7B,$$

which is the required result.

2.3. *Proof of the left-hand inequality in* (1.41). We have

$$|f(z)| = |P(z)||E'(z)||E''(z)||\Phi(z)|, \tag{2.31}$$

where $P(z)$ and $\Phi(z)$ are defined as before, and

$$\log E''(z) = c_\rho z^\rho + \frac{1}{\rho}\sum_1^N \frac{z^\rho}{z_n^\rho},$$

$$\log E'(z) = \sum_{p=1}^{\rho-1} c_p z^p + \sum_{p=1}^{\rho-1}\sum_{n=1}^{N} \frac{z^p}{p z_n^p}.$$

It follows from (2.28) that

$$\log|E'(re^{i\theta})| < K_6(B+\epsilon)\rho r^\rho$$

for $Rk^{-2} \leqslant r \leqslant Rk^{-1}$. It also follows from (2.22) and (2.222) that

$$\log|\Phi(re^{i\theta})| < K_8(B+\epsilon)\rho r^\rho$$

for $r \leqslant Rk^{-1}$, and by Lemma B

$$\log|P(re^{i\theta})| < K_9(B+\epsilon)\rho r^\rho,$$

provided that $Rk^{-2} \leqslant r \leqslant Rk^{-1}$. Hence, combining these results with (2.31), we have

$$\log|f(re^{i\theta})| < \log|E''(re^{i\theta})| + (K_6+K_8+K_9)(B+\epsilon)\rho r^\rho$$

for $Rk^{-2} \leqslant r \leqslant Rk^{-1}$. Dividing by r^ρ we have

$$\frac{\log|f(re^{i\theta})|}{r^\rho} < S'(R)\cos\rho\theta - S''(R)\sin\rho\theta + (K_6+K_8+K_9)(B+\epsilon)\rho \tag{2.32}$$

for every $\epsilon > 0$ and every $R > R_0(\epsilon, k)$. Making $R \to \infty$ we have the required result.

2.4. *Proof of Corollary* 2. Corollary 1 follows immediately, and then it only remains to construct a series $\Sigma 1/z_n^\rho$ which oscillates between $\pm A\rho$ and for which $(r_n/n) \to \infty$ with n.

Suppose for simplicity that $\rho = 1$, $A = 1$; there is no essential difference in the other cases. Suppose that all the zeros are real so that $S''(R) \equiv 0$, and let $r_1 = 1$, $r_n = n\log n$ for $n > 1$. Consider

$$S_N = \sum_1^N \frac{1}{n\log n};$$

this is a function of N increasing steadily to ∞, and $S_N - S_{N-1} \to 0$ as $N \to \infty$. Hence it is easy to see that we may choose a sequence of numbers $N_1, N_2, \ldots, N_m \to \infty$ such that $S_{N_m} - S_{N_{m-1}} \to 2$ as $m \to \infty$, and

$$S_{N_1} - (S_{N_2} - S_{N_2}) - \ldots + (-1)^{m-1}(S_{N_m} - S_{N_{m-1}}) = (-1)^{m-1} + \xi,$$

where $|\xi|<\epsilon$ for $m>m_0(\epsilon)$. Then, putting

$$z_n=(-1)^{m-1}n\log n$$

for $N_{m-1}<n\leqslant N_m$, we have

$$S'(R_m)=\sum_1^{N_m}\frac{1}{z_n}=(-1)^{m-1}+\xi.$$

It follows from Corollary 1 that $h(\theta)=|\cos\theta|$, and, in particular,

$$h(0)=h(\pi)=1=\overline{\lim_{R\to\infty}}\,|S'(R)|.$$

2.4. *Proof of Theorem* V. We have

$$S'(R)=S'(r)+\frac{1}{\rho}\sum_{r<r_n\leqslant R}\Re\frac{1}{z_n^\rho}=S'(r)+I, \tag{2.41}$$

where

$$|I|\leqslant N/\rho r^\rho<\epsilon k^{2\rho}<2\epsilon$$

for every $\epsilon>0$ and every $r>r_0(\epsilon)$, provided that k is sufficiently near 1 and that $Rk^{-2}\leqslant r\leqslant Rk^{-1}$. A similar result holds for $S''(R)$ and part (1) follows at once from (2.32).

Part (2) requires more delicate treatment of $P(z)$. We write

$$P(z)=P_1(z)P_2(z),$$

where

$$P_1(z)=\prod_{r_n\leqslant Rk^{-3}}\left(1-\frac{z}{z_n}\right),$$

$$P_2(z)=\prod_{Rk^{-3}<r_n\leqslant R}\left(1-\frac{z}{z_n}\right).$$

Then

$$\log|P_1(re^{i\theta})|\geqslant n(Rk^{-3})\log(k-1)\geqslant\epsilon R^\rho k^{-3}\log(k-1)>-\epsilon' r^\rho$$

for every $\epsilon'>0$ and every $r>r_0(\epsilon')=r_0(\epsilon,k)$. We also have*

$$\log|P_2(re^{i\theta})|>-\epsilon\gamma^\rho \tag{2.42}$$

in $Rk^{-2}\leqslant r\leqslant Rk^{-1}$, except in a set of measure less than $\eta Rk^{-1}(1-k^{-1})$, for every $\epsilon>0$, $\eta>0$, every $k>1$ and $R>R_0(\epsilon,k,\eta)$. Now using (2.41) it follows as in (2.29) that (1.42) holds in $Rk^{-2}\leqslant r\leqslant Rk^{-1}$, except in a set of measure less than $\eta Rk^{-1}(1-k^{-1})$. Repeating this argument for the intervals (Rk^{-3}, Rk^{-2}), (Rk^{-4}, Rk^{-3}), ..., $(0, Rk^{-q})$,

* See E. C. Titchmarsh, *Proc. London Math. Soc.* (2), 26 (1927), 193, 194.

we find that (1.42) holds for $0 < r < R$ except in a set of measure less than ηR, which is the required result.

If $f(z)$ has no zeros for which $\alpha < \theta_n < \beta$, we have

$$(2.43) \qquad \log|P_2(re^{i\theta})| > N \log\{2(1-\cos\delta)\} > \epsilon r^\rho \log\delta > -\epsilon' r^\rho$$

uniformly for $\alpha+\delta \leqslant \theta \leqslant \beta-\delta$, $Rk^{-2} \leqslant r \leqslant Rk^{-1}$, provided that $R > R_0(\epsilon', k, \delta)$. Using (2.43) instead of (2.42) we have part (3).

Proofs of Theorems VI, VII, VIII, *and* IX.

3.1. We now require a lemma of a different type. For $\rho = 1$ the following formula is a combination of Carleman's* formulae for the right and left half planes; and for $\rho > 1$, it is a combination of the appropriate generalizations due to R. Nevanlinna†.

LEMMA C. *Suppose that $F(z)$ is regular for $|z| \leqslant R$, and that $F(0) = 1$. Let $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$, ..., where $0 < r_1 \leqslant r_2 \leqslant \ldots$, be the zeros of $F(z)$ in the circle $|z| \leqslant R$, and let*

$$\log F(z) = \sum_{m=1}^{\infty} (a_m + ib_m) z^m,$$

where a_m and b_m are real, for $|z| < r_1$. Then

$$(3.11) \qquad T'(R, \rho) = a_\rho + \frac{1}{\rho} \sum_{r_n < R} \frac{\cos\rho\theta_n}{r_n^\rho}\left(1 - \frac{r_n^{2\rho}}{R^{2\rho}}\right)$$

$$= \frac{1}{\pi R^\rho} \int_{-\pi}^{\pi} \log|F(Re^{i\theta})| \cos\rho\theta \, d\theta,$$

where ρ is a positive integer.

Consider the integral‡

$$\frac{1}{2\pi i} \int_C \log F(z) \left(\frac{1}{z^\rho} + \frac{z^\rho}{R^{2\rho}}\right) \frac{dz}{z}$$

taken round the closed contour C consisting of the circle $|z| = R$ and loops round the zeros. The loops are arranged so that $F(z)$ has no zeros inside C, and therefore $\log F(z)$ is regular inside and on the boundary of C. Apply Cauchy's theorem, and take the real part. The result follows as in the proof of Carleman's formula.

* T. Carleman, *Arkiv för Mat.*, 17, 9 (1923), 5. See also G. Pólya and G. Szegö, *Aufgaben und Lehrsätze*, 1 (1925), 310 (solution 240).

† R. Nevanlinna, *Acta Soc. Sci. Fennicae* 50, 12 (1925).

‡ Compare G. Pólya and G. Szegö, *loc. cit.*

3.2. *Proof of Theorem* V. As we have already seen, we may replace (1.51) by (1.511). Then, given any $\epsilon > 0$, we can choose $r_0 = r_0(\epsilon)$ so that

$$-2\epsilon < \int_{-\pi}^{\pi} \left\{ h(\theta) + \epsilon - \frac{\log|f(re^{i\theta})|}{r^\rho} \right\} d\theta < 2\epsilon,$$

and also so that the integrand is positive for $r > r_0(\epsilon)$. Hence

$$-2\epsilon < \int_{-\pi}^{\pi} \left\{ h(\theta) + \epsilon - \frac{\log|f(re^{i\theta})|}{r^\rho} \right\} \cos\rho\theta\, d\theta < 2\epsilon$$

for $r > r_0(\epsilon)$, and therefore

$$\lim_{r\to\infty} \int_{-\pi}^{\pi} \frac{\log|f(re^{i\theta})|}{r^\rho} \cos\rho\theta\, d\theta = \int_{-\pi}^{\pi} h(\theta)\cos\rho\theta\, d\theta = H_1. \tag{3.21}$$

When $|z|$ is very small, *i.e.* $|z| = r \leqslant r_1 k^{-1}$, it follows from (2.22) that we may write

$$\begin{aligned} \log f(z) &= c_1 z + c_2 z^2 + \ldots + c_\rho z^\rho + O(r^{\rho+1}) \\ &= \sum_1^\rho (a_m + ib_m) z^\rho + O(r^{\rho+1}). \end{aligned}$$

Hence*, applying Lemma C to $f(z)$, we have

$$T'(R, \rho) = \frac{1}{\pi R^\rho} \int_{-\pi}^{\pi} \log|f(Re^{i\theta})| \cos\rho\theta\, d\theta.$$

Make $R \to \infty$; then it follows from (3.21) that

$$\lim_{R\to\infty} T'(R, \rho) = H_1.$$

Put $R^\rho = m/H\rho$, where m is an integer. Then

$$|m - n(R)| = o(m)$$

as $m \to \infty$, so that

$$\lim_{m\to\infty} \frac{1}{\rho} \sum_{n=1}^{m} \frac{\cos\rho\theta_n}{r_n^\rho} \left(1 - \frac{r_n^{2\rho} H^2 \rho^2}{m^2}\right) = H_1 - a_\rho.$$

Let $$\sigma_m = \sum_{n=1}^{m} \frac{\cos\rho\theta_n}{r_n^\rho} \left(1 - \frac{r_n^{2\rho} H^2 \rho^2}{m^2}\right).$$

Then

$$(m+1)^2\sigma_{m+1} - m^2\sigma_m = (2m+1) \sum_{n<m} \frac{\cos\rho\theta_n}{r_n^\rho} + \frac{\cos\rho\theta_m}{r_m^\rho} \{(m+1)^2 - H^2\rho^2 r_m^{2\rho}\},$$

so that $$\sum_1^m \frac{\cos\rho\theta_n}{r_n^\rho} = \frac{(m+1)^2\sigma_{m+1} - m^2\sigma_m}{2m+1} + o(1).$$

* From this point the proof follows Titchmarsh's work very closely.

Hence

$$\frac{1}{p}\sum_{m=1}^{p}\sum_{n=1}^{m}\frac{\cos\rho\theta_n}{r_n^\rho}=\frac{1}{p}\sum_{m=1}^{p}\frac{(m+1)^2\sigma_{m+1}-m^2\sigma_m}{2m+1}+o(1)$$

$$=\frac{1}{p}\sum_{m=1}^{p}\frac{2m^2\sigma_m}{4m^2-1}+\frac{(p+1)^2}{p(2p+1)}\sigma_{p+1}+o(1).$$

When $p\to\infty$, this tends to the same limit as σ_p, and therefore

$$\frac{1}{\rho}\sum\frac{\cos\rho\theta_n}{r_n^\rho} \tag{3.22}$$

is summable $(C, 1)$ to sum H_1-a_ρ. But $n(r)<(H+\epsilon)\rho r^\rho$, *i.e.* $r_n>n/\rho(H+\epsilon)$, so that the n-th term is $O(1/n)$. Hence (3.22), which is the real part of (1.21), is convergent to the sum H_1-a_ρ.

For the imaginary part of (1.21) we replace (3.11) by a similar formula deduced from the integral

$$\frac{1}{2\pi i}\int\log F(z)\left(\frac{1}{z^\rho}-\frac{z^\rho}{R^{2\rho}}\right)\frac{dz}{z}.$$

3.3. *Proof of Theorem* VII. Consider the function

$$f(z)=\psi(z)\sinh(\tfrac{1}{2}\pi Bz^\rho),$$

where $\psi(z)$ is a function of the type used in Theorem IV, Corollary 2. More precisely we suppose that (i) $n'(r)=o(r^\rho)$, where $n'(r)$ is the number of zeros of $\psi(z)$, (ii) the zeros of $\psi(z)$ are all real, (iii) $\sum_1^{n'}{}' 1/z_n^\rho$, where Σ' denotes summation over the zeros of $\psi(z)$ only, oscillates between $\pm\frac{1}{2}\pi H'$, where $H'>0$.

Now

$$\sum_{r_n<R}\frac{1}{z_n^\rho}=\sum_{r_n<R}{}'\frac{1}{z_n^\rho};$$

for the partial sum taken over the zeros of $\sinh(\frac{1}{2}\pi Hz^\rho)$ cancels out. We have therefore constructed a function for which

$$n(r)\sim B\rho r^\rho,$$

and yet $\Sigma 1/z_n^\rho$ oscillates between $\pm\frac{1}{2}\pi H'$. It remains to show that, for this function, $H=B+H'$.

It follows from Theorem IV, Corollary 1, that

$$\log|\psi(re^{i\theta})|<\{\tfrac{1}{2}\pi H'|\cos\rho\theta|+\epsilon\}r^\rho.$$

Hence

$$h(\theta)=\overline{\lim_{r\to\infty}}\left\{\frac{\log|\psi(re^{i\theta})|+\log|\sinh(\tfrac{1}{2}\pi Hre^{i\theta})|}{r^\rho}\right\}$$

$$=\tfrac{1}{2}\pi(H'+B)|\cos\rho\theta|,$$

and therefore $$H = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta)\,d\theta = H' + B,$$

which completes the proof.

3.4. *Proof of Theorem* VIII. By (3.11) and the corresponding formula for the imaginary part of the mean value we have

$$T(R,\rho) = c_\rho + \frac{1}{\rho}\sum_{r_n<R}\frac{1}{z_n^\rho}\left(1-\frac{r_n^{2\rho}}{R^{2\rho}}\right) = \frac{1}{\pi R^\rho}\int_{-\pi}^{\pi}\log|f(Re^{i\theta})|\,e^{-i\theta\rho}\,d\theta.$$

Let $T(R,\rho) = e^{i\alpha}|T(R,\rho)|$, then

$$|T(R,\rho)| = \frac{1}{\pi R^\rho}\int_{-\pi}^{\pi}\log|f(Re^{i\theta})|\,e^{-i(\alpha+\theta\rho)}\,d\theta \tag{3.41}$$

$$= \frac{1}{\pi R^\rho}\int_{-\pi}^{\pi}\log|f(Re^{i\theta})|\cos(\alpha+\rho\theta)\,d\theta.$$

For $|T(R,\rho)|$ must be real. By Jensen's formula

$$\int_0^R \frac{n(u)}{u}\,du = \frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f(Re^{i\theta})|\,d\theta.$$

Dividing this by $\frac{1}{2}R^\rho$ and adding it to (3.41) we have*

$$|T(R,\rho)| + \frac{2n_1(R)}{R^\rho} = \frac{1}{\pi R^\rho}\int_{-\pi}^{\pi}\log|f(Re^{i\theta})|\,\{1-\cos(\alpha+\rho\theta)\}\,d\theta.$$

Since $1-\cos(\alpha+\rho\theta)$ is positive, we have

$$|T(R,\rho)| + \frac{2n_1(R)}{R^\rho} \leqslant \frac{(A+\epsilon)}{\pi}\int_{-\pi}^{\pi}\{1-\cos(\alpha+\rho\theta)\}\,d\theta$$

$$\leqslant 2(A+\epsilon)$$

for every $\epsilon > 0$ and every $R > R_0(\epsilon)$. Making $R\to\infty$, we have

$$\overline{\lim_{R\to\infty}}\{|T(R,\rho)| + 2n_1(R)/R^\rho\} \leqslant 2A,$$

or

$$\overline{\lim_{R\to\infty}}|T(R,\rho)| \leqslant 2(A-B_1'). \tag{3.42}$$

3.5. *Proof of Theorem* IX. It follows from (3.42) that

$$\left|c_\rho + \frac{1}{\rho}\sum_{r_n<R}\frac{1}{z_n^\rho}\left(1-\frac{r_n^{2\rho}}{R^{2\rho}}\right)\right| \leqslant 2(A-B_1')+\epsilon$$

* This argument is due to M. Riesz.

for every $R > R_0(\epsilon)$, and

$$\left| \frac{1}{\rho} \sum_{r_n < R} \frac{1}{z_n^\rho} \frac{r_n^{2\rho}}{R^{2\rho}} \right| \leqslant \frac{1}{\rho} \sum_{r_n < R} \frac{r_n^\rho}{R^{2\rho}} \leqslant \frac{1}{\rho} \sum_1^N \frac{1}{R^\rho} \leqslant B + \epsilon.$$

Hence $$S(R) \leqslant 2(A - B_1') + B + \epsilon.$$

Making $R \to \infty$ we have the first inequality and the others follow at once.

4. It may be of some interest to see how these mean values are connected formally with the canonical product for $f(z)$. Suppose that $f(z)$ has no zeros on the circle $|z| = r$, and let

$$|G(z)| = |e^{g(z)}| \prod_{r_n < r} \left| 1 - \frac{z}{z_n} \right| |E_n(z, \rho)|.$$

Then $$\log |e^{g(z)}| = \mathfrak{R}\{g(z)\} = \sum_1^\rho (a_p \cos p\theta - b_p \sin p\theta) r^p,$$

and

$$\log \left| 1 - \frac{z}{z_n} \right| |E_n(z, \rho)| = \log \frac{r}{r_n} + \log \left| 1 - \frac{z_n}{z} \right| + \mathfrak{R}\left\{ \frac{z}{z_n} + \ldots + \frac{z^\rho}{\rho z_n^\rho} \right\}$$

$$= \log \frac{r}{r_n} - \mathfrak{R}\left\{ \sum_1^\infty \frac{z_n^p}{p z^p} \right\} + \mathfrak{R}\left\{ \frac{z}{z_n} + \ldots + \frac{z^\rho}{\rho z_n^\rho} \right\}$$

$$= \log \frac{r}{r_n} + \sum_{p=1}^\rho \frac{\cos p(\theta - \theta_n)}{p r_n^p} r^p \left(1 - \frac{r_n^{2p}}{r^{2p}} \right) - \sum_{\rho+1}^\infty \frac{\cos p(\theta - \theta_n)}{p} \left(\frac{r_n}{r} \right)^p.$$

Hence

$$\log |G(z)| = \log \frac{r^n}{r_1 r_2 \ldots r_n} + \sum_{p=1}^\rho \{T'(r, p) \cos p\theta - T''(r, p) \sin p\theta\} r^p$$

$$+ \sum_{p=\rho+1}^\infty \frac{\cos p(\theta - \theta_n)}{p} \left(\frac{r_n}{r} \right)^p,$$

where $T(r, p) = T'(r, p) + iT''(r, p)$. If $r_n > r$ we have

$$\log \left| 1 - \frac{z}{z_n} \right| |E_n(z, \rho)| = \sum_{\rho+1}^\infty \frac{1}{p} \mathfrak{R} \left(\frac{z}{z_n} \right)^p.$$

Hence

$$\log |f(z)| = \int_0^r \frac{n(u)}{u} du + \sum_{p=1}^\rho \{T'(r, p) \cos p\theta + T''(r, p) \sin p\theta\} r^p$$

$$- \sum_{p=\rho+1}^\infty \frac{1}{p} \left\{ \sum_{r_n < r} \cos p(\theta - \theta_n) \left(\frac{r_n}{r} \right)^p - \sum_{r_n > r} \cos p(\theta - \theta_n) \left(\frac{r}{r_n} \right)^p \right\}.$$

FORMULAE CONNECTING DIFFERENT CLASSES OF SELF-RECIPROCAL FUNCTIONS

By G. H. Hardy *and* E. C. Titchmarsh.

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1. This note originates from a remark of Ramanujan*, that the function

$$f(x) = \frac{1}{x} + \frac{1}{x} + \frac{2}{x} + \frac{3}{x} + \cdots,$$

which may also be expressed in the forms

$$f(x) = \int_0^\infty \frac{e^{-xt}}{\cosh at}\, dt = 2\left(\frac{1}{x+a} - \frac{1}{x+3a} + \frac{1}{x+5a} - \ldots\right),$$

where $a = \sqrt{(\frac{1}{2}\pi)}$, is its own sine transform. Proofs have been given by Phillips and Watson†. An examination of the second proof shows that it rests on the fact that sech ax is its own cosine transform, and suggests the general rule

I. *If $f(x)$ is its own cosine (sine) transform, then*

$$h(x) = \int_0^\infty e^{-xt} f(t)\, dt$$

is its own sine (cosine) transform.

We shall use the following language. If $f(x)$ is its own Hankel transform of order μ, so that

$$f(x) = \int_0^\infty \sqrt{(xt)}\, J_\mu(xt)\, f(t)\, dt,$$

we shall say that $f(x)$ belongs to R_μ; and we shall write R_C for $R_{-\frac{1}{2}}$ and R_S for $R_{\frac{1}{2}}$. Our rule may then be stated in the form: *if $f(x)$ belongs to R_C (R_S), then $h(x)$ belongs to R_S (R_C).* We shall also express this by saying that *the kernel e^{-xt} transforms R_C (R_S) into R_S (R_C).*

* Ramanujan, **5**, 334.

† See Phillips, **4**.

The lines of the proof are as follows. Supposing, for example, that $f(x)$ belongs to R_C, we have

$$\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty h(t)\sin xt\,dt = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \sin xt\,dt\int_0^\infty e^{-ty}f(y)\,dy$$
$$= \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty f(y)\,dy\int_0^\infty e^{-yt}\sin xt\,dt$$
$$= \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \frac{x}{x^2+y^2}f(y)\,dy.$$

Now $\sqrt{\left(\frac{2}{\pi}\right)}\frac{x}{x^2+y^2}$ is the cosine transform of e^{-xy}, and $f(y)$ is its own cosine transform. Hence Parseval's theorem for cosine transforms gives

$$\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \frac{x}{x^2+y^2}f(y)\,dy = \int_0^\infty e^{-xy}f(y)\,dy = h(x), \tag{1.1}$$

which establishes the rule formally.

2. We shall justify this argument under appropriate conditions later. First, however, we obtain a more general formal rule of the same character.

II. *The kernel* $(xt)^{\frac{1}{2}(\mu+\nu+1)}K_{\frac{1}{2}(\nu-\mu)}(xt)$

transforms R_μ *into* R_ν.

The rule may be established (when its form has been found) by transformations similar to those of § 1. It is more instructive to proceed as follows. A general formula for functions of R_μ is*

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\,\Gamma(\tfrac{1}{2}s+\tfrac{1}{2}\mu+\tfrac{1}{4})\,\psi(s)\,x^{-s}\,ds, \tag{2.1}$$

where $0 < c < 1$ and

$$\psi(s) = \psi(1-s). \tag{2.2}$$

Hence we obtain

$$h(x) = \int_0^\infty (xt)^{\frac{1}{2}(\mu+\nu+1)}K_{\frac{1}{2}(\nu-\mu)}(xt)\,f(t)\,dt$$
$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\,\Gamma(\tfrac{1}{2}s+\tfrac{1}{2}\mu+\tfrac{1}{4})\,\psi(s)\,ds\int_0^\infty (xt)^{\frac{1}{2}(\mu+\nu+1)}K_{\frac{1}{2}(\nu-\mu)}(xt)\,t^{-s}\,dt;$$

* Hardy and Titchmarsh, **1** (1.58).

and, on inserting the value of the inner integral*,

$$(2.3)\quad h(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}(\mu+\nu-s-1)}\,\Gamma(\tfrac{1}{2}s+\tfrac{1}{2}\mu+\tfrac{1}{4})\,\Gamma(\tfrac{1}{2}\mu+\tfrac{3}{4}-\tfrac{1}{2}s) \times \Gamma(\tfrac{1}{2}\nu+\tfrac{3}{4}-\tfrac{1}{2}s)\,\psi(s)\,x^{s-1}\,ds$$

$$= \frac{1}{2\pi i}\int_{1-c-i\infty}^{1-c+i\infty} 2^{\frac{1}{2}w}\,\Gamma(\tfrac{1}{2}w+\tfrac{1}{2}\nu+\tfrac{1}{4})\,\psi_1(w)\,x^{-w}\,dw,$$

where

$$\psi_1(w) = 2^{\frac{1}{2}\mu+\frac{1}{2}\nu-1}\,\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\mu-\tfrac{1}{2}w)\,\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}w)\,\psi(w).$$

It may be verified at once that $\psi_1(w) = \psi_1(1-w)$; and then, comparing (2.3) and (2.1), we see that $h(x)$ belongs to R_ν.

Taking $\mu = \nu$, we see that

III. *The kernel* $(xt)^{\mu+\frac{1}{2}}K_0(xt)$ *transforms* R_μ *into itself; in particular,* $K_0(xt)$ *transforms* R_C *into itself.*

3. Rule I is the case $\mu = -\frac{1}{2}$, $\nu = \frac{1}{2}$ of rule II (or the converse case). So far as rigorous analysis is concerned, we confine ourselves to rule I. There are two things in the analysis of § 1 which require justification, the inversion of the integrations and the appeal to Parseval's theorem.

It is easy to show, first, that the inversion is legitimate whenever the integrals

$$(3.1)\qquad \int_0^1 |f(y)|\,dy, \qquad \int_1^\infty \frac{|f(y)|}{y^2}\,dy$$

are finite. It is obvious that we may invert the integrations over the ranges (τ, T) of t and $(0, \infty)$ of y; and it is therefore sufficient to prove that the integrals

$$I_1 = \int_0^\infty f(y)\,dy\int_0^\tau e^{-yt}\sin xt\,dt, \qquad I_2 = \int_0^\infty f(y)\,dy\int_T^\infty e^{-yt}\sin xt\,dt$$

exist, and tend to zero when $\tau \to 0$ and $T \to \infty$. But

$$|I_1| \leqslant x\int_0^Y |f(y)|\,dy\int_0^\tau t\,dt + x\int_Y^\infty |f(y)|\,dy\int_0^\infty te^{-yt}\,dt$$

$$= \tfrac{1}{2}x\tau^2\int_0^Y |f(y)|\,dy + x\int_Y^\infty \frac{|f(y)|}{y^2}\,dy < \epsilon$$

* Watson, **6**, § 13.21 (8).

for $Y = Y(\epsilon)$, $\tau < \tau_0(Y, \epsilon) = \tau_0(\epsilon)$; and

$$|I_2| \leqslant \int_0^\infty |f(y)| \left| \frac{x\cos Tx + y\sin Tx}{x^2+y^2} \right| e^{-Ty}\,dy \leqslant \frac{1}{x}\int_0^\infty |f(y)|\,e^{-Ty}dy$$

$$\leqslant \frac{1}{x}\int_0^\eta |f(y)|\,dy + \frac{e^{-\frac{1}{2}T\eta}}{x}\int_\eta^\infty |f(y)|\,e^{-\frac{1}{2}Ty}\,dy < \epsilon$$

for $\eta = \eta(\epsilon)$ and $T > T_0(\eta, \epsilon) = T_0(\epsilon)$. Hence I_1 and I_2 tend to zero.

4. It remains to consider the appeal to Parseval's theorem. For this, we may use the criteria enumerated in our note **2**. Here $g(y) = e^{-xy}$ and its transform $G(y)$ satisfy any of the conditions there imposed on either $f(x)$ or $g(x)$, so that the theorem is true whenever our present $f(y)$ satisfies any of these conditions. The situation is also simplified by the fact that $f(y)$ is its own transform. Thus our rule I is applicable whenever $f(x)$ belongs to L (in which case it is also bounded), or to any L^p for which $1 < p \leqslant 2$ (in which case it also belongs to $L^{p'}$ and to all intervening Lebesgue classes); or when it is of bounded variation in $(0, \infty)$, and tends to zero when $x \to \infty$; or when it is bounded, and its cosine integral is uniformly convergent over any finite range of the variable. It will be found, for example, that if we take $f(x)$ to be any of the special functions enumerated in our paper **1**, there is no difficulty in picking out applicable criteria. It will, however, sometimes be necessary to write $f(x)$ as the sum of two functions and use different criteria for them; and sometimes our criteria will show only that one of the integrals of the theorem exists in some generalised sense, and it will be necessary to appeal to the special form of $f(x)$ to see that this integral is in fact convergent.

We may illustrate these remarks by considering two particular examples.

(i) Suppose that $f(x) = x^{-\frac{1}{2}}$. We write $f = f_1 + f_2$, where $f_1 = 0$ in $(1, \infty)$ and $f_2 = 0$ in $(0, 1)$. Then criterion **4** shows that

$$\int_0^\infty e^{-xy} f_1(y)\,dy = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \frac{x}{x^2+y^2} F_1(y)\,dy,$$

where F_1 is the transform of f_1, and criterion **6** gives the corresponding equation for f_2. Adding the equations, and observing that $F_1 + F_2 = F = f$, we obtain (1.1).

(ii) Suppose that $f(x) = x^{\frac{1}{2}} J_{-\frac{1}{4}}(\frac{1}{2}x^2)$. Here $f(x)$ does not belong to any Lebesgue class, nor is it of bounded variation. But $f(x)$ is bounded,

and $$\int_0^\infty t^{\frac{1}{2}} J_{-\frac{1}{4}}(\tfrac{1}{2}t^2)\cos xt\,dt$$

is uniformly convergent in any finite interval of x, so that we can apply criterion **8**. We observe finally that the integrals (1.1) are, in fact, absolutely convergent.

5. We conclude the paper by determining a number of further rules for transformation of the classes R_μ. Our point of view here is purely formal; the analysis of §§ 3–4 will be sufficient to show how the validity of a rule may be discussed when its form is known.

It is natural to ask: *what is the most general kernel $k(xt)$ which transforms R_C into R_S?* Let us suppose that $f(x)$ belongs to R_C, and write

$$h(x) = \int_0^\infty k(xt)f(t)\,dt, \tag{5.1}$$

$$K(s) = \int_0^\infty k(u)u^{s-1}\,du. \tag{5.2}$$

Taking $\mu = -\frac{1}{2}$ in (2.1), and substituting for $f(t)$ from (2.1) into (5.1), we obtain

$$\begin{aligned} h(x) &= \int_0^\infty k(xt)\,dt\,\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\psi(s)t^{-s}\,ds \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\psi(s)\,ds\int_0^\infty k(xt)t^{-s}\,dt \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\psi(s)K(1-s)x^{s-1}\,ds \\ &= \frac{1}{2\pi i}\int_{1-c-i\infty}^{1-c+i\infty} 2^{\frac{1}{2}w}\Gamma(\tfrac{1}{2}w+\tfrac{1}{2})\psi_1(w)x^{-w}\,dw, \end{aligned}$$

where $$\psi_1(w) = 2^{\frac{1}{2}-w}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}w)}{\Gamma(\frac{1}{2}w+\frac{1}{2})}K(w)\psi(1-w).$$

The condition that $h(x)$ should belong to R_S is that $\psi_1(w) = \psi_1(1-w)$.

Inverting (5.2) by Mellin's formula, we obtain

$$k(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} K(s)x^{-s}\,ds = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\Gamma(s)\chi(s)x^{-s}\,ds, \tag{5.3}$$

where

$$\chi(s) = 2^{s-\frac{1}{2}}\frac{\Gamma(\frac{1}{2}s+\frac{1}{2})}{\Gamma(s)\Gamma(\frac{1}{2}-\frac{1}{2}s)}\frac{\psi_1(s)}{\psi(1-s)} = \frac{\sqrt{(2\pi)}}{\Gamma(\frac{1}{2}s)\Gamma(\frac{1}{2}-\frac{1}{2}s)}\frac{\psi_1(s)}{\psi(1-s)},$$

so that

$$\chi(s) = \chi(1-s). \tag{5.4}$$

Hence we derive the rule

IV. *The most general kernel $k(xt)$ which transforms R_C into R_S is that defined by* (5.3), *where $\chi(s)$ is a solution of* (5.4).

If we take $\chi(s) = 1$, we find that $k(x) = e^{-x}$. Taking

$$\chi(s) = \frac{\sqrt{\pi}}{\Gamma(\frac{1}{2}+\frac{1}{2}s)\,\Gamma(1-\frac{1}{2}s)}, \quad \frac{2\sqrt{\pi}}{\Gamma(\frac{1}{2}s)\,\Gamma(\frac{1}{2}-\frac{1}{2}s)}, \quad \frac{\sqrt{3}}{2\sqrt{\pi}}\,\Gamma(s-\tfrac{1}{3})\,\Gamma(\tfrac{2}{3}-s),$$

we obtain*

$$k(x) = J_0(x), \quad xJ_0(x), \quad x^{-\frac{1}{2}}e^{\frac{1}{2}x}K_{\frac{1}{3}}(\tfrac{1}{2}x).$$

Thus

V. *The kernels $J_0(xt)$, $xtJ_0(xt)$, $(xt)^{-\frac{1}{2}}e^{\frac{1}{2}xt}K_{\frac{1}{3}}(\frac{1}{2}xt)$ transform R_C into R_S.*

6. Further rules may be found by substituting $f(t)$, in one of the rules given already, its expression as its own transform; or, again, by iterating a rule, or combining two different rules. Applying the first process to rule I, we find

$$h(x) = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty e^{-xt}\,dt\int_0^\infty f(t)\cos ty\,dy = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \frac{x}{t^2+x^2}f(t)\,dt.$$

If we replace the e^{-xt} of rule I by the $J_0(xt)$ of rule V, we find

$$h(x) = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^x \frac{f(t)}{\sqrt{(x^2-t^2)}}\,dt.$$

If we iterate rule I, we obtain

$$g(x) = \int_0^\infty e^{-xy}h(y)\,dy = \int_0^\infty e^{-xy}\,dy\int_0^\infty e^{-yt}f(t)\,dt = \int_0^\infty \frac{f(t)}{t+x}\,dt.$$

Here $f(t)$ belongs to R_C, $h(y)$ to R_S, and $g(x)$ to R_C (or conversely). If we replace one of the exponentials by a J_0, we obtain

$$g(x) = \int_0^\infty \frac{f(t)}{\sqrt{(t^2+x^2)}}\,dt.$$

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* See Watson, **6**, 192–193.

Thus

VI. *The kernels*

$$\frac{x}{t^2+x^2}; \quad \frac{1}{\sqrt{(x^2-t^2)}} \quad (t<x), \quad 0 \quad (t>x)$$

transform R_C *into* R_S. *The kernels*

$$\frac{1}{t+x}, \quad \frac{1}{\sqrt{(t^2+x^2)}}$$

transform R_C *into itself.*

7. It is plain that rules of this character might be multiplied indefinitely, and our object is merely to indicate a few of the simplest. We may observe finally that it is interesting to apply them to a "singular" example of a function of R_C given in our paper **1**. This function was defined conventionally* by

$$f(x) = 1 \infty, \quad f(x) = -(2\pi)^{-\frac{1}{2}}, \tag{7.1}$$

according as x is or is not a positive integral multiple of $\sqrt{(2\pi)}$; an integral $\int \phi(x) f(x)\, dx$ is to be interpreted as a Stieltjes integral $\int \phi(x)\, df_1(x)$, where

$$f_1(x) = \left[\frac{x}{\sqrt{(2\pi)}}\right] - \frac{x}{\sqrt{(2\pi)}}.$$

Applying rules I and V, we find that

$$h(x) = \int_0^\infty e^{-xt}\, df_1(t) = \sum_1^\infty e^{-nx\sqrt{(2\pi)}} - \frac{1}{\sqrt{(2\pi)}} \int_0^\infty e^{-xt}\, dt$$

$$= \frac{e^{-x\sqrt{(2\pi)}}}{1-e^{-x\sqrt{(2\pi)}}} - \frac{1}{x\sqrt{(2\pi)}},$$

and†

$$h(x) = \int_0^\infty J_0(xt)\, df_1(t) = \sum_1^\infty J_0\{nx\sqrt{(2\pi)}\} - \frac{1}{\sqrt{(2\pi)}} \int_0^\infty e^{-xt}\, dt$$

$$= \sqrt{\left(\frac{2}{\pi}\right)} \sum_{n<x/\sqrt{(2\pi)}} \frac{1}{\sqrt{(x^2-2n^2\pi)}} - \tfrac{1}{2} - \frac{1}{x\sqrt{(2\pi)}}$$

belong to R_S: the first of these results is well known. We cannot discuss such applications of our rules on the lines of §§ 3–4 without exten-

* See Hardy and Titchmarsh, **1**, § 5.5.

† Nielsen, **3**, 336.

sions of the Parseval formula to Stieltjes integrals. These would carry us beyond our present object; but they are, in any case, unnecessary here, because, when $f(x)$ is the function (7.1), the Parseval formula reduces to "Poisson's summation formula".

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ELECTROMAGNETIC INDUCTION IN A UNIFORM PERMEABLE CONDUCTING SPHERE

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1. *Introduction.*

In a previous paper* formulae were obtained for the distribution and magnetic field of electric currents induced in a uniformly conducting sphere, of unit permeability, by a varying magnetic field of an assigned type. This inducing field possessed symmetry about a diameter of the sphere, and its potential was supposed analysed into zonal harmonics, the coefficients of which were functions of the time t, expressible in the form $\Sigma a \exp(-\lambda t)$. The main object was to build up a solution satisfying prescribed initial conditions.

A more general problem is now considered. The magnetic permeability of the sphere is not restricted to be unity, but has any positive value. The inducing field has any spatial distribution consistent with an external origin; its potential is supposed analysed into spherical harmonics, the coefficients of which are continuous functions of t. Actually we consider only one of these harmonics, since the solution for the general case is obtainable by superposing the solutions for the separate harmonics in the given potential.

Using the Heaviside operational calculus, a "symbolic" solution of the problem is first obtained (§ 3). From this the explicit solution is derived in two distinct forms, one of which is convenient for calculation when t is large or moderate (§ 4), and the other when t is small (§ 6). The special case, when the coefficient (e, say) of the spherical harmonic in the inducing field has the simple form $1-\exp(-\lambda t)$, is treated first, and the general nature of the induced field for this case is briefly examined (§ 5). The same method gives the solution when e is any simple analytic function of t

* *Proc. London Math. Soc.* (2), 31 (1930), 217.

for $t > 0$ and is zero for $t \leqslant 0$. The solution for the general case, when e is any continuous function of t for $t > 0$ and zero for $t \leqslant 0$, is then shown to be obtainable from the special case when e is the Heaviside unit function, by means of a definite integral formula due to J. R. Carson* (§ 7).

This problem arose in an investigation by Prof. S. Chapman and the writer† on the electric and magnetic state of the interior of the earth, as indicated by the analysis of terrestrial magnetic variations; numerical calculations of various formulae obtained in the present paper will be found therein.

2. *The differential equations of the problem.*

Let the conductivity and permeability of the material inside the sphere of radius a cms. have the constant values κ and μ respectively; for the medium immediately surrounding the sphere, let these values be zero and unity respectively. Let $\mathbf{A}$, $\mathbf{H}$, and $\mathbf{i}$ denote the magnetic vector potential, the magnetic force, and the current density respectively. The vector $\mathbf{A}$ satisfies the well-known equations

$$\nabla^2 \mathbf{A} = 4\pi\kappa\mu \, \partial\mathbf{A}/\partial t \quad (r < a), \tag{1}$$

$$\nabla^2 \mathbf{A} = 0 \quad (r > a), \tag{2}$$

with

$$\operatorname{div} \mathbf{A} = 0 \tag{3}$$

everywhere. Also $\mathbf{H}$ and $\mathbf{i}$ can be determined from $\mathbf{A}$ by means of the relations

$$\nabla^2 \mathbf{A} = 4\pi\mu\mathbf{i} \quad (r < a), \tag{4}$$

$$\operatorname{curl} \mathbf{A} = \mu\mathbf{H} \quad (r < a), \tag{5}$$

$$\operatorname{curl} \mathbf{A} = \mathbf{H} \quad (r > a). \tag{6}$$

The problem thus reduces to finding a vector function $\mathbf{A}$ which (i) satisfies the above differential equations, (ii) satisfies the boundary conditions at $r = a$, and (iii) is consistent with the given magnetic field of external origin and with the given initial conditions. Since we are concerned solely with the effects of the inducing field, we take as the initial condition that $\mathbf{A}$ is zero everywhere.

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* J. R. Carson, *Electric circuit theory*, 44.

† *Phil. Trans. Royal Soc.* (A), 229 (1930), 427.

3. *The symbolic solution.*

Outside the sphere, $\nabla^2 \mathbf{A}$ is always zero, so that $\mathbf{H}$ can be derived from a scalar potential, Ω say. At any time t, let Ω be analysed into spherical harmonic terms of the form

$$\Omega_n = a(\rho^n e + \rho^{-n-1} i) S_n \quad (r > a), \tag{7}$$

where

$$\rho = r/a, \tag{8}$$

while S_n is any surface spherical harmonic of degree n, and e and i are functions of t (only); e will be a known function of t since it relates to the given inducing field of external origin, while i, corresponding to the induced field, is as yet undetermined.

It is easily verified that the term in the vector potential $\mathbf{A}$, corresponding to Ω_n, is

$$\mathbf{A}_n = a(\mathbf{r} \wedge \operatorname{grad} S_n)\{e\rho^n/(n+1) - i\rho^{-n-1}/n\}, \tag{9}$$

where $\wedge$ denotes a vector product.

Inside the sphere, $\mathbf{A}$ satisfies equation (1); if we write

$$\zeta^2 = 4\pi\kappa\mu a^2 \partial/\partial t, \tag{10}$$

the auxiliary Heaviside equation* becomes

$$(\nabla^2 - \zeta^2/a^2)\mathbf{A} = 0 \quad (r < a), \tag{11}$$

which we proceed to solve as if ζ were simply a constant; the solution is to be regarded as an operator, the operand being $H(t)$, the Heaviside unit function.

Lamb† has shown that a solution of (11), when ζ is a constant, is

$$\mathbf{A}_n = a(\mathbf{r} \wedge \operatorname{grad} S'_n)\rho^n c R_n(-\zeta^2\rho^2) \quad (r < a), \tag{12}$$

where

$$R_n(z^2) = 1 - \frac{z^2}{2(2n+3)} + \frac{z^4}{2\,.\,4\,.\,(2n+3)(2n+5)} - \ldots = \frac{\Gamma(n+\frac{3}{2})}{(\frac{1}{2}z)^{n+\frac{1}{2}}} J_{n+\frac{1}{2}}(z), \tag{13}$$

and where c is any constant‡ and S'_n any surface spherical harmonic of degree n.

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* O. Heaviside, *Electromagnetic theory*, 2, 13.

† H. Lamb, *Phil. Trans. Royal Soc.* (A), 180 (1889), 467. A second type of solution of (11) exists but does not satisfy other conditions of the problem.

‡ *I.e.* a constant with respect to the space variables; when the symbolic solution is interpreted, c is a function of the time.

At the surface of the sphere, we have the conditions that $\mathbf{A}$ is continuous and div $\mathbf{A}$ is zero. When these conditions are applied to the expressions (9) and (12) for $\mathbf{A}$, the product $cR_n(-\zeta^2\rho^2)$ in (12) must be regarded as a function of t, and therefore corresponds to the time-functions e and i in (9). The boundary conditions are found to be satisfied for the portions $\mathbf{A}_n$ of the vector potential given by (9) and (12), if S_n' is identical with S_n and if the following identities* hold between the time functions e, i, and $cR_n(-\zeta^2\rho^2)$:

$$i = \frac{n}{n+1}\left\{1 - \frac{(2n+1)\mu R_n(-\zeta^2)}{(2n+1)R_{n+1}(-\zeta^2)+n(\mu-1)R_n(-\zeta^2)}\right\}e, \tag{14}$$

$$cR_n(-\zeta^2\rho^2) = \frac{(2n+1)\mu R_n(-\zeta^2\rho^2)}{(n+1)\{(2n+1)R_{n-1}(-\zeta^2)+n(\mu-1)R_n(-\zeta^2)\}}e. \tag{15}$$

These relations express the required time functions i and $cR_n(-\zeta^2\rho^2)$ as the results of certain operations on the known time function e. Hence, together with (9) and (12), they constitute the complete *symbolic* solution of our problem. It is now only necessary to interpret this symbolic solution for any given form of the function e.

4. *The explicit solution for the case when the inducing field varies as* $1-\exp(-\lambda t)$.

If the time function e of the inducing field has the simple form

$$\begin{aligned} e &= 1-\exp(-\lambda t) \quad (t > 0), \\ &= 0 \qquad\qquad\quad (t \leqslant 0), \end{aligned} \tag{16}$$

we may write it in operational form as

$$e = \frac{\lambda}{\sigma+\lambda}H(t) = \frac{\beta^2}{\zeta^2+\beta^2}H(t), \tag{17}$$

where

$$\sigma \equiv \partial/\partial t \tag{18}$$

and

$$\beta^2 = 4\pi\kappa\mu a^2\lambda. \tag{19}$$

* It is obvious from equation (10) that the formal relations between e, i, and $cR_n(-\zeta^2\rho^2)$ will be identical with those which are found when $\mathbf{A}$ is varying as $\exp(lt)$, where $l = \zeta^2/(4\pi\kappa\mu a^2)$; for the derivation of the relations on this assumption see Lamb, *loc. cit.*

From (14) and (15) it will be seen that the function i is readily derived from the value of cR_n when $\rho = 1$; hence it is only necessary to consider the expression (15) for cR_n. When e, given by (17), is substituted in (15), the latter expression can be interpreted as an explicit function of the time by applying the Heaviside "partial fraction rule*"; this states that

$$\frac{f(\sigma)}{F(\sigma)} H(t) = \frac{f(0)}{F(0)} + \Sigma \frac{f(a) \exp(at)}{aF'(a)}, \tag{20}$$

where the summation on the right extends over all the zeros (a) of $F(\sigma)$.

In the present case we note that the zeros of the denominator of (15), with the above value of e inserted, and *considered as a function of* ξ, are purely imaginary [*cf.* (13)]; hence, from (10) and (18), the corresponding a's in formula (20) will all be real and negative. This, of course, is in agreement with physical considerations.

It is easily seen that

$$\frac{f(0)}{F(0)} = \frac{(2n+1)\mu}{(n+1)(n\mu+n+1)}, \tag{21}$$

while the term corresponding to $\sigma = -\lambda$ (or $\xi^2 = -\beta^2$) in formula (20) is

$$\frac{f(-\lambda) \exp(-\lambda t)}{-\lambda F'(-\lambda)} = -\frac{(2n+1)\mu R_n(\beta^2\rho^2) \exp(-\lambda t)}{(n+1)\{(2n+1)R_{n-1}(\beta^2) + n(\mu-1)R_n(\beta^2)\}}. \tag{22}$$

The remaining terms in (20) correspond to the separate positive real roots of the equation

$$(2n+1)R_{n-1}(x^2) + n(\mu-1)R_n(x^2) = 0. \tag{23}$$

This equation may be written in terms of Bessel functions in the form

$$x^{-n-\frac{1}{2}}\{xJ'_{n+\frac{1}{2}}(x) + (n\mu+\tfrac{1}{2})J_{n+\frac{1}{2}}(x)\} = 0, \tag{24}$$

where the dash denotes differentiation with respect to x.

Let the roots of this equation be arranged in ascending order of magnitude, and let x_s denote the s-th root. Let the corresponding zero of $F(\sigma)$ in (20) be denoted by $-l_s$, so that, from (10) and (18),

$$l_s = x_s^2/(4\pi\kappa\mu a^2). \tag{25}$$

* Heaviside, *loc. cit.*, 2, 126.

Then, from (13) and (15), it is easily shown that

$$-l_s F'(-l_s) = -x_s^2(\beta^2 - x_s^2)\left\{\tfrac{1}{2}R_n(x_s^2) + \frac{n(\mu-1)}{2(2n+3)}R_{n+1}(x_s^2)\right\}$$

$$= -\tfrac{1}{2}(n+1)(\beta^2 - x_s^2)\{x_s^2 + n(\mu-1)(u\mu+n+1)\}R_n(x_s^2). \tag{26}$$

The term in (20) corresponding to $\sigma = -l_s$ is therefore

$$\frac{f(-l_s)}{-l_s F'(-l_s)} = \frac{-2(2n+1)\mu\beta^2 R_n(x_s^2\rho^2)\exp(-l_s t)}{(n+1)(\beta^2 - x_s^2)\{x_s^2 + n(\mu-1)(n\mu+n+1)R_n(x_s^2)\}'} \tag{27}$$

Hence, when $e = 1 - \exp(-\lambda t)$, we have, on collecting our results,

$$cR_n(-\xi^2\rho^2) = \frac{(2n+1)\mu}{n+1}\left\{\frac{1}{n\mu+n+1} - \frac{R_n(\beta^2\rho^2)\exp(-\lambda t)}{(2n+1)R_{n-1}(\beta^2) + n(\mu-1)R_n(\beta^2)}\right.$$

$$\left. - \sum_{s=1}^{\infty}\frac{2\beta^2 R_n(x_s^2\rho^2)\exp(-l_s t)}{(\beta^2 - x_s^2)\{x_s^2 + n(\mu-1)(n\mu+n+1)R_n(x_s^2)\}}\right\}. \tag{28}$$

Let the function of t on the right of (28) be denoted by $\psi(t, \rho)$; then, from (14), we have

$$i = \{n/(n+1)\}\{1 - \exp(-\lambda t) - (n+1)\psi(t, 1)\}. \tag{29}$$

These equations enable us to express the vector and scalar potentials of the magnetic field, given in (12) and (7) respectively, as explicit functions of the time; the solution is therefore analytically complete for this particular case.

It is desirable to verify that the above solution is correct, since a general proof of the validity of the partial fraction rule, when applied to continuous systems, does not appear to have been constructed. We note first that, when (28) and (29) are substituted in (12) and (9) respectively, every *separate* exponential term of the resulting expressions satisfies the differential equations and the boundary conditions of the problem; this necessarily follows from the form of the partial fraction expansion, and the method of derivation of the symbolic solution [*cf.* especially the foot-note on the identities (14) and (15)]; it can also, of course, be verified by direct substitution. The only remaining condition which the solution must satisfy is that **A** is zero everywhere when t is zero. This condition will be satisfied if

$$\psi(0, \rho) = 0 \quad \text{when} \quad 0 < \rho \leqslant 1^*, \tag{30}$$

* It will be noted that $\psi(0, \rho)$ is not zero when $\rho = 0$, since the infinite series for $\psi(0, \rho)$ becomes oscillatory as $\rho \to 0$. The corresponding value of **A** at $\rho = 0$ *is*, however, zero, in virtue of the factor ρ^n in the expression (12) for **A**.

which implies that

$$(31)\quad \frac{R_n(\beta^2\rho^2)}{(2n+1)\,R_{n-1}(\beta^2)+n(\mu-1)\,R_n(\beta^2)} = \frac{1}{n\mu+n+1} - \sum_{s=1}^{\infty} \frac{2\beta^2 R_n(x_s^2\rho^2)}{(\beta^2-x_s^2)\{x_s^2+n(\mu-1)(n\mu+n+1)\,R_n(x_s^2)\}}$$

over the same range of values of ρ.

This result is most readily proved to be correct by treating the expression on the left as a *function of* β^2 and expanding it by Mittag-Leffler's theorem* in a series of fractions of the form $\beta^2/(\beta^2-x_s^2)$, where the constants x_s^2 are the simple poles of the function expanded.

Alternately, (31) may be expressed in terms of Bessel functions; on multiplying by $\rho^{n+\frac{1}{2}}$, it becomes

$$(32)\quad \frac{J_{n+\frac{1}{2}}(\beta\rho)}{\beta J'_{n+\frac{1}{2}}(\beta)+(n\mu+\frac{1}{2})\,J_{n+\frac{1}{2}}(\beta)} = \frac{\rho^{n+\frac{1}{2}}}{n\mu+n+1} - \sum_{s=1}^{\infty} \frac{2\beta^2 J_{n+\frac{1}{2}}(x_s\rho)}{(\beta^2-x_s^2)\{x_s^2+n(\mu-1)(n\mu+n+1)\,J_{n+\frac{1}{2}}(x_s)\}}.$$

When in this form, the expansion on the right is seen to be a Fourier-Bessel series of the type investigated by Dini. Hence the result can now be proved by expanding the *function of* ρ on the left by means of the Dini expansion theorem† for an arbitrary function of ρ in a series of Bessel functions of the type $J_{n+\frac{1}{2}}(x_s\rho)$. This, in fact, would be the form in which the series would present itself, and also the method by which it would be obtained, if the usual Fourier procedure of building up the solution of the general problem from the elementary exponential solutions were adopted. It may be remarked, however, that, in the actual working, the Dini expansion theorem proves much less convenient to apply to this particular problem than the Heaviside partial-fraction rule.

5. *The general character of the induced field.*

Let the time factor (i) in the potential of the induced magnetic field for the case when $e = 1-\exp(-\lambda t)$ be denoted by $\phi(t)$; substituting from

* *Cf.* Whittaker and Watson, *Modern analysis*, 4th ed., 134.

† *Cf.* Watson, *Theory of Bessel functions* (1922), 600.

(28) in (29) and using (31), we get

$$\phi(t) = \frac{n}{n+1}\left[2(2n+1)\,\mu\beta^2 \sum_{s=1}^{\infty} \frac{\exp(-l_s t)-\exp(-\lambda t)}{(\beta^2-x_s^2)\{x_s^2+n(\mu-1)(n\mu+n+1)\}} - \frac{(\mu-1)(n+1)}{n\mu+n+1}\{1-\exp(-\lambda t)\}\right]. \tag{33}$$

In order to see the general nature of this function, it is useful to examine first the limiting case when λ becomes infinite, *i.e.* the case when the time factor e in the potential of the inducing field reduces to the Heaviside unit function $H(t)$; the limiting value of $\phi(t)$ is given by

$$\Phi(t) = \frac{n}{n+1}\left\{2(2n+1)\,\mu \sum_{s=1}^{\infty} \frac{\exp(-l_s t)}{x_s^2+n(\mu-1)(n\mu+n+1)} - \frac{(n+1)(\mu-1)}{n\mu+n+1}\right\}. \tag{34}$$

Now from (31) on putting $\rho = 1$ and making $\beta \to \infty$, we have*

$$\sum_{s=1}^{\infty} \frac{1}{x_s^2+n(\mu-1)(n\mu+n+1)} = \frac{1}{2(n\mu+n+1)}. \tag{35}$$

Putting $t = 0$ in (34) and using (35), we find that the initial value of $\Phi(t)$ is simply

$$\Phi(0) = \frac{n}{n+1}, \tag{36}$$

a result independent both of μ and κ.

It is now obvious from (34) that $\Phi(t)$ decreases continuously from $n/(n+1)$ to $-n(\mu-1)/(n\mu+n+1)$ as t increases from zero to infinity. Returning to the more general case, the following properties of $\phi(t)$ are readily verified:

(i) $$\phi'(0) = n\lambda/(n+\lambda),$$

which is essentially positive;

(ii) $$\phi(t) \to -n(\mu-1)/(n\mu+n+1) \quad \text{as} \quad t \to \infty;$$

(iii) $\phi(t)$ has only one maximum, which is attained at the value of t which satisfies the equation $\phi(t) = \Phi(t)$.

Hence the time factor $\phi(t)$ in the potential of the induced field is, for all values of λ, initially of the same sign as the factor $1-\exp(-\lambda t)$ in the potential of the inducing field. It rises to a maximum, which is always

* A special case (viz. when $\mu = 0$ and $n = 1$) of this relation between the roots of equation (24) was noted by Rayleigh, *Theory of sound*, 2, 270.

less than $n/(n+1)$, and then decreases, becomes negative, and tends to the ultimate steady value $-n(\mu-1)/(n\mu+n+1)$.

The last term in the expression (33) for $\phi(t)$ represents the contribution to the induced field from the induced magnetism of the permeable sphere, for it is zero when $\mu = 1$ and is independent of κ. This term is always negative and bears a constant ratio to the time factor of the inducing field.

The remaining terms in (33) represent the contribution to the induced field from the electric currents induced in the conducting sphere. All these terms are positive, since $l_s/x_s = \lambda/\beta$; further, each tends to zero as t tends to infinity. Thus the magnetic field of the induced currents is represented by an "internal" harmonic, *having the same sign* as the external harmonic, which represents the inducing field, and decays ultimately to zero.

Since $\phi'(0)$ is always positive, the field of the induced currents is at first always more important than that of the induced magnetism, but, as these currents decay, the field of the induced magnetism finally predominates.

6. *Alternative form of the solution.*

The normal function solution given by (28), (29) is rapidly convergent when t is large, but it is not very convenient for numerical calculations when t is small, owing to the slow convergence of the series in (28) and the consequent necessity* of calculating a considerable number of the roots (x_s) of the equation (24).

An alternative form of the solution, rapidly convergent for small values of t, may be obtained by expanding† the operator in (15) in a series of exponential functions of ξ, and interpreting term by term.

From (13) we have

$$R_n(-\xi^2\rho^2) = \frac{\Gamma(n+\frac{3}{2})}{(\frac{1}{2}\xi\rho)^{n+\frac{1}{2}}}\, I_{n+\frac{1}{2}}(\xi\rho), \tag{37}$$

where I denotes the Bessel function of imaginary argument, so that‡

$$I_{n+\frac{1}{2}}(z) = \sqrt{\left(\frac{1}{2\pi z}\right)}\left\{e^{z}u_n(z)+(-)^{n+1}e^{-z}u_n(-z)\right\}, \tag{38}$$

where

$$u_n(z) = \sum_{r=0}^{n} \frac{(-1)^r(n+r)!}{(2z)^r\, r!\,(n-r)!}. \tag{39}$$

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* Except when $n = \mu = 1$, in which case $x_s = s\pi$.

† Heaviside, *loc. cit.*, 2, 69.

‡ See Watson, *loc. cit.*, 80.

Hence (15) can be put in the form

$$cR_n(-\zeta^2\rho^2) = [f\exp\{-\xi(1-\rho)\} + g\exp\{-\xi(1+\rho)\}]\{1-h\exp(-2\xi)\}^{-1}H(t), \tag{40}$$

where

$$f = \frac{(2n+1)\mu\rho^{-n-1}\beta^2 u_n(\xi\rho)}{(n+1)(\xi^2+\beta^2)\{\xi u_{n-1}(\xi)+n(\mu-1)u_n(\xi)\}}, \tag{41}$$

$$g = \frac{(-1)^{n+1}(2n+1)\mu\rho^{-n-1}\beta^2 u_n(-\xi\rho)}{(n+1)(\xi^2+\beta^2)\{\xi u_{n-1}(\xi)+n(\mu-1)u_n(\xi)\}}, \tag{42}$$

$$h = \frac{(-1)^{n+1}\{\xi u_{n-1}(-\xi)-n(\mu-1)u_n(-\xi)\}}{\xi u_{n-1}(\xi)+n(\mu-1)u_n(\xi)}. \tag{43}$$

If (40) is now expanded by the binomial theorem, it becomes

$$cR_n(-\zeta^2\rho^2) = [f\exp\{-\xi(1-\rho)\}+g\exp\{-\xi(1+\rho)\} + fh\exp\{-\xi(3-\rho)\}+\ldots]H(t). \tag{44}$$

The first term of (44) may be interpreted* as the direct wave in the vector potential **A**, transmitted from the surface of the sphere; the remaining terms represent the successive reflected waves, the reflections occurring at the surface and the centre of the sphere. The exponential functions are the "attenuation-factors" of these waves and the coefficients of these functions are the "reflection coefficients".

For sufficiently small values of t, the magnitudes of the reflected waves, owing to attenuation, will be small compared with that of the direct wave (except for the first reflected wave, near $\rho = 0$), and may be neglected. This would be roughly equivalent to neglecting $\{\exp(-2\xi)\}H(t)$ compared with unity, which is permissible for results required to four significant figures† if

$$t < 4\pi\mu\kappa a^2/10. \tag{45}$$

When this condition is satisfied, we need consider only the first two terms in (44). These may be evaluated by expanding f and g in descending powers of ξ and interpreting the resulting expansions term by term; the terms will be of the form $A\xi^{-m}\exp\{\xi(1-\rho)\}$, which can be interpreted by known methods‡.

To illustrate this with a simple case, we obtain the expression for the potential of the induced magnetic field outside the sphere, when $\mu = n = 1$.

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* Heaviside, *loc. cit.*, 25, 69.

† *Cf.* Bromwich, *Proc. London Math. Soc.* (2), 31 (1930), 209.

‡ See, for example, H. Jeffreys, *Operational methods in mathematical physics* (1927), ch. 5.

Some numerical calculations of the *normal-function form* of the solution for this case have already been made*, so that it will be possible to test the new form of the solution numerically.

When ρ, μ, and n are all unity, and t is small enough to make $\{\exp(-2\zeta)\}H(t)$ neglible, the expression (44) for cR_n reduces to $\psi_1(t)$, where

$$\psi_1(t) = \frac{3\beta^2(\zeta-1)}{2\zeta^2(\beta^2+\zeta^2)}\, H(t). \tag{46}$$

When this is expanded in descending powers of ζ by the binomial theorem, it is readily interpreted in the form

$$\psi_1(t) = \frac{3\{1-\lambda t-\exp(-\lambda t)\}}{2\beta^2} + \frac{2}{\beta}\sqrt{\left(\frac{\lambda^3 t^3}{\pi}\right)}\left\{1-\frac{2}{5}\lambda t+\frac{2\cdot 2}{5\cdot 7}(\lambda t)^2-\frac{2\cdot 2\cdot 2}{5\cdot 7\cdot 9}(\lambda t)^3\ldots\right\}. \tag{47}$$

From (29), the time factor, $\phi_1(t)$, say, in the potential of the induced magnetic field is then given by

$$\phi_1(t) = \tfrac{1}{2}\{1-\exp(-\lambda t)-2\psi_1(t)\}. \tag{48}$$

If t satisfies the inequality (45), this expression for $\phi_1(t)$ should give results accurate to four significant figures. This has been tested by comparison with the normal-function solution; the expression for $\phi_1(t)$ in this form of the solution is obtained from (33), by putting μ and n each equal to unity and x_s equal to $s\pi$. We find that

$$\phi_1(t) = \frac{3\beta^2}{\pi^2}\sum_{s=1}^{\infty}\frac{\exp(-s^2\pi^2\lambda t/\beta^2)-\exp(-\lambda t)}{s^2(\beta^2-s^2\pi^2)}. \tag{49}$$

In the actual problem for which numerical values were required, the value of $4\pi\kappa a^2$ was approximately $1{\cdot}7\times10^6$; hence (48) should give results, accurate to four significant figures, if $t < 1{\cdot}7\times10^5$ secs., or about 2 days (actually just the period of importance in the problem studied). To test this, the values of ϕ_1 were calculated from (48) for the case when $\beta = \pi$ and $t = 1$, 12, 24, and 48 hours respectively; these values, to four significant figures, were found to agree exactly with those previously obtained from (49).

The wave-expansion solution (48) takes a specially simple form when the change in the inducing field is instantaneous, *i.e.* when e becomes

* Chapman and Price, *loc. cit.*

$H(t)$*, for, denoting the corresponding value of $\phi_1(t)$ by $\Phi_1(t)$, we have, from (46) and (48),

$$\Phi_1(t) = \tfrac{1}{2} - \frac{3(\zeta-1)}{2\zeta^2} H(t) \tag{50}$$

$$= \tfrac{1}{2} - 3\sqrt{\left(\frac{t}{4\pi^2\kappa a^2}\right)} + \frac{3t}{8\pi\kappa a^2}.$$

From (49) the normal function form of $\Phi_1(t)$ is

$$\Phi_1(t) = \frac{3}{\pi^2} \sum_{s=1}^{\infty} \frac{\exp(-s^2\pi t/4\kappa a^2)}{s^2}, \tag{51}$$

so that, if $t < 4\pi\kappa a^2$, the sum of this infinite series will be given correct to four figures by the simple expression (50). In fact, on writing

$$\alpha = \pi t/4\kappa a^2,$$

we have the interesting result

$$\sum_{s=1}^{\infty} \frac{\exp(-s^2\alpha)}{s^2} = \left\{\frac{\pi^2}{6} - \sqrt{(\pi\alpha)} + \frac{\alpha}{2}\right\}(1+\epsilon), \tag{52}$$

where $\epsilon < 10^{-4}$ if $0 \leqslant \alpha \leqslant 1$.

7. *The general case with the inducing field any continuous function of the time.*

We consider finally the determination of i, given by (14), as a function of t, when e is any given continuous function of t for $t > 0$, and is zero for $t \leqslant 0$. This function e may be written in the operational form†

$$e(t) = E(\sigma) H(t), \tag{53}$$

where

$$E(\sigma) = \int_0^{\infty} e(u) \exp(-\sigma u)\, \sigma\, du.$$

Let the coefficient of e in (14) be denoted by $F(\sigma)$, so that

$$i = F(\sigma)\, e = F(\sigma)\, E(\sigma)\, H(t). \tag{54}$$

When e is simply $H(t)$, *i.e.* when $E(\sigma) \equiv 1$, the function i reduces to $\Phi(t)$, which can be calculated either from (34) or from formulae similar to

* This case is important, because the solution when e is *any* continuous function of t can be derived from it (§ 7).

† Heaviside, *loc. cit.*, 3, 236.

(50). Hence we can write

$$\Phi(t) = F(\sigma)\, H(t), \tag{55}$$

where $\Phi(t)$ is a known function of t.

It has been shown by J. R. Carson (*loc. cit.*) that, if an operator can be factorized as in (54), and the separate operators interpreted as in (53) and (55), the operand in each case being $H(t)$, then the result of the combined operation is given by

$$i = \frac{d}{dt}\int_0^t e(t-u)\,\Phi(u)\,du, \tag{56}$$

$$= e(0)\,\Phi(t) + \int_0^t e'(t-u)\,\Phi(u)\,du. \tag{57}$$

By integrating by parts, we can transform this into

$$i = e(t)\,\Phi(0) + \int_0^t e(t-u)\,\Phi'(u)\,du, \tag{58}$$

$$= \frac{n}{n+1}\,e(t) + \int_0^t e(t-u)\,\Phi'(u)\,du, \tag{59}$$

from (36). Since $\Phi'(u)$ can be obtained immediately from (36) or from the corresponding result in the wave-expansion solution, such as (50) for $\Phi_1(t)$, the above expression for i can be readily evaluated by numerical or graphical integration. A knowledge of i is sufficient to determine the induced magnetic field outside the sphere; to obtain the magnetic field inside the sphere, and also the current distribution, it is only necessary to apply the same methods to the function $cR_n(-\xi^2\rho^2)$ given by equation (15).

A NOTE ON CERTAIN APPROXIMATE SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER (2)

By S. GOLDSTEIN.

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1. The differential equation

$$\frac{d^2y}{dx^2}-(h^2\chi_0+h\chi_1)y=0, \tag{1}$$

where χ_0 and χ_1 are functions of x, and h is a parameter independent of x, is approximately satisfied, for sufficiently large values of h, by*

$$y=\chi_0^{-\frac{1}{4}}\exp(\pm M), \tag{2}$$

where

$$M=\int^x(h\chi_0^{\frac{1}{2}}+\tfrac{1}{2}\chi_1/\chi_0^{\frac{1}{2}})dx. \tag{3}$$

The approximations break down at a zero of χ_0, and, if we take any approximate solution on one side of a zero, the problem arises of finding its continuation on the other side of the zero. Jeffreys (*loc. cit.*) gave a method of doing this when χ_1 is absent, and χ_0 has a simple zero. The results for any multiple zero of χ_0, with χ_1 still absent, were given in *Proc. London Math. Soc.* (2), 28 (1928), 81†. If χ_0 has a simple zero, the presence of χ_1 will not affect the result‡, but if χ_0 has a double zero, then the result is modified by the presence of χ_1.

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* Jeffreys, *Proc. London Math. Soc.* (2), 23 (1925), 428 *et. seq.*

† There is an unfortunate misprint on p. 84 of this paper. In equations (5) and (6), g_1 and G_1 should be interchanged. On p. 86, in the second of equations (7), read "$\sin(k\cos x)$" instead of "$\cos(k\cos x)$".

‡ If $h^2\chi_0+h\chi_1=h^2\chi$, and χ_0 has a simple zero at the origin, χ has a simple zero of order h^{-1}.

We may, without loss of generality, suppose the double zero to be at $x = 0$. When x is negative, we put $-x = x_1$, and take

$$\left.\begin{aligned} M &= \int_0^x h\chi_0^{\frac{1}{2}}\,dx + \int_\epsilon^x \frac{\chi_1}{2\chi_0^{\frac{1}{2}}}\,dx \quad (x > 0) \\ &= \int_x^0 h\chi_0^{\frac{1}{2}}\,dx + \int_x^{-\epsilon} \frac{\chi_1}{2\chi_0^{\frac{1}{2}}}\,dx \quad (x < 0) \end{aligned}\right\}, \tag{4}$$

in which $\arg \chi_0^{\frac{1}{2}}$ is to be zero or $\frac{1}{2}\pi$ according as χ_0 is positive or negative, and ϵ is small, and, say, $o(h^{-\frac{2}{5}})$. We take $\arg \chi_0^{-\frac{1}{4}}$ as 0 or $-\frac{1}{4}\pi$, and put

$$\left.\begin{aligned} \chi_0^{-\frac{1}{4}} \exp M &= G \quad (x > 0) & &= G_1 \quad (x < 0) \\ \chi_0^{-\frac{1}{4}} \exp(-M) &= g & &= g_1 \end{aligned}\right\}. \tag{5}$$

2. Let us first suppose that χ_0 is negative on both sides of the origin. For small values of x,

$$\left.\begin{aligned} \chi_0 &= -\lambda^2 x^2, \\ \chi_1 &= A, \end{aligned}\right\} \tag{6}$$

where

$$-\lambda^2 = \tfrac{1}{2}\left(\frac{\partial^2 \chi_0}{\partial x^2}\right)_{x=0}, \tag{7}$$

$$A = (\chi_1)_{x=0}. \tag{8}$$

We suppose that A is not zero.

Substituting from (6) in (4) and (5), we find that

$$G = \lambda^{-\frac{1}{2}} \exp\left(\frac{Ai}{2\lambda} \log \epsilon - \tfrac{1}{4}\pi i\right) x^{-\frac{1}{2}} \exp\left(\tfrac{1}{2} i\lambda h x^2 - \frac{Ai}{2\lambda} \log x\right) \tag{9}$$

$$g = \lambda^{-\frac{1}{2}} \exp\left(-\frac{Ai}{2\lambda} \log \epsilon - \tfrac{1}{4}\pi i\right) x^{-\frac{1}{2}} \exp\left(-\tfrac{1}{2} i\lambda h x^2 + \frac{Ai}{2\lambda} \log x\right), \tag{10}$$

and G_1 and g_1 are the same functions of x_1 as G and g of x. In the neighbourhood of the origin the differential equation (1) becomes

$$y'' + (h^2\lambda^2 x^2 - Ah)y = 0. \tag{11}$$

We have here neglected a multiple of h^2x^3y and retained a multiple of hy, so that hx^3 must be small.

If we put

$$p = e^{\frac{1}{4}\pi i}(2\lambda h)^{\frac{1}{4}}, \tag{12}$$

$$z = px, \tag{13}$$

$$m + \tfrac{1}{2} = -Ah/p^2 = \tfrac{1}{2}Ai/\lambda, \tag{14}$$

equation (11) becomes

$$\frac{d^2y}{dz^2}+(m+\tfrac{1}{2}-\tfrac{1}{4}z^2)y=0. \tag{15}$$

This is Weber's equation (Whittaker and Watson's *Modern analysis*, p. 347), which is satisfied by the parabolic cylinder functions $D_m(z)$ and $D_{-m-1}(e^{-\frac{1}{2}\pi i}z)$. The argument of z is $\frac{1}{4}\pi$, and that of $e^{-\frac{1}{2}\pi i}z$ is $-\frac{1}{4}\pi$. Hence

$$D_m(z)\sim e^{-\frac{1}{4}z^2}z^m, \tag{16}$$

$$D_{-m-1}(e^{-\frac{1}{2}\pi i}z)\sim e^{\frac{1}{4}z^2}(e^{-\frac{1}{2}\pi i}z)^{-m-1}. \tag{17}$$

We here suppose that $|z|$ is large, so that hx^2 is large. We have previously assumed that hx^3 is small. Both these conditions are satisfied by taking $x=O(h^{-\frac{2}{3}})$, for example. We may also note that the condition for the expressions (2) to be approximate solutions of the differential equation when x is small is also that hx^2 must be large. [*Cf.* Jeffrey, *loc. cit.*, p. 429, equations (2) to (5). The process there described, when applied to equation (11) above, gives the usual asymptotic expansions of the parabolic cylinder functions.]

Also, if $z_1=-z$,

$$D_m(z)=e^{-m\pi i}D_m(z_1)+\frac{(2\pi)^{\frac{1}{2}}}{\Gamma(-m)}e^{-\frac{1}{2}(m+1)\pi i}D_{-m-1}(e^{-\frac{1}{2}\pi i}z_1), \tag{18}$$

$$D_{-m-1}(e^{-\frac{1}{2}\pi i}z)=\frac{(2\pi)^{\frac{1}{2}}}{\Gamma(m+1)}e^{-\frac{1}{2}m\pi i}D_m(z_1)-e^{-m\pi i}D_{-m-1}(e^{-\frac{1}{2}\pi i}z_1). \tag{19}$$

If now we take the values of G and g as given by (9) and (10), and substitute from (12), (13), and (14) in (16) and (17), we find

$$\begin{aligned}
G &\sim \lambda^{-\frac{1}{2}}\exp\left[\left(\frac{Ai}{2\lambda}+\tfrac{1}{2}\right)(\tfrac{1}{2}\log 2\lambda h-\tfrac{1}{4}\pi i)-\tfrac{1}{4}\pi i+\frac{Ai}{2\lambda}\log\epsilon\right]D_{-m-1}(e^{-\frac{1}{2}\pi i}z)\\
&=\lambda^{-\frac{1}{2}}\exp\left[\left(\frac{Ai}{2\lambda}+\tfrac{1}{2}\right)(\tfrac{1}{2}\log 2\lambda h-\tfrac{1}{4}\pi i)-\tfrac{1}{4}\pi i+\frac{Ai}{2\lambda}\log\epsilon\right]\\
&\qquad\times\left[\frac{(2\pi)^{\frac{1}{2}}}{\Gamma(m+1)}e^{-\frac{1}{2}m\pi i}D_m(z_1)-e^{-m\pi i}D_{-m-1}(e^{\frac{1}{2}\pi i}z_1)\right]\\
&\sim\frac{(2\pi)^{\frac{1}{2}}}{\Gamma(m+1)}e^{-\frac{1}{2}m\pi i}\exp\left[\frac{Ai}{2\lambda}\log 2\lambda h-\tfrac{1}{4}\pi i+\frac{Ai}{\lambda}\log\epsilon\right]g_1-e^{-m\pi i}G_1\\
&=\frac{(2\pi)^{\frac{1}{2}}}{\Gamma[\frac{1}{2}+\frac{1}{2}Ai/\lambda]}e^{A\pi/4\lambda}\exp\left[\frac{Ai}{2\lambda}\log(2\lambda h\epsilon^2)\right]g_1-ie^{\frac{1}{2}A\pi/\lambda}G_1.
\end{aligned} \tag{20}$$

Similarly

$$g \sim \lambda^{-\frac{1}{2}} \exp\left[\left(-\frac{Ai}{2\lambda}+\tfrac{1}{2}\right)(\tfrac{1}{2}\log 2\lambda h+\tfrac{1}{4}\pi i)-\tfrac{1}{4}\pi i-\frac{Ai}{2\lambda}\log\epsilon\right]D_m(z)$$

$$= \lambda^{-\frac{1}{2}} \exp\left[\left(-\frac{Ai}{2\lambda}+\tfrac{1}{2}\right)(\tfrac{1}{2}\log 2\lambda h+\tfrac{1}{4}\pi i)-\tfrac{1}{4}\pi i-\frac{Ai}{2\lambda}\log\epsilon\right]$$

$$\times\left[e^{-m\pi i}D_m(z_1)+\frac{(2\pi)^{\frac{1}{2}}}{\Gamma(-m)}e^{-\frac{1}{2}(m+1)\pi i}D_{-m-1}(e^{-\frac{1}{2}\pi i}z_1)\right]$$

$$\sim e^{-m\pi i}g_1+\frac{(2\pi)^{\frac{1}{2}}}{\Gamma(-m)}e^{-\frac{1}{2}(m+1)\pi i}\exp\left[-\frac{Ai}{2\lambda}\log 2\lambda h+\tfrac{1}{4}\pi i-\frac{Ai}{\lambda}\log\epsilon\right]G_1$$

$$= ie^{\frac{1}{2}A\pi/\lambda}g_1+\frac{(2\pi)^{\frac{1}{2}}}{\Gamma\{\frac{1}{2}-\frac{1}{2}Ai/\lambda\}}e^{\frac{1}{4}A\pi/\lambda}\exp\left[-\frac{Ai}{2\lambda}\log(2\lambda h\epsilon^2)\right]G_1. \tag{21}$$

Equations (20) and (21) give the continuations of G and g.

If we put

$$\chi_0 = e^{i\pi}\psi_0, \tag{22}$$

$$M = iL, \tag{23}$$

$$H = \psi_0^{-\frac{1}{4}}\cos L, \tag{24}$$

$$h = \psi_0^{-\frac{1}{4}}\sin L, \tag{25}$$

then

$$H = \tfrac{1}{2}e^{\frac{1}{4}\pi i}(G+g) \tag{26}$$

$$h = \frac{1}{2i}e^{\frac{1}{4}\pi i}(G-g), \tag{27}$$

and we easily find that

$H \longleftrightarrow e^{\frac{1}{2}A\pi/\lambda}h_1+(2\pi)^{\frac{1}{2}}e^{\frac{1}{4}A\pi/\lambda}$ times the real part of

$$[H_1-ih_1]\left\{\Gamma\left(\tfrac{1}{2}+\frac{Ai}{2\lambda}\right)\right\}^{-1}\exp\left(\frac{Ai}{2\lambda}\log(2\lambda h\epsilon^2)\right), \tag{28}$$

and

$h \longleftrightarrow -e^{\frac{1}{2}A\pi/\lambda}H_1+(2\pi)^{\frac{1}{2}}e^{\frac{1}{4}A\pi/\lambda}$ times the imaginary part of

$$[H_1-ih_1]\left\{\Gamma\left(\tfrac{1}{2}+\frac{Ai}{2\lambda}\right)\right\}^{-1}\exp\left(\frac{Ai}{2\lambda}\log(2\lambda h\epsilon^2)\right), \tag{29}$$

where H_1 and h_1 are formed from G_1 and g_1 in the same way as H and h from G and g.

If χ_1 is absent, so that A is zero, these reduce to

$$H \longleftrightarrow h_1+2^{\frac{1}{2}}H_1 \tag{30}$$

$$h \longleftrightarrow -H_1-2^{\frac{1}{2}}h_1, \tag{31}$$

in agreement with *Proc. London Math. Soc.* (2), 28 (1928), 83, equations (12) and (13).

3. Let us now suppose that χ_0 is positive on both sides of the origin. In place of (6) and (7) we have

$$\chi_0 = \lambda^2 x^2, \tag{32}$$

$$\lambda^2 = \tfrac{1}{2}\left(\frac{\partial^2 \chi_0}{\partial x^2}\right)_{x=0}. \tag{33}$$

G and g are now given by

$$G = \lambda^{-\frac{1}{2}} \epsilon^{-\frac{1}{2}A/\lambda} x^{-\frac{1}{2}} \exp\left\{\tfrac{1}{2}\lambda x^2 + \frac{A}{2\lambda}\log x\right\}, \tag{34}$$

$$g = \lambda^{-\frac{1}{2}} \epsilon^{\frac{1}{2}A/\lambda} x^{-\frac{1}{2}} \exp\left\{-\tfrac{1}{2}\lambda x^2 - \frac{A}{2\lambda}\log x\right\}. \tag{35}$$

In the neighbourhood of the origin the differential equation reduces to

$$y'' - (h^2\lambda^2 x^2 + Ah)y = 0, \tag{36}$$

and with

$$p = (2\lambda h)^{\frac{1}{2}}, \tag{37}$$

$$z = px, \tag{38}$$

and

$$m + \tfrac{1}{2} = -Ah/p^2 = -\tfrac{1}{2}A/\lambda, \tag{39}$$

solutions are $D_m(z)$ and $D_{-m-1}(e^{-\frac{1}{2}\pi i}z)$, with asymptotic approximations given by (16) and (17). But m and z are now real; $D_m(z)$ is exponentially small compared with $D_{-m-1}(e^{-\frac{1}{2}\pi i}z)$, and if we add any multiple of $D_m(z)$ to $D_{-m-1}(e^{-\frac{1}{2}\pi i}z)$ its asymptotic approximation remains unaltered.

With the values of G and g in (34) and (35), we find, substituting from (37), (38), and (39) in (16) and (17),

$$\begin{aligned} g &\sim \lambda^{-\frac{1}{2}} p^{-m} \epsilon^{\frac{1}{2}A/\lambda} D_m(z) \\ &= \lambda^{-\frac{1}{2}} p^{-m} \epsilon^{\frac{1}{2}A/\lambda}\left[e^{-m\pi i} D_m(z_1) + \frac{(2\pi)^{\frac{1}{2}}}{\Gamma(-m)}\, e^{-\frac{1}{2}(m+1)\pi i} D_{-m-1}(e^{-\frac{1}{2}\pi i} z_1)\right] \\ &\sim e^{-m\pi i} g_1 + p^{-(2m+1)} \epsilon^{A/\lambda} \frac{(2\pi)^{\frac{1}{2}}}{\Gamma(-m)}\, G_1, \end{aligned} \tag{40}$$

but we may add any multiple of $D_m(z_1)$ to $D_{-m-1}(e^{-\frac{1}{2}\pi i} z_1)$ in the second line or any multiple of g_1 to G_1 in the third line. We may, however,

neglect g_1 compared with G_1, and, if m is not a positive integer, we have

$$g \longleftrightarrow p^{-(2m+1)}\epsilon^{A/\lambda}\frac{(2\pi)^{\frac{1}{2}}}{\Gamma(-m)}G_1$$

$$= \frac{(2\lambda h\epsilon^2)^{\frac{1}{2}A/\lambda}(2\pi)^{\frac{1}{2}}}{\Gamma\{\frac{1}{2}+\frac{1}{2}A/\lambda\}}G_1. \tag{41}$$

If m is a positive integer, $\{\Gamma(-m)\}^{-1}$ is zero, and

$$g \longleftrightarrow (-1)^m g_1. \tag{42}$$

Similarly, if C is an arbitrary constant,

$$G \sim \lambda^{-\frac{1}{2}}e^{-\frac{1}{2}(m+1)\pi i}p^{(m+1)}\epsilon^{-\frac{1}{2}A/\lambda}[D_{-m-1}(e^{-\frac{1}{2}\pi i}z)+CD_m(z)]$$

$$= \lambda^{-\frac{1}{2}}e^{-\frac{1}{2}(m+1)\pi i}p^{(m+1)}\epsilon^{-\frac{1}{2}A/\lambda}\Big[\frac{(2\pi)^{\frac{1}{2}}}{\Gamma(m+1)}e^{-\frac{1}{2}m\pi i}D_m(z_1)-e^{-m\pi i}D_{-m-1}(e^{-\frac{1}{2}\pi i}z_1)$$

$$+C\Big\{e^{-m\pi i}D_m(z_1)+\frac{(2\pi)^{\frac{1}{2}}}{\Gamma(-m)}e^{-\frac{1}{2}(m+1)\pi i}D_{-m-1}(e^{-\frac{1}{2}\pi i}z_1)\Big\}\Big]. \tag{43}$$

We may add any multiple of $D_m(z_1)$ to $D_{-m-1}(e^{-\frac{1}{2}\pi i}z_1)$, so that, if m is not a positive integer, there is a complete arbitrariness in the connections. If G is an approximation to a solution of the differential equation, G plus any multiple of g is also an approximation to the same solution, since g/G is exponentially small. But if m is not a positive integer, the continuation of g involves G_1, so that the continuation of G plus an arbitrary multiple of g will involve an arbitrary multiple of G_1. In addition, since g_1/G_1 is exponentially small, any multiple of g_1 may be added to G_1. This explains the complete arbitrariness in the connections. We can make the connections definite by neglecting g compared with G, and postulating that the continuation of G is to be a multiple of g_1. The relation between g_1 and G must then be the same as between g and G_1 in (41). This is easily verified. Since the right-hand side of (43) is not to contain $D_{-m-1}(e^{-\frac{1}{2}\pi i}z_1)$, we must have

$$C = (2\pi)^{-\frac{1}{2}}\Gamma(-m)e^{-\frac{1}{2}(m-1)\pi i}, \tag{44}$$

so that, since

$$\Gamma(-m)\Gamma(1+m) = -\pi\operatorname{cosec} m\pi, \tag{45}$$

we have

$$\frac{(2\pi)^{\frac{1}{2}}}{\Gamma(1+m)}e^{-\frac{1}{2}m\pi i}+Ce^{-m\pi i} = \frac{\Gamma(-m)}{(2\pi)^{\frac{1}{2}}}e^{\frac{1}{2}(m+1)\pi i}. \tag{46}$$

Hence

$$G \sim \lambda^{-\frac{1}{2}} e^{-\frac{1}{2}(m+1)\pi i} p^{(m+1)} \epsilon^{-\frac{1}{2}A/\lambda} \left[\frac{(2\pi)^{\frac{1}{2}}}{\Gamma(m+1)} e^{-\frac{1}{2}m\pi i} + Ce^{-m\pi i} \right] D_m(z_1)$$

$$= \lambda^{-\frac{1}{2}} p^{(m+1)} \epsilon^{-\frac{1}{2}A/\lambda} \Gamma(-m)(2\pi)^{-\frac{1}{2}} D_m(z_1)$$

$$= p^{(2m+1)} \epsilon^{-A/\lambda} \Gamma(-m)(2\pi)^{-\frac{1}{2}} g_1. \qquad (47)$$

Therefore

$$G \longleftrightarrow (2\lambda h\epsilon^2)^{-\frac{1}{2}A/\lambda} \Gamma\left(\tfrac{1}{2} + \frac{A}{2\lambda}\right)(2\pi)^{-\frac{1}{2}} g_1. \qquad (48)$$

If, however, m is a positive integer, the multiple of $D_{-m-1}(e^{-\frac{1}{2}\pi i} z_1)$ on the right of (43) is determinate, and if we neglect $D_m(z_1)$ compared with $D_{-m-1}(e^{-\frac{1}{2}\pi i} z_1)$, *i.e.* g_1 compared with G_1, then

$$G \longleftrightarrow (-1)^{m+1} G_1.$$

When A is zero, (41) and (48) agree with equations (5) and (6) on p 85 of *Proc. London Math. Soc.* (2), 28 (1928).

ON EXTENDED CONGRUENCES CONNECTED WITH THE FOUR-COLOUR MAP THEOREM

By P. J. HEAWOOD.

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Some years ago I showed† that the successful colouring of a map in four colours depends on the solution, without zeros, of a set of linear congruences to modulus 3 equal in number to the divisions of the map‡. Supposing variables x_r, ... assigned one to each corner point, where boundaries meet, we write down for each division or "face" a congruence

$$x_r + x_s + x_t + \ldots \equiv 0 \pmod 3,$$

where x_r, x_s, ... are the variables corresponding to the points round that face. If and only if such a system can be satisfied by assigning values $+1$ or -1 to each of the x's, the map can be coloured in four colours§. But this particular set of congruences is only one of a greatly extended system, differing from the above only in having on the right 0 or 1 or 2, and not necessarily zero. Moreover, any distribution of $+1$ and -1 at the corner points of the map will necessarily correspond to the solution of *one* such system of congruences. It is such extended systems of congruences, which incidentally put the map colour theorem in a new light, that it is the object of the present paper to discuss. It will be shown in

† *Quart. J. of Math.*, 29 (1897), 270–285.

‡ The map is supposed to lie either on a plane or (more symmetrically) to cover the surface of a sphere. For maps on multiply-connected surfaces the proof of the corresponding theorem may be completed, as was shown by the author, *Quart. J. of Math.*, 24 (1890), 332.

§ Apparently Cayley, in 1878, proposed to the London Mathematical Society the question whether a general proof was known of the proposition that any such map can be successfully coloured in four colours and so brought the question into prominence. Such a proof has not, it seems, even now been reached, though the subject has of late engaged the attention of various continental and American mathematicians, and it has apparently been shown that failure could not possibly occur for a map having fewer than twenty-eight divisions. A bibliography and general notes are to be found in a *Thesis for a Doctorate* by A. Errera of Brussels (Paris, Gauthier-Villars et Cie, 1921), and in an article by the same author in the *Periodico di Matematiche* (4), 7 (1927), 20–41.

typical instances how few are the cases out of the immense number of such systems related to a given map in which solution is impossible; and some grounds are shown for believing that, when there are more than a very limited number of divisions, such exceptions do not occur at all.

2. It may be premised that, in dealing with the map-colour theorem, we may restrict ourselves to maps in which only three boundaries meet at any point. Should there be a point at which more boundaries meet, say five separating five divisions a, b, c, d, e (taken in order round the point), these five must not all be considered as touching each other. If we choose to suppose that b touches e, that debars a from touching c or d. In other words, we treat such a point as a condensation of two or more ordinary points:

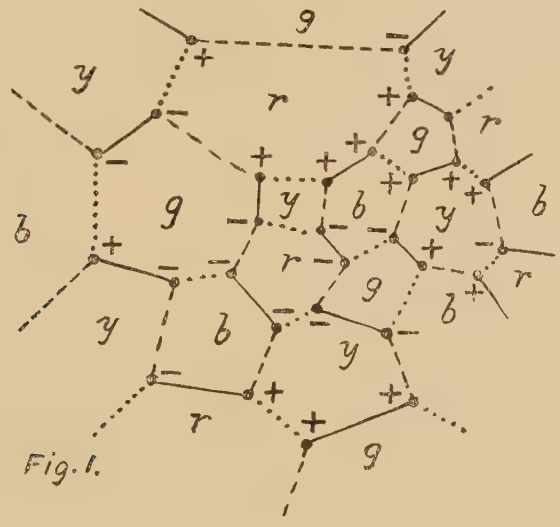
We may therefore assume that only triple points occur. Suppose that such a map *can* be successfully coloured with (say) red, blue, yellow, and green, and that we mark with a continuous line ——— the boundaries between r–b and y–g, those between b–y or r–g by — — — —, and those between y–r or b–g by ..... Then the three lines which meet at any point will be all different; and if we mark by + the points where the three types of boundary, in some specified order, occur in one sense of rotation, and by − the points where they come in the opposite sense, the balance of +'s over −'s, or of −'s over +'s, as we go round any "face" must be a multiple of 3, for only so do we return to a boundary of the same type as that from which we started. (See Fig. 1.) This is the same thing as saying that the system of congruences originally specified can be solved with values +1 or −1 for the x's. Conversely, if such a system of congruences can be solved, the result will be the key to a system of boundary lines which join up properly, and these again give the clue to an available disposition of colours; or the scheme of colouring may be derived from the system of +'s and −'s at once, without actually inserting the lines.

3. In the more extended systems of congruences with which we have to deal, we assign variables to the corner points as before; but we also associate a number α, which may be 0 or 1 or 2, with each face, and take

for the congruence corresponding to it (say)

$$x_r + x_s + x_t + \ldots \equiv a \pmod 3.$$

Supposing the map to cover the surface of a sphere† completely, by addition of all the congruences we get $\Sigma a \equiv 0$, since each x, being associated with three faces, occurs precisely three times. This necessary condition being satisfied, the number of independent congruences, n, must be one fewer than the number of faces F. To the network of boundaries involved, the fundamental formula for polyhedra, $S+F=E+2$, is applicable. With three edges meeting at each point, and each edge



belonging to two corner points, we have also $3S = 2E$, so that

$$\frac{E}{3} = \frac{S}{2} = \frac{E-S}{1} = F-2 = n-1.$$

Therefore $\qquad E = 3(n-1), \quad S = 2(n-1).$

Since the number of independent congruences is n, and each may have 0 or 1 or 2 on the right, there are 3^n possible sets of congruences in our extended system. Since the number S of variables is equal to $2n-2$, each of which may have the value $+1$ or -1, the number of possible ways of assigning values to the variables is 4^{n-1}. Moreover, each of these satisfies some *one* of the possible sets of congruences. With $F=6$, $n=5$, and

† We have only to suppose an extra division added to an ordinary, limited, map to bring it to this standard type.

then $3^n = 243$, $4^{n-1} = 256$, so that already the number of ways of assigning variables exceeds the number of sets of congruences. As F increases, the divergence will be still more marked in the same direction, since the ratio $4^{n-1}/3^n$ increases rapidly with n.

Further, in the map-colour problem we have to concern ourselves only with maps where no division is touched by fewer than five others; for so long as there are divisions with only three or four sides the map can be "reduced", *i.e.* its colouring made to depend on that of one with fewer divisions. For a "triangle", a fourth colour is available if the map can be done when the triangle is shrunk to a point, so with a "quadrilateral" we can "reduce" the map by throwing into one this and two divisions abutting on it on opposite sides: if one pair of opposite divisions touch elsewhere, the other pair is available. A map with no divisions lower than "pentagons" is called a "normal" map (*carte normale*). We limit ourselves, in discussing the extended systems of congruences, to those which are related to normal maps.

4. The first case which presents itself is that of twelve pentagonal divisions†, the regular *dodecahedron*, if we use the term familiar in the theory of polyhedra. Here $n = 11$, so that $3^n = 177147$, the number of possible sets of independent congruences; while $4^{n-1} = 1048576$, the number of ways of assigning values to the variables. The ratio of the latter number to the former is about 5·95, so that there are enough possible arrangements of variables to satisfy all the sets of congruences nearly six times over. The crucial point is whether they can all be solved. The map colour set, with zeros on the right, can of course be satisfied, but that is only one of nearly two hundred thousand systems.

The twelve congruences, all to modulus 3, of which eleven are independent, for constants and variables assigned as shown in Fig. 2, are:

$$u_1+u_2+u_3+u_4+u_5 \equiv \alpha \quad (1) \qquad v_1+v_2+v_3+v_4+v_5 \equiv \beta \quad (2)$$

$$\left.\begin{aligned} u_5+u_1+x_5+x_1+y_1 &\equiv \alpha_1 \\ u_1+u_2+x_1+x_2+y_2 &\equiv \alpha_2 \\ u_2+u_3+x_2+x_3+y_3 &\equiv \alpha_3 \\ u_3+u_4+x_3+x_4+y_4 &\equiv \alpha_4 \\ u_4+u_5+x_4+x_5+y_5 &\equiv \alpha_5 \end{aligned}\right\} \text{(i)}, \qquad \left.\begin{aligned} v_1+v_2+y_1+y_2+x_1 &\equiv \beta_1 \\ v_2+v_3+y_2+y_3+x_2 &\equiv \beta_2 \\ v_3+v_4+y_3+y_4+x_3 &\equiv \beta_3 \\ v_4+v_5+y_4+y_5+x_4 &\equiv \beta_4 \\ v_5+v_1+y_5+y_1+x_5 &\equiv \beta_5 \end{aligned}\right\} \text{(ii)},$$

† No normal map can have fewer than twelve.

where the constants (each 0, 1, or 2) must satisfy the condition

$$\alpha+\Sigma\alpha_r+\beta+\Sigma\beta_s\equiv 0.$$

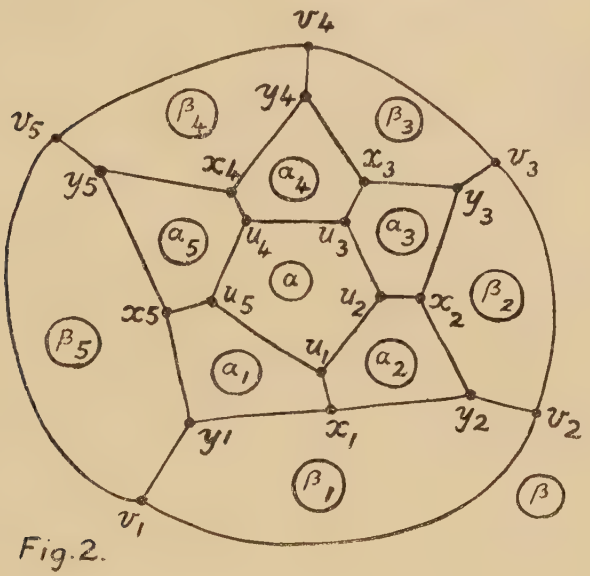


Fig. 2.

I. Adding together all the congruences in the first column, (1) and (i), the u's disappear, and, since $\Sigma 2x\equiv\Sigma -x$, we have the relation

$$\Sigma y-\Sigma x\equiv\alpha+\Sigma\alpha_r;$$

Similarly, from (2) and (ii),

$$\Sigma x-\Sigma y\equiv\beta+\Sigma\beta_s,$$

The two agree in virtue of the condition above.

II. Adding and subtracting alternately in (i), we have

$$2u_5+2x_5+y_1-y_2+y_3-y_4+y_5\equiv\alpha_1-\alpha_2+\alpha_3-\alpha_4+\alpha_5.$$

Now write

$$\alpha_1-\alpha_2+\alpha_3-\alpha_4+\alpha_5\equiv A_1,$$

$$\alpha_2-\alpha_3+\alpha_4-\alpha_5+\alpha_1\equiv A_2,$$

... ...

and similarly

$$y_1-y_2+y_3-y_4+y_5 \equiv Y_1, \quad \ldots,$$

$$\beta_1-\beta_2+\beta_3-\beta_4+\beta_5 \equiv B_1, \quad \ldots,$$

$$x_1-x_2+x_3-x_4+x_5 \equiv X_1, \quad \ldots.$$

We have, from (i),

$$u_5 \equiv Y_1-x_5-A_1 \equiv \xi_1-A_1 \quad \text{(say)},$$

$$u_1 \equiv Y_2-x_1-A_2 \equiv \xi_2-A_2,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

Similarly, from (ii),

$$v_1 \equiv X_1-y_1-B_1 \equiv \eta_1-B_1,$$

$$v_2 \equiv X_2-y_2-B_2 \equiv \eta_2-B_2,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

The result of I suggests a fundamental distinction of cases according as

$$a+\Sigma a_r \equiv \Sigma y-\Sigma x \equiv 0,\ 1,\ 2.$$

In the first case

either (*a*) $\Sigma x \equiv 0, \quad \Sigma y \equiv 0$;

or (*b*) $\Sigma x \equiv 1, \quad \Sigma y \equiv 1$;

or (*c*) $\Sigma x \equiv 2, \quad \Sigma y \equiv 2$.

The values of x, y available do not involve other constants, and may be tabulated once for all. Then possible X's and Y's follow, to be combined, Y's with x's and X's with y's to form ξ's and η's in accordance with II.

In the first instance, we do not count as distinct x's or y's which differ only by cyclic rotation; but when we come to combine (say) x's 12121, X's 20110 with y's 21112, Y's 01110, they give rise to five quite distinct sets of ξ's and η's, as the y's and Y's are rotated cyclically relatively to the x's and X's. For the ξ's and η's separately it will be found that all values are possible, subject to $\Sigma\xi \equiv \Sigma y-\Sigma x$, $\Sigma\eta \equiv \Sigma x-\Sigma y$; but with a given set of ξ's only a limited number of η's are possible.

Now consider any particular sets of A's and B's depending on given constants in the original congruences. In order that there may be possible

values for the u's and the v's there must be related ξ's and η's, such that the ξ's "avoid" the A's and the η's the B's; *i.e.* such that each ξ is different from the corresponding A and each η different from the corresponding B.

Briefly, we can construct a Table giving the ξ's available with every possible set of A's; this will be called Table I. Then, knowing the η's consistent with these ξ's, we can construct a Table giving the η's which will go with the A's (Table II). Table I will itself give the η's available with any given set of B's. Solution will be possible if the η's shown under the A's in Table I and under the B's in Table II include a common set of values. To determine what sets of congruences are possible we have to compare the η's under each possible set of A's with the η's under each possible set of B's. We need not tabulate separately for A's or B's which differ only by cyclic rotation, but this must be taken account of in comparing the results. The procedure thus outlined for $\alpha+\Sigma a_r \equiv 0$ must be repeated for $\alpha+\Sigma a_r \equiv 1$. That is sufficient, since the results for $\alpha+\Sigma a_r \equiv 2$ are complementary and can be inferred. Some details of the work are given in §§ 5, 6.

5. In the first case, when $\Sigma\xi \equiv 0 \equiv \Sigma\eta$, we adopt the following abridged notation for sets of ξ's or η's : [0] for 00000, a_1 for 01110, b_1 for 10101, c_1 for 12000, d_1 for 10200, e_1 for 12120, f_1 for 12210, g_1 for 11220, h_1 for 11112; with the understanding that then a_2 stands for 00111, a_3 for 10011, and so on, with cyclic rotation of the digits. Further, a_1' stands for 02220, with complementary digits, and so throughout. Then the following table (where, for shortness, a_{135} is put for $a_1a_3a_5$, and so on) shows the η's consistent with different sets of ξ's.

ξ's	η's	ξ's	η's
a_1	$a_{135}a_3'h_1h_1'$	e_1	e_{15}'
b_1	$b_{135}b_{24}'d_3d_5'f_1$	f_1	$b_5d_4d_3'f_3$
c_1	c_{15}'	g_1	g_{15}'
d_1	$b_3b_1'd_{24}f_2f_3'$	h_1	$a_5a_3'h_{15}$
[0]	[0]		

Only the ξ's denoted by $a_1b_1\ldots$ are shown, it being understood that if a_1 (ξ's) goes with a_3' (η's), a_2 will go with a_4' and so on; and, further, a_1' with a_3, a_2' with a_4, ...

We next construct Table I, showing the ξ's which will not clash with given A's. Then the above table, of η's consistent with ξ's, enables us

to construct Table II showing the η's that will go with given A's; while we have in Table I the η's that will go with given B's. A specimen column of each table is given below :

From Table I

	A's [or B's] 12121
a	
b	—
	1
c	24
	13
d	4
	5
[ ]	[0]
	2
e	1
	4
f	—
g	
h	

[This shows ξ's available with A's or η's with B's].

From Table II

	A's 20002
a	1−5
	1−5
b	15
	—
c	
d	45
	34
[ ]	
e	12
	15
f	34
	—
g	12
	15
h	1−5
	1

[This shows the η's consistent with given A's].

Under any set of A's [or B's] suffixes only are shown, in double lines, opposite to the letters a, b, c, ..., the first line referring to the letter shown, the second line to that letter dashed. Since, *e.g.*, 4 occurs opposite to d under 12121 in Table I and under 20002 in Table II (as seen above), the coincidence of d_4 in the two indicates a solution for the case where the A's are 20002 and the B's 12121, *i.e.* the α_r's 10012 and the β_s's 00001 (with $\alpha \equiv 2$ and $\beta \equiv 2$); the corresponding η's being 20010, given by d_4. The presence of e_1' in each column points to another solution. To cover all possible combinations of A's and B's, however, we have to compare each column of Table I with each column of Table II not only as they stand, but as they would stand if the A's were cyclically rotated. It is unnecessary to tabulate for such A's separately, since a shift of one place in the A's merely means the addition of 1 to each suffix; if $d_4 d_5$ are available when the A's are 20002, $d_5 d_1$ will be possible for A's 22000, and so on. Thus we have to compare the columns in five different ways, or rather in ten different ways if, remembering that complementary A's will go with

complementary η's, we refrain from tabulating such complementary values. [Obvious exceptions are A's 00000, which is unique, and A's 22222 (which, with the complementary, counts for two), these being unaltered by rotation.]. Thus understood, twenty-six columns account for the 243 possible sets of five A's. In the columns given above there are several coincidences; but, if the suffixes under 12121 in I are all advanced by 2, there are no longer any coincidences; nor are there when in this new result the indications for dashed and undashed letters are transposed. Thus neither 21121 nor 12212 are possible B's to go with A's 20002; *i.e.* we have cases of failure

(1) when $\alpha_1\alpha_2\ldots$ are 10012, $\beta_1\beta_2\ldots 01000$ $(\alpha \equiv 2,\ \beta \equiv 2)$;

(2) $\alpha_1\alpha_2\ldots$ 10012, $\beta_1\beta_2\ldots 02000$ $(\alpha \equiv 2,\ \beta \equiv 1)$.

These cases are shown in Fig. 3.

Sometimes careful comparison is needed, but sometimes success is obvious at a glance; if, *e.g.*, a column of II has a_{1-5} (as in the instance

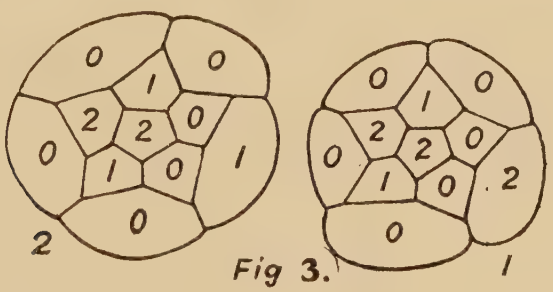


Fig 3.

shown), while a column of I contains a_4 and a'_5, there is certainly one coincidence between the two when the latter column is taken in each of the ten ways.

When these comparisons are completed, we have to deal in precisely the same way with the class of cases in which $\alpha+\Sigma a_r \equiv 1$. If done completely, this will include, as complementaries, cases where $\alpha+\Sigma a_r \equiv 2$; but we now require fifty-one columns, not twenty-six, in Table II. We now have $\Sigma\xi \equiv 2$, $\Sigma\eta \equiv 1$, with new relations between ξ's and η's; but the old notation for ξ's or η's may still be used with new interpretations. For ξ's a_1 may stand for 12221 instead of 01110, while for η's it stands for 20002; similarly, a'_1 will stand for 10001 or 21112, according as it refers to ξ's or η's. With this understanding the old Table I, with its twenty-

six columns, will still apply, with new alternative headings, but Table II must be worked out anew.

6. The object of the scheme is to detect failures, not to supply solutions in possible cases. We briefly indicate how the latter may be found. A coincidence between two columns indicates possible η's, so in the columns shown above the presence of d_4 in both shows that η's may be 20010. These are consistent with values of ξ's given by b_2, b_5', d_1, d_3, f_1, f_2', but a reference to the 20002 column of Table I† would show that f_1 alone goes with these A's; the ξ's are therefore 12210. Then ξ_2-A_2, ξ_3-A_3, ... give u's 22112; similarly η_1-B_1, η_2-B_2, ... give v's 11222. To work back to the x's and y's is less direct, but we have

$$Y_1+X_1+X_5 \equiv \xi_1, \quad Y_2+X_2+X_1 \equiv \xi_2, \quad \ldots,$$

$$X_1+Y_1+Y_2 \equiv \eta_1, \quad X_2+Y_2+Y_3 \equiv \eta_2, \quad \ldots,$$

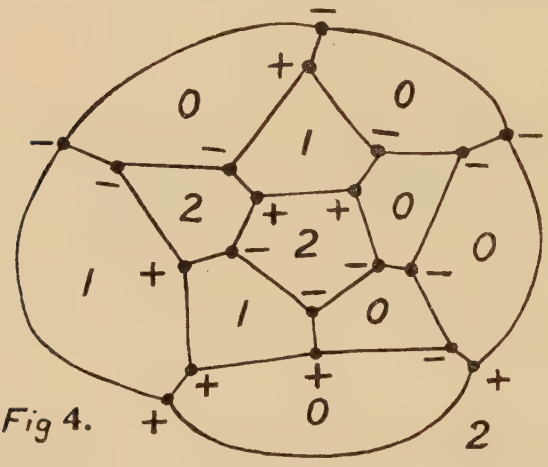


Fig 4.

ten congruences, equivalent to nine only, since

$$\Sigma\xi \equiv \Sigma Y - \Sigma X \equiv -\Sigma\eta,$$

so that their solution is not strictly determinate; but, expressing the Y's in terms of the X's, we have

$$X_5+X_1+X_2 \equiv \gamma_3, \quad X_1+X_2+X_3 \equiv \gamma_4, \quad \ldots,$$

† Not given here.

where the γ's are known constants. Hence

$$\gamma_1-\gamma_2 \equiv X_3-X_1 \equiv x_1-x_2,$$

$$\gamma_2-\gamma_3 \equiv X_4-X_2 \equiv x_2-x_3,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

Thus the x's differ from the γ's by a constant (possibly zero) which, unless the γ's are all equal, can be chosen in only one way to get results different from zero. When the x's are obtained we have the X's, and the y's follow, since $y_r = X_r - \eta_r$. Applying this to the example above, we have γ's, 01110; x's, 12221; X's, 02222; y's, 12212. The result is shown in Fig. 4.

7. When the tabulations and comparisons outlined in § 4, and more fully explained in §§ 5 and 6, are completed, we find in the first class sixty-eight cases of failure (thirty-four being merely complementary to the thirty-four first obtained). Counting as distinct those derived by cyclic rotation of α's and β's we have 340. In the second class there are thirty-two, making 160 by rotation, with a like number in the third class with complementary values. Thus of the 177147 possible sets of values of α's and β's, as reckoned in the first instance, the number for which solution is impossible is 660, or ·37 per cent. Owing, however, to the symmetry of the dodecahedron, very few of these are essentially distinct. The six cases shown in Fig. 5, together with those complementary to the last two, eight in all, exhaust the list. The first four are self-complementary, *i.e.* essentially the same when 1 is put for 2 and 2 for 1. Since the total number of essentially distinct arrangements of constants possible may be

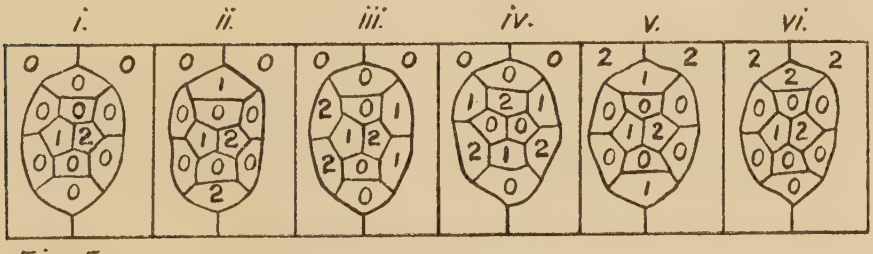


Fig. 5

reckoned at 1835, the percentage of failures from this point of view (eight out of 1835) is ·44, not a very different figure from that obtained above. It is interesting to notice that Class I includes all the essentially distinct

cases of failure. Classes II and III merely yield repetitions, but are important as checking and confirming the results obtained for Class I.

8. The next higher case of a "normal" map is that with fourteen divisions, consisting of two hexagons opposite to each other, each surrounded by six pentagons, as in Fig. 6. The number of sets of congruences to be considered is now $3^{13} = 1594323$, while the number of possible assignments of values to the twenty-four variables involved is $4^{12} = 16777216$. The ratio of the latter number to the former is 10·523...,

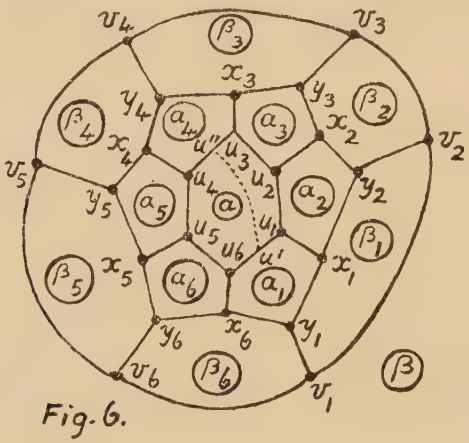


Fig. 6.

so that on the average each set of fourteen congruences (thirteen independent) can be satisfied in ten or eleven different ways. The possibility suggests itself that here there may be no failures, but that can only be settled by detailed analysis.

With constants and variables assigned as shown in Fig. 6, we write the fourteen congruences :

$$u_1+u_2+u_3+u_4+u_5+u_6 \equiv \alpha \quad (1) \qquad v_1+v_2+v_3+v_4+v_5+v_6 \equiv \beta \quad (2)$$

$$\left.\begin{aligned} u_6+u_1+x_6+x_1+y_1 &\equiv \alpha_1 \\ u_1+u_2+x_1+x_2+y_2 &\equiv \alpha_2 \\ u_2+u_3+x_2+x_3+y_3 &\equiv \alpha_3 \\ u_3+u_4+x_3+x_4+y_4 &\equiv \alpha_4 \\ u_4+u_5+x_4+x_5+y_5 &\equiv \alpha_5 \\ u_5+u_6+x_5+x_6+y_6 &\equiv \alpha_6 \end{aligned}\right\} \text{(i)}, \qquad \left.\begin{aligned} v_1+v_2+y_1+y_2+x_1 &\equiv \beta_1 \\ v_2+v_3+y_2+y_3+x_2 &\equiv \beta_2 \\ v_3+v_4+y_3+y_4+x_3 &\equiv \beta_3 \\ v_4+v_5+y_4+y_5+x_4 &\equiv \beta_4 \\ v_5+v_6+y_5+y_6+x_5 &\equiv \beta_5 \\ v_6+v_1+y_6+y_1+x_6 &\equiv \beta_6 \end{aligned}\right\} \text{(ii)},$$

where, as before, the constants (each 0, 1, or 2) must satisfy the condition

$$\alpha+\Sigma\alpha_r+\beta+\Sigma\beta_s\equiv 0,$$

and (1) may be omitted, as deducible from the rest.

As compared with the dodecahedron, where the quantities involved ran in sets of five (a prime number), the fact that they here run in sets of six (a composite number) makes a material difference in the methods that are applicable. With five a's (*e.g.*) the functions

$$a_1-a_2+a_3-\ldots, \quad a_2-a_3+\ldots, \quad \ldots$$

take five distinct values, on which the a's depend. With six a's there are but two values, differing only in sign.[1] Certain tentative methods, which in the end succeeded in proving that cases of failure do exist, were finally superseded by the following :

I. Let $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6$ be such that

$$\gamma_6+\gamma_1\equiv -a_1, \quad \gamma_1+\gamma_2\equiv -a_2, \quad \gamma_2+\gamma_3\equiv -a_3,$$

$$\gamma_3+\gamma_4\equiv -a_4, \quad \gamma_4+\gamma_5\equiv -a_5.$$

[One γ is arbitrary; then the rest are determinate.]

We then have

$$\begin{aligned}\gamma_5+\gamma_6 &\equiv -a_1+a_2-a_3+a_4-a_5\\ &\equiv -a_6+h, \text{ if } h\equiv -a_1+a_2-a_3+a_4-a_5+a_6.\end{aligned}$$

The congruences (i) may now be written :

$$-y_1\equiv(u_6+x_6+\gamma_6)+(u_1+x_1+\gamma_1)$$

$$-y_2\equiv(u_1+x_1+\gamma_1)+(u_2+x_2+\gamma_2)$$

$$-y_3\equiv(u_2+x_2+\gamma_2)+(u_3+x_3+\gamma_3)$$

$$-y_4\equiv(u_3+x_3+\gamma_3)+(u_4+x_4+\gamma_4)$$

$$-y_5\equiv(u_4+x_4+\gamma_4)+(u_5+x_5+\gamma_5)$$

$$-y_6\equiv(u_5+x_5+\gamma_5)+(u_6+x_6+\gamma_6)-h.$$

II. If now we write R_1 for $u_1+x_1+\gamma_1$, R_2 for $u_2+x_2+\gamma_2$, ... the original congruences may be replaced by

$$\left.\begin{aligned} -y_1 &\equiv R_6+R_1 \\ -y_2 &\equiv R_1+R_2 \\ -y_3 &\equiv R_2+R_3 \\ -y_4 &\equiv R_3+R_4 \\ -y_5 &\equiv R_4+R_5 \\ -y_6 &\equiv R_5+R_6-h \end{aligned}\right\}, \quad (A) \qquad \left.\begin{aligned} -x_1 &\equiv v_1+v_2-R_6+R_1-R_2-\beta_1 \\ -x_2 &\equiv v_2+v_3-R_1+R_2-R_3-\beta_2 \\ -x_3 &\equiv v_3+v_4-R_2+R_3-R_4-\beta_3 \\ -x_4 &\equiv v_4+v_5-R_3+R_4-R_5-\beta_4 \\ -x_5 &\equiv v_5+v_6-R_4+R_5-R_6-\beta_5+h \\ -x_6 &\equiv v_6+v_1-R_5+R_6-R_1-\beta_6+h \end{aligned}\right\}, \quad (B)$$

$$\left.\begin{aligned} u_1 &\equiv v_1+v_2-R_6-R_1-R_2-\beta_1-\gamma_1 \\ u_2 &\equiv v_2+v_3-R_1-R_2-R_3-\beta_2-\gamma_2 \\ u_3 &\equiv v_3+v_4-R_2-R_3-R_4-\beta_3-\gamma_3 \\ u_4 &\equiv v_4+v_5-R_3-R_4-R_5-\beta_4-\gamma_4 \\ u_5 &\equiv v_5+v_6-R_4-R_5-R_6-\beta_5-\gamma_5+h \\ u_6 &\equiv v_6+v_1-R_5-R_6-R_1-\beta_6-\gamma_6+h \end{aligned}\right\}, \quad (C)$$

and also

$$v_1+v_2+v_3+v_4+v_5+v_6 \equiv \beta. \quad (D)$$

It is easy to tabulate the v's consistent with D; they depend on β only. So the R's which will give non-zero values to the y's, in accordance with A, depend on h only. From the former may be derived the values of

$$V_1, \ V_2, \ \ldots \equiv -v_1-v_2, \ -v_2-v_3, \ \ldots;$$

from the latter the values of

$$P_1, \ P_2, \ \ldots \equiv R_6+R_1+R_2, \ R_1+R_2+R_3, \ \ldots,$$

and of $$Q_1, \ Q_2, \ \ldots \equiv R_6-R_1+R_2, \ R_1-R_2+R_3, \ \ldots.$$

If then we denote by $p_1, p_2, p_3, \ldots, p_6$ the numbers $-\beta_1-\gamma_1$, $-\beta_2-\gamma_2$, $-\beta_3-\gamma_3$, $-\beta_4-\gamma_4$, $-\beta_5-\gamma_5+h$, $-\beta_6-\gamma_6+h$; and by $q_1, q_2, q_3, \ldots, q_6$ the numbers $-\beta_1$, $-\beta_2$, $-\beta_3$, $-\beta_4$, $-\beta_5+h$, $-\beta_6+h$, which depend

only on the constants of the congruences, the final conditions for solution, in accordance with B, C, are that among possible sets of V's and P's and Q's there should be at least one set such that

$$P_1+V_1,\ P_2+V_2,\ \ldots\ \text{"avoid"}\ p_1,\ p_2,\ \ldots,$$

$$Q_1+V_1,\ Q_2+V_2,\ \ldots\ \text{"avoid"}\ q_1,\ q_2,\ \ldots.$$

In carrying this out, we make the value of β the primary basis of division. According as $\beta \equiv 0$ or $\beta \equiv 1$, the possible values of the v's and V's are those given in the following table, with those derived from them by cyclic rotation :

$\beta \equiv 0$

v's	V's	
111111	111111	(1)
222222	222222	(1)
121212 } 212121 }	000000	(1)
111222	110220	(6)
112122	100020	(6)
112212	102000	(6)

$\beta \equiv 1$

v's	V's	
111112	111100	(6)
112222	102220	(6)
121222	000220	(6)
122122	020020	(3)

There are thus exactly twenty-one forms available for the V's in either case.

For the R's, if we put $R_2 \equiv \sigma$, $R_3 \equiv \tau$, all the primary conditions for the R's will be satisfied if $R_1 \equiv b-\sigma$, $R_6 \equiv a-b+\sigma$, $R_4 \equiv c-\tau$, $R_5 \equiv d-c+\tau$, where a, b, c, d are each either 1 or 2; provided further that $\sigma+\tau \equiv \rho$ is not congruent to zero, and also not congruent to $-a+b+c-d+h \equiv \Delta+h$, say; it is only so that h comes in. These conditions secure that there are suitable values for the y's [Equations A]. In terms of the above quantities the P's take the values

$$a+\sigma,\quad b+\tau,\quad c+\sigma,\quad d+\tau,\quad a-b+d+\sigma,\quad a-c+d+\tau,$$

or (say) $\quad P_1'+\sigma,\ P_2'+\tau,\ P_3'+\sigma,\ P_4'+\tau,\ P_5'+\sigma,\ P_6'+\tau$;

while the Q's take the values

$$a+b,\quad b+\rho,\quad c+\rho,\quad c+d,\quad a-b-c-d+\rho,\quad d-b-c-a+\rho,$$

or (say) $\quad Q_1',\ Q_2'+\rho,\ Q_3+\rho,\ Q_4',\ Q_5'+\rho,\ Q_6'+\rho,$

where the P''s and Q''s are entirely independent of any given constants and, like the V's, may be tabulated once for all. There are just sixteen sets available. Then, combining results, we may tabulate once for all the possible values of

$$V_1+P_1', \quad V_2+P_2', \quad \ldots,$$

$$V_1+Q_1', \quad V_2+Q_2', \quad \ldots,$$

and the complete values V_1+P_1, etc., will be got by adding as above σ, τ, or ρ, these being subject to the conditions stated. Since our main problem is not to find solutions in specified cases, but to track cases where no solutions are possible, our task is to discover among all possible p's and q's what sets there are which are *not* avoided by any related values of $P-V$'s and $Q-V$'s respectively. Such must be noted as cases of failure. We are able to limit the sets of p's to be considered to certain standard forms, owing to equivalences due to cyclical rotation or reversal. For a standard form we examine the possible $P-V$'s that are consistent with it, and tabulate the corresponding $Q-V$'s. Then we classify and examine the results to discover whether there are any q's with which none of the $Q-V$'s available is consistent; such a set of q's with the p's in question will correspond to a set of constants for which no solution of the congruences is possible. Further details of the work are explained in §§ 9–11.

9. Owing to the different ways in which P's and Q's are affected by the values of ρ, σ, τ, different notations are desirable. We take the P's in two triplets in the order $P_1P_3P_5$, $P_6P_4P_2$, and adopt for the triplets the following special notation :

$$\omega_0 \equiv 000, \quad \omega_1 \equiv 111, \quad \omega_2 \equiv 222; \quad \theta_0 \equiv 012, \quad \theta_1 \equiv 120, \quad \theta_2 \equiv 201;$$

$$\theta_0' \equiv 021, \quad \theta_1' \equiv 102, \quad \theta_2' \equiv 210; \quad \lambda_0 \equiv 211, \quad \lambda_1 \equiv 022, \quad \lambda_2 \equiv 100;$$

$$\lambda_0' \equiv 122, \quad \lambda_1' \equiv 200, \quad \lambda_2' \equiv 011; \quad \mu_0 \equiv 112, \quad \mu_1 \equiv 220, \quad \mu_2 \equiv 001;$$

$$\mu_0' \equiv 221, \quad \mu_1' \equiv 002, \quad \mu_2' \equiv 110; \quad \nu_0 \equiv 121, \quad \nu_1 \equiv 202, \quad \nu_2 \equiv 010;$$

$$\nu_0' \equiv 212, \quad \nu_1' \equiv 020, \quad \nu_2' \equiv 101.$$

If $P_1'P_3'P_5'$, $P_6'P_4'P_2'$ are expressed in this notation, the additions of σ, τ, which will result in the full values $P_1P_3P_5$, $P_6P_4P_2$, are made by merely adding σ, τ to the respective suffixes. The Q's we take in three doublets

in the order $Q_1 Q_4$, $Q_2 Q_6$, $Q_3 Q_5$, the notation for the doublets being

$$a_0 \equiv 00, \quad a_1 \equiv 11, \quad a_2 \equiv 22;$$

$$b_0 \equiv 12, \quad b_1 \equiv 20, \quad b_2 \equiv 01;$$

$$c_0 \equiv 21, \quad c_1 \equiv 02, \quad c_2 \equiv 10.$$

If the Q''s are thus expressed, the additions of ρ, to get the Q's, is made by adding ρ to the last *two* suffixes of the three.

In the resulting $P-V$ forms types recur; they include only twenty-seven of the eighty-one possible collocations of two of the nine distinct triplet symbols. In the $Q-V$ forms the second suffix is higher than, equal to, or lower than the third (in cyclic order), according as the first letter is b, a, or c; this helps to check results. Take an instance of the formation of corresponding $P-V$'s and $Q-V$'s; possible P''s and Q''s, in the appropriate notations, are $\mu_2' \mu_0$ and $c_1 b_1 c_2$. V's 102000 are expressed in the two notations by $\theta_1 \omega_0$, $c_2 a_0 b_1$. These combine with the above P''s and Q''s to give $\lambda_1' \mu_0$ and $b_0 b_1 a_0$. Since $\Delta \equiv 1$, ρ must be 1 if $h \equiv 1$, 2 if $h \equiv 0$; but if $h \equiv 2$, $\Delta + 2 \equiv 0$, and ρ may take either value. In other words, $\rho \equiv 1$ goes with $h \equiv 1$ or 2, $\rho \equiv 2$ with $h \equiv 0$ or 2. Accordingly, the complete $Q-V$ forms are $b_0 b_2 a_1$ and $b_0 b_0 a_2$ in the two cases; for the corresponding $P-V$'s we may write $(\lambda' \mu)_2$, $(\lambda' \mu)_0$, denoting by a suffix outside the bracket the sum of the suffixes. The σ and τ to be added to the separate suffixes are simply governed by $\sigma + \tau \equiv \rho$, so that it is only the sum which is relevant. When the combinations have been effected we tabulate under the twenty-seven possible $P-V$ forms the $Q-V$'s which go with them, according to the values of h and the suffix sums of the $P-V$ forms in question (Table I). A small portion of the result *under the heading* $h \equiv 0$ *or* 2 is given below, showing the value of ρ used for completion; it includes the values $(\lambda' \mu)_0$ and $b_0 b_0 a_2$ referred to above.

P—V form	ρ	*suffix-sum* 0	*suffix-sum* 1	*suffix-sum* 2
$\lambda' \mu$	1	$a_2 c_0 a_0$	$b_1 c_1 a_0$	$c_0 c_2 a_0$
	2	$c_2 c_2 c_0$	$b_1 c_2 c_1$	$a_0 c_2 c_2$
	2	$b_0 b_0 a_2$	$a_2 b_0 a_0$	$c_1 b_0 a_1$
$\lambda' \nu$	1	$c_1 a_0 a_1$	$c_0 a_2 a_0$	$c_2 a_1 a_2$
	2		$c_1 c_0 a_1$	$c_2 a_2 c_0$
	2		$a_1 c_2 a_2$	$a_2 a_1 c_1$
	2		$b_1 c_1 a_0$	$b_2 a_0 c_2$.

We have next to see what $P-V$ forms "avoid" the various sets of p's. The p sets may include any six digits (each 0, 1, or 2), but the number that need be tried is reduced owing to the practical identity of sets of congruences which differ only by cyclic rotation of the constants, *i.e.* in which $a_1, a_2, \ldots$ and $\beta_1, \beta_2, \ldots$ are replaced by $a_2, a_3, \ldots$ and $\beta_2, \beta_3, \ldots$, and so on. This does not involve mere cyclical rotation of the p's, except when $h \equiv 0$. $p_1, p_2, p_3, p_4, p_5, p_6$ are replaced by $p_2, p_3, p_4, p_5-h, p_6-h, p_1-h$; then, as h changes sign, by $p_3, p_4, p_5-h, p_6, p_1, p_2+h$, and so on. Again, $p_4 p_3 p_2 p_1 p_6 p_5$ corresponds to an equivalent arrangement of the constants in the opposite cyclic order. Taking all this into account, the p's to be examined may be reduced to fourteen standard types represented by $\omega\omega$, $\omega\theta$, $\theta\theta$, $\theta\theta'$, $\omega\nu$ $(\omega\nu')$, $\theta\nu$ $(\theta'\nu')$, $\mu\mu$ $(\mu'\mu')$, $\mu\lambda$ $(\mu'\lambda')$, $\lambda\lambda'$, $\mu\mu'$; and for $\beta \equiv 0$ those in brackets may be omitted, since each of these is merely complementary to the preceding, and the types reduce to ten. Table II gives under each of the twenty-seven possible $P-V$ forms the suffix sum or sums, if any, for which it avoids each of the standard p forms, with suffix sum zero. Thus under $\lambda\nu'$ we find 2 opposite $\theta_0\nu_0$; $\lambda_2\nu_0'$ $(\equiv 100/212)$ escapes any coincidences with $\theta_0\nu_0$ $(\equiv 012/121)$. For other than zero suffixes in the p form we have merely to advance the other suffixes by a like amount; $\lambda_0\nu_1'$ will avoid $\theta_1\nu_1$. There may be blanks; $\omega\theta$ will not avoid $\theta\nu$ with any suffixes. Sometimes all suffix sums are possible.

10. Table I and Table II together enable us to discover what $Q-V$ forms are possible in conjunction with any of the standard p forms; for the latter gives the $P-V$ forms that will go with the p form, while the former gives the $Q-V$ forms consistent with these $P-V$ forms. Thus, for p form $\mu_0\mu_0$, Table II gives under $\lambda'\mu$ "2 or 0". These are the suffix sums for which the $P-V$ form $\lambda'\mu$ will avoid $\mu\mu$ when the suffix sum of the latter is zero. The small portion of Table I quoted in the last section shows that, for $h = 0$ or 2, the corresponding $Q-V$ forms which are possible are $c_0 c_2 a_0$, $a_0 c_2 c_2$, $c_1 b_0 a_1$; $a_2 c_0 a_0$, $c_2 c_2 c_0$, $b_0 b_0 a_2$. For $\mu\mu$ with suffix sum 1, the suffix sums for $\lambda'\mu$ would be 0 or 1, and the $Q-V$'s would be $a_2 c_0 a_0$, $c_2 c_2 c_0$, $b_0 b_0 a_2$; $b_1 c_1 a_0$, $b_1 c_2 c_1$, $a_2 b_0 a_0$. Proceeding thus, we tabulate the sum total of the $Q-V$ forms available with any p form, according as the suffix sum, t, of the latter is 0, 1, or 2; for $h \equiv 0$ or 1, $k \equiv 0$ or 2, $h \equiv 1$ or 2 respectively. Thus, for $\mu\mu$, the complete result under the headings $h \equiv 0$ or 2, $t \equiv 1$ is given by the following, arranged under the initial letters of the $Q-V$ forms included; $a_0 b_0$ in the column

headed a_0 meaning $a_0a_0b_0$, and so on :

$\mu\mu$
$h \equiv 0$ or 2,
$t \equiv 1$

a_0	a_1	a_2	b_0	b_1	b_2	c_0	c_1	c_2
a_0b_0	b_0b_0	a_0b_0	c_2c_1	$\mathbf{c_2c_1}$	a_0c_2	c_1c_2	c_1b_2	c_1a_2
c_0c_0	a_1a_1	$\mathbf{c_0a_0}$	$\mathbf{b_0a_2}$	c_2a_1	a_0b_2	b_2c_0	a_2c_0	b_1c_2
a_0a_0		$\mathbf{b_0a_0}$	c_1a_0	b_0a_2	b_1b_0	a_0b_1	b_0b_1	b_2a_0
c_1b_1		b_0b_0		a_0c_2	c_1c_0	b_0b_1	a_0c_1	a_2b_0
b_1c_1		a_1a_1		c_1c_0	b_1a_0		a_0b_1	$\mathbf{c_2c_0}$
a_1b_1		a_1b_1		b_1a_0	c_1a_0		b_0b_1	c_2b_0
c_1c_1		c_2a_2		$\mathbf{c_1a_0}$				b_0b_1
c_1a_1		b_2b_2						c_0c_1
b_1b_1		a_2b_2						
c_2a_2		c_2c_2						

Repetitions occur owing to the same $Q-V$ forms coming from two different available $P-V$ forms. The heavy type simply calls attention to forms whose origin has been noted in the text above. The complete "sectional" table for $\mu\mu$ includes nine compartments like that here given, arranged in columns under $t \equiv 0$, $t \equiv 1$, $t \equiv 2$.

11. Having completed a sectional table, our final task is to see whether there are any q forms not provided for by the $Q-V$ forms available. As the 729 possible q forms have to be tested for each value of h and t, one section covers 6561 cases. In many sections, as, in fact, in the $\mu\mu$ section under $\beta \equiv 0$, there are no failures at all; but each case has to be tested. A large number, however, may be disposed of by help of a diagram or diagrams constructed as follows. We select a column, under any t, where under any initial letter there are numerous entries. In the $\mu\mu$ section which we have been using as an illustration the complete a_2 column, under $t \equiv 1$, comprises :

a_2		a_2		a_2	
$h \equiv 0$ or 1	a_0a_0	$h \equiv 0$ or 2	a_0b_0	$h \equiv 1$ or 2	a_1a_1
	b_0a_0		c_0a_0		c_1c_1
	a_0c_0		b_0a_0		b_2a_2
	c_0c_0		b_0b_0		c_2a_2
	a_1b_1		a_1a_1		
	c_1a_1		a_1b_1		
			c_2a_2		
			b_2b_2		
			a_2b_2		
			c_2c_2		

For the corresponding diagram we take nine larger squares marked $a_0 a_1 a_2 b_0 b_1 b_2 c_0 c_1 c_2$, each divided into nine smaller squares, the rows being numbered 0, 1, 2, and the columns headed a, b, c. For any entry in the column selected, *e.g.* $a_0 b_0$, we choose the four larger squares marked with letters representing doublets which "avoid" the doublet given by the first of the pair, *i.e.* such that the doublet indicated by one of these letters has no coincidences with the doublet in question. For a_0 these will be $a_1 a_2 b_0 c_0$, involving the same letter with a different suffix or a different letter with the same suffix. In each of these four larger squares we place, in the *compartment* corresponding to the final letter (which for b_0 is the top middle compartment), a mark corresponding to the available values of h. We use $*$ for an entry for which *any* value of h is available (*e.g.*, for $a_1 b_1$, which comes under $h \equiv 0$ or 1 and under $h \equiv 0$ or 2), and $|$, $-$, $+$ for $h \equiv 0$ or 1, $h \equiv 0$ or 2, and $h \equiv 1$ or 2 respectively. Thus $-$ is the mark for $a_0 b_0$ or $b_0 b_0$; and this explains its presence in the top middle compartments of the squares headed $a_1 a_2 b_0 c_0$, $a_0 c_0 b_1 b_2$ in the completed diagram. Other marks are omitted where an asterisk occurs, but otherwise there may be repetitions :

	a_0			a_1			a_2			b_0			b_1			b_2			c_0			c_1			c_2		
	a	b	c	a	b	c	a	b	c	a	b	c	a	b	c	a	b	c	a	b	c	a	b	c	a	b	c
0	*	−	|	|	−	|	|	−	|	|	−	||	*	−		*	−		*	=	|	−		|	−		|
1	*	*		|		+	*	*					*	*	+				|		+	*	*		|		+
2		−			−		*	−	−	+	−		+	−		*	−	−	*		−	*		−	+	=	
				(1) b_2						(1) b_1						(1) $b_1 c_1$						(1) c_0			(0) b_2		
				(2) c_1						(2) c_1															(2) c_1		

Such a diagram is based on $Q-V$ forms, all beginning with the same letter—in this case a_2; q forms, which will be "covered" by them, will be forms beginning with a_0, a_1, b_2, c_2. Whether any such form will be completely covered depends upon its second and third letters. Here the diagram comes in. Such a form will be provided for if, in the larger square headed by its *second* letter, the compartment corresponding to its *third* letter has in the same row or column with it an appropriate h symbol (leaving out of account any symbol *in* the compartment in question); for owing to the structure of the diagram this shows that with the proper h there is a $Q-V$ form whose third letter avoids the third letter of the q form, while its second letter avoids the letter which heads the square, which is the second letter of the q form. For instance, $a_0 a_1 a_2$ is covered (for $h = 2$) because it begins with a_0 (one of the proper initial letters), while in the larger square headed by a_1 the bottom left-hand compartment,

which corresponds to a_2, has the mark — in line with it. The $Q-V$ form which led to this mark was $a_2 a_2 b_2$. So * in the top left-hand corner of the square headed by b_2 shows (without reference to any other symbol) that, for all values of h, every q form is covered whose *initial* letter is a_0, a_1, b_2, or c_2, middle letter b_2, and final letter a_1, a_2, b_0, or c_0. That single symbol accounts for forty-eight cases, the Q form from which it arises being $a_2 b_0 a_0$. In practice it is easy to see almost at a glance whether in any big square all the compartments are covered for all values of h, or whether any cases must be reserved for further consideration. Thus three *'s in a row, horizontal or vertical, cover all cases for all values of h. Similarly for other relevant symbols so far as they go. Thus, in the a_2 square, * — — in the last line covers all cases for $h \equiv 0$ or 2, while | * * in the first column covers those for which $h \equiv 0$ or 1; so all are provided for. Since four initial letters are available, one large square may thus cover 108 q sets. Below the squares in the above diagram are noted the only cases not provided for, the number in brackets being the value of h, and the letter the third in a q form, of which that which heads the square is the second. Here all the deficiencies are easily supplied otherwise; *e.g.*, $a_1 a_1 b_2$ ($h \equiv 1$) is covered by $c_1 c_1 c_2$ under $h \equiv 0$ or 1, $a_1 a_1 c_1$ ($h \equiv 2$) by $a_0 b_1 b_1$ under $h \equiv 0$ or 2, and so on. It is convenient to use a set of three such diagrams, based on columns headed by three letters, such as $a_2 b_2 c_2$, which between them cover all possible initial letters. Should there be a failure for a particular square in only one of these diagrams, *e.g.* the a_2 diagram (the others being b_2, c_2), the only q sets which need to be specially examined are those beginning with a_0 and a_1. Where, for any value of t, in any section, three (or four) such diagrams are judiciously chosen, the residual cases to be examined form in general a very small proportion of the whole.

The main object is to discover cases of failure rather than to find detailed solutions where solution is possible; but these are easily worked out from the $P-V$'s and $Q-V$'s, which are found to fit with given p's and q's. The connection of the latter with the β's and γ's, if we know the value of h, is given in the preliminary work (§ 8), and the γ's determine the α's. Taken in the natural order, the p's *minus* the $P-V$'s give the u's; the $Q-V$'s *minus* the q's give the x's. If we form

$$u_1+x_1,\ u_2+x_2,\ \ldots \equiv z_1,\ z_2,\ \ldots \quad \text{(say)},$$

and then

$$z_6+z_1,\ z_1+z_2,\ \ldots,$$

the original congruences show that $a_1-(z_6+z_1)$, $a_2-(z_1+z_2)$, ... give the y's. Finally, forming y_1+y_2, y_2+y_3, ..., adding x_1, x_2, ..., and subtract-

ing $\beta_1, \beta_2, \ldots$ we have the V's, and hence the v's. Thus all the variables are determined.

12. When, in the search for failures, the whole of the ten sections under $\beta \equiv 0$ have been disposed of by the methods explained in detail in §§ 9–11, the fourteen sections necessary for $\beta \equiv 1$ must be treated in the same manner. Since the V forms are different, the available $P-V$ and $Q-V$ forms will be different, and new Tables I and II have to be constructed, the one to give the $Q-V$'s consistent with the various $P-V$'s, of which there are still twenty-seven distinct types, the other to show the suffix sums for which these $P-V$ forms avoid the fourteen standard p forms. By means of these tables the results for each of fourteen sections have to be worked out, just as before when $\beta \equiv 0$.

When this has been done the cases of failure which reveal themselves may be tabulated as follows :

$\beta \equiv 0$	i	ii	iii	iv	v	vi	vii	viii
p form in special notation	$\omega_0\theta_1$	$\omega_0\theta_1$	$\omega_0\theta_1$	$\omega_0\theta_1$	$\omega_0\theta_2$	$\theta_0\nu_2$	$\mu_0\lambda_1$	$\lambda_0\lambda_0'$
h	1	1	1	2	2	2	1	0
α	0	0	0	0	0	1	2	0
q form in special notation	$a_0b_2b_2$	$b_2b_2b_2$	$c_1b_2b_2$	$a_0b_2c_1$	$a_0b_0c_1$	$a_0c_1c_1$	$a_0b_2b_2$	$c_2c_1a_0$
α_r's	000210	000100	000020	000200	000010	101210	000001	000000
β_s's	000000	000200	000100	000001	020000	000000	000000	200001

$\beta \equiv 1$	I	II	III	IV	V	VI	VII	VIII
p form in special notation	$\omega_0\omega_0$	$\theta_0\theta_1'$	$\theta_0\theta_2'$	$\omega_0\nu_1$	$\omega_0\nu_2$	$\omega_0\nu_2'$	$\theta_0'\nu_2'$	$\lambda_0\lambda_2'$
h	0	2	1	0	0	0	1	1
α	2	2	2	0	0	1	1	2
q form in special notation	$a_0a_0a_0$	$a_2a_0c_1$	$a_2b_2a_1$	$c_0a_0a_2$	$b_2a_0a_0$	$b_2a_1a_2$	$a_0c_0a_0$	$b_2a_0c_0$
α_r's	000000	200210	210200	000000	000000	001001	021212	200201
β_s's	000000	100102	102100	101210	000200	021212	010010	001201

These, however, are not all independent. Some differ from others merely in the direction of rotation of α's and β's ["reverses"], or in having α's and β's interchanged (with or without change of direction of rotation) ["opposites"]. Thus i and viii, ii and iii, iv and v are opposites; so are

vi and IV, VI and VII. II, III are identical, and VIII is merely the reverse of these. V is the *complementary opposite* of vii. I stands apart, its complementary being the same as its opposite, and so, omitting those which are merely complementaries of others (*i.e.* differing by interchange of 1 and 2), we have just *eight* distinct forms. The duplication noted forms an important check on the accuracy of the results. If we include complementaries as distinct, only three more have to be added, bringing the number up to eleven; it is only in the case of vi, vii, and VI that complementaries give really different forms. The *eight* forms are shown below (Fig. 7). They correspond as follows with the failures tabulated above under Roman figures, those being starred of which the complementaries are different: 1 (I), 2 (i, vii), 3 (ii, iii), 4* (vii, V), 5 (iv, v), 6* (vi, IV), 7 (II, III, VIII), 8* (VI, VII). Each of these, however

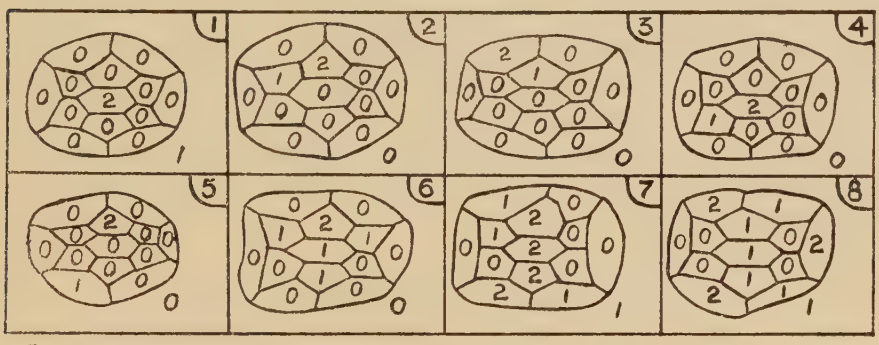


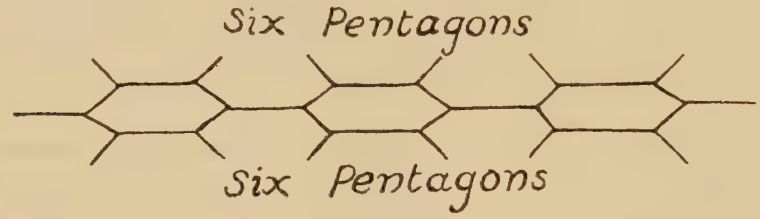
Fig. 7.

(except 1), with reverses, opposites, complementaries—seldom all distinct—and cyclical rotations of these, counts for twenty-four out of the million and a-half or so of possible assignments of values to the individual constants of the original congruences; (1) counts for two only. The total number of assignments of values of constants which lead to failure is therefore $2+24\times 7=170$ out of 1594323, *i.e.* ·010663 per cent. If we compare the really distinct cases of failure with the really distinct arrangements of constants, the percentage is somewhat higher, eleven out of 68,369 giving ·0161 per cent., and eight out of 34,508 giving ·0232 per cent., according as complementaries are treated as distinct, or omitted. But either way the drop in percentages as compared with the dodecahedron is striking; the ·01 per cent. *here* corresponds to ·37 per cent., similarly reckoned, in the other case. Although it has been quite beyond the scope of the investigation to discover how many solutions there may be in individual (possible) cases, a certain number with unique solutions have incidentally come to light.

The average number of solutions, it may be remembered, is between ten and eleven.

If the figures of the failures are examined one point of interest emerges. In five of the eight there is a single 2 and a single 1, with all the other constants zero, and in three of these the 2 and 1 are adjacent; in fact, all possible arrangements of 2 and 1 in adjacent divisions with the other constants zero lead to failures. This is the more striking when we note that for the dodecahedron also a like arrangement involved failure; that, however, included only one distinct arrangement.

13. The next higher cases to be investigated would be normal maps of fifteen and sixteen divisions, for which complete analysis would be a very formidable task. One such fifteen-face figure (possibly not unique) can be derived from the fourteen-face figure by dividing one of the hexagons into two pentagons by a line joining, say, u' between u_6 and u_1 to u'' between u_3 and u_4 (Fig. 6). The resulting figure possesses a symmetry, not apparent from its origin; it consists essentially of three hexagons, as below, filled in with twelve pentagons, six on each side of the string of hexagons, three of the six touching two hexagons, while three touch one only:



In the derived map let a', a'' be the constants of the divisions into which the a face is divided. Any set of constants for the new map will imply one of *four* distributions of constants for the old according as we suppose that $u' \equiv \pm 1$, $u'' \equiv \pm 1$; but it is not sufficient that one of these sets of constants should be possible for the old map. That secures the correct value for $a'+a''$, with the appropriate values of u' and u''; but the proper distribution of this sum between a' and a'' depends on $u_1+u_2+u_3$, $u_4+u_5+u_6$ separately having suitable values. We may use our former investigation as a basis, if we examine all possible solutions for the corresponding cases of the fourteen-face figure and sort them out according to the values that they give to $u_1+u_2+u_3$ and $u_4+u_5+u_6$. Considering the peculiar interest which attaches to arrangements with 1 and 2 assigned to two adjacent divisions and zeros to all the rest, inasmuch

as these have so far led to failures, it seemed worth while to apply the method to the examination of such cases here. In the fifteen-face figure, however, there are *five* distinct ways in which an adjacent pair may be selected, two involving hexagon and pentagon, three involving two pentagons. The pairs actually chosen for 1 and 2 in the new map were (1) a_1, a'; (2) a_1, a_2; (3) a', a''; (4) a', a_2; (5) a_2, a_3. It is almost startling to find failures in all five. The fourteen-face solutions which give the proper values to $a'+a''$ and the other constants are in each of the five cases consistent with any distribution of values between a' and a'' *except* that which would result in the proposed arrangement. That is in every case impossible.

The result is still more striking when a like method is applied to the two distinct sixteen-face figures which can be derived from the fourteen-face by dividing both hexagons. If the a face is divided as above, the two distinct figures may be got by joining (1) v' between v_2 and v_3 to v'' between v_5 and v_6; (2) v' between v_1 and v_2 to v'' between v_4 and v_5 In the former, owing to its greater symmetry, there are but three distinct ways of choosing an adjacent pair of the divisions; in the latter there are nine. The pairs actually chosen for assignments of constants 1 and 2 (all the others being zero) were, for (1), $a'a_1$, $a'a_2$, $a'a''$; for (2), $a_1\beta_1$, $a'a_1$, a_1a_2, a_6a_1, $a'a''$, $a'a_2$, $a'a_3$, a_2a_3, β_2a_3. In each case there are now *sixteen* possible distributions of constants in the old map according to the values assigned to u', u'', v', v''. We examine all solutions of these, 1801 in number, which give appropriate values, in the different cases, to $a'+a''$ and $\beta'+\beta''$, but always with the same result. In every instance all distributions of values between a' and a'', β' and β'' are possible, except those which would lead to such arrangements as are in question. Through all variations, the assignment of constants 1 and 2 to two adjacent divisions of such a sixteen-face map with all the others zero leads to failure. What other cases of failure there may be, if any, we do not know.

14. The results of the preceding sections naturally suggest the question whether the failures in the cases we have been able to examine are more numerous than we might expect merely as a matter of average distribution. For the fourteen-face figure, for instance, we have 16777216 possible arrangements of + and − at the corner points; each of these corresponds to one of 1594323 possible assignments of 0 or 1 or 2 to the different faces for the congruence constants. There are enough arrangements of + and − possible to cover all possible systems of constants more than ten times over. From the point of view merely of random distribution, is the number

of failures more than we might reasonably expect? If from a bag containing 1594323 balls 16777216 drawings were made, the ball being replaced after each drawing, with ideal arrangements for shuffling, what would be the chance that something like 170 balls should fail to appear? Even apart from possible impediments to certain types of arrangement, there are obvious differences between the two problems. On the one hand, the same ball might theoretically be drawn every time, while the various arrangements of + and − must give a very great variety of constants; on the other, our cases of failure are not entirely disconnected. One implies another, and we may expect a somewhat larger proportion of failures than if they were quite isolated events. But the possibility of a bar to certain kinds of distribution only adds interest to the question as to what the chance might be if the results were isolated independent occurrences. Consider the question generally for the map of F faces, where the number of independent congruences is $F-1$, or n. The number M of such sets of congruences is 3^n, while the number N of assignments of + and − is 4^{n-1}; we shall write μM for N, so that both M and μ tend to infinity with n, though in the cases with which we are most concerned M is very large, while μ is comparatively small. When $n=13$, M is over a million and a-half, while μ is about 10·52.

Now if we regard the problem as a case of N drawings from a bag containing M balls, the chance that r *specified* balls should fail to appear is $\{(M-r)/M\}^N$, since there are $M-r$ balls available at each drawing. Call this chance P_r. We next observe that if, of a number of events which may happen, S_r denotes the sum of the chances for all *specified* sets of r to happen, the chance that *some* r events exactly should happen is given by the formula

$$c_0 S_r + c_1 S_{r+1} + c_2 S_{r+2} + \ldots,$$

where c_0, c_1, c_2, ... are the coefficients in the expansion of $(1+x)^{-r-1}$. To verify this result, observe that, if some s exactly of the events should happen, they will include ${}_sC_r$ different sets of r, and so this case will be counted ${}_sC_r$ times over in the reckoning for S_r, ${}_sC_{r+1}$ times in the reckoning for S_{r+1}, and so on; and therefore it will be counted

$$c_0\,{}_sC_r + c_1\,{}_sC_{r+1} + c_2\,{}_sC_{r+2} + \ldots$$

times in all in the reckoning of the expression given above. Now this sum is equal to 1 when $s=r$ and to zero when $s \neq r$; consequently the formula stated gives the chance for exactly r events.

In the problem under discussion $S_m = {}_MC_m P_m$, since there are ${}_MC_m$ sets of m, for each of which the chance is P_m. When we substitute, we

find that the chance of exactly r failures is

$$\sum_{s=0}^{M-r} c_s {}_M C_{r+s} \left(1-\frac{r+s}{M}\right)^N = {}_M C_r \sum_{s=0}^{M-r} (-1)^s {}_{M-r} C_s \left(1-\frac{r+s}{M}\right)^{\mu M},$$

and we wish to determine approximately the value of this sum when M is large for various values of r.

We shall write

$$\Omega(n, r) \equiv {}_M C_r \sum_{s=0}^{M-r} (-1)^s {}_{M-r} C_s \left(1-\frac{r+s}{M}\right)^{\mu M}.$$

The terms in this sum alternate in sign and, for the value of n in which we are most interested ($n = 13$), their numerical values are not monotonic; though, for any fixed value of r, their numerical values are monotonic for all sufficiently large values of n.

So long as $(r+s)^2$ is small compared with M^2/N, $e^{-\mu(r+s)}$ is a good approximation to $\{1-(r+s)/M\}^N$, and this suggests that, if

$$\Omega_0(n, r) \equiv {}_M C_r \sum_{s=0}^{M-r} (-1)^s {}_{M-r} C_s e^{-\mu(r+s)},$$

then $\Omega_0(n, r)$ may be a good approximation to $\Omega(n, r)$. The suggestion, however, needs careful examination, because the errors in the individual terms may be large compared with the sum $\Omega_0(n, r)$, and it will have to be proved that the combined effect of the errors in all the terms is negligible compared with $\Omega_0(n, r)$.

It tends to confirm the suggestion that $\Omega_0(n, r)$ is a valid approximation to $\Omega(n, r)$ to observe that

$$\begin{aligned}\Omega_0(n, r) &= e^{-\mu r} {}_M C_r (1-e^{-\mu})^{M-r} \\ &= (1-e^{-\mu})^M {}_M C_r (e^{\mu}-1)^{-r},\end{aligned}$$

and therefore

$$\sum_{r=0}^{M} \Omega_0(n, r) = (1-e^{-\mu})^M \sum_{r=0}^{M} {}_M C_r (e^{\mu}-1)^{-r} = 1,$$

so that the sum of the approximate chances is exactly equal to the sum of the actual chances, but we must examine the suggestion further.

From one point of view† we may observe that, when $r = 0$, it amounts to saying, with the notation of the Difference Calculus, that $(1-e^{-\mu})^M$,

† Prof. G. N. Watson has suggested an alternative treatment of the problem by the "method of steepest descents" and has sent me some of his work; according to him the problem affords a fairly straightforward example of the method, which should not present any difficulty in working to those who are familiar with it.

which is $\Omega_0(n, 0)$, is under suitable conditions a good approximation to $\Delta^M O^N/M^N$ when M and N are large. This depends on a more extended formula

$$\frac{\Delta^M O^N}{M^N} = (1-e^{-\mu})^M \left\{1+\frac{N}{2M^2}Mk-\frac{N}{2M^2}M(M-1)k^2\ldots\right\},$$

where $\mu = N/M$ and $k = (e^{\mu}-1)^{-1}$; the conditions being, as the form suggests, (1) that N/M^2 should be small, (2) that N should be comparable with $M \log M$†. If $N \geqslant M \log M$, $e^{\mu} \geqslant M$ and $(M-1)k \leqslant 1$. Apart from the second condition, the largeness of the factors Mk, $M(M-1)k^2$, ... in the successive terms might neutralise the smallness of N/M^2. Now it is remarkable that these conditions apply to the numbers with which we are dealing from about $n = 13$ onwards. Since $N = 4^{n-1}$, $M = 3^n$, N/M^2 is necessarily small for any considerable values of n; also $N/(M \log M) = 4^{n-1}/(n3^n \log 3)$, which at first diminishes as n increases. but increases from $n = 4$ onwards. For $n = 13$, 14, 15, 16, the values are ·737, ·912, 1·135, 1·419, so that from $n = 15$ onwards the conditions are strictly fulfilled. For $n = 13$, Mk is as large as 42·9; but, even so, the extreme smallness of N/M^2 sufficiently compensates for the size of the other factor. For $n = 14$, $Mk = 3·86$, and from that point onward Mk is less than unity.

Again, the above formula, with $M' = M-r$ in place of M, will serve to verify the degree of approximation of $\Omega_0(n, r)$ to $\Omega(n, r)$, the precise chance of r failures, for a considerable range of r's. In the difference notation,

$$\Omega(n, r) = {}_MC_r \frac{\Delta^{M'} 0^N}{M^N} = \left(\frac{M'}{M}\right)^N {}_MC_r \frac{\Delta^{M'} 0^N}{M'^N}.$$

For this, by the same approximate formula, we may write, for moderate values of r,

$$\Omega(n, r) = \left(1-\frac{r}{M}\right)^N {}_MC_r(1-e^{-\mu'})^{M'}\left\{1+\frac{N}{2M'^2}M'k'-\frac{N}{2M'^2}M'(M'-1)k'^2\ldots\right\},$$

† Cayley gives a somewhat different formula, in correction of a result given by Laplace [*Proc. Royal Soc. Edinburgh*, 14 (1887), 149–153, *Papers*, 12 (Cambridge, 1897), 412–415.] In terms of M, N as above, and a new symbol u, his result is

$$\frac{\Delta^M 0^N}{M^N} = e^{-u}\left\{1+u\frac{N}{2M^2}-\frac{u^2}{2}\left(\frac{1}{M}+\frac{N}{M^2}\right)+\ldots\right\}, \quad \text{where} \quad u = Me^{-N/M}.$$

He notes the condition that N must be comparable with $M \log M$, which secures for him that u is not very different from unity. His result confirms the value given above but is not so well adapted to our purpose nor so capable of extension. He only carries his results in more general form to about four terms. For the complete expression, in terms Mk, M_2k^2, M_3k^3, etc., see a note at the end of this paper.

where the dashed letters represent the quantities which result when M' takes the place of M.

Since $\Omega_0(n, r) = e^{-\mu r}\,{}_MC_r(1-e^{-\mu})^{M'}$, we have, for the ratio of Ω to Ω_0, very nearly

$$e^{\mu r}\left(1-\frac{r}{M}\right)^N\{(1-e^{-\mu'}) \div (1-e^{-\mu})\}^{M'}\left\{1+\frac{N}{2M^2}Mk-\frac{N}{2M^2}M(N-1)k^2\ldots\right\},$$

if in the small terms of the last bracket we may take the original in place of the modified values.

Taking logarithms we have :

(1) $$\log\{e^{\mu r}(1-r/M)^N\} = -Nr^2/2M^2 \quad \text{very nearly.}$$

(2) $$M'\{\log(1-e^{-\mu'})-\log(1-e^{-\mu})\}$$

$$= M'\{e^{-\mu}+\tfrac{1}{2}e^{-2\mu}-e^{-\mu'}-\tfrac{1}{2}e^{-2\mu'}\} \quad \text{very nearly,}$$

$$= M'(e^{-\mu}-e^{-\mu'})\left(1+\frac{e^{-\mu}+e^{-\mu'}}{2}\right),$$

$$= M'k(1-e^{\mu-\mu'}) \quad \text{very nearly,}$$

$$= M'k(\mu'-\mu) \quad \text{very nearly, } \mu'-\mu \text{ being small,}$$

$$= M'k\left(\frac{N}{M'}-\frac{N}{M}\right) = \frac{Nrk}{M}.$$

Allowing also for the small terms inside the final bracket, the divergence from unity of the ratio we are considering is about

$$-\frac{N}{2M^2}\{r^2-2Mrk+M^2k^2-Mk\}.$$

This, for moderate values of r, is a very small quantity, which vanishes when $r = Mk \pm \sqrt{(Mk)}$.

$\Omega_0(n, r)$ is therefore, for a considerable range of r's, a good approximation to $\Omega(n, r)$ for the values of n which we are considering.

15. For $n = 13$, $\frac{1}{2}N/M^2 = \cdot0000033$, $Mk = 42{\cdot}9$, $\sqrt{(Mk)} = 6{\cdot}55$ nearly, so that from about $r = 25$ to $r = 62$ the approximation should be very good indeed, the error as above estimated, even at the extremes of this range, being only about one-thousandth part of the true value. Between

these limits we have the following table of chances as given by the approximate formula :

Number of failures	Chance	Number of failures	Chance	Number of failures	Chance
25	·0010	38	·0483	51	·0271
26	·0016	39	·0531	52	·0224
27	·0026	40	·0569	53	·0181
28	·0039	41	·0596	54	·0144
29	·0058	42	·0609	55	·0112
30	·0083	43	·0607	56	·0086
31	·0115	44	·0592	57	·0065
32	·0154	45	·0564	58	·0048
33	·0200	46	·0526	59	·0035
34	·0252	47	·0480	60	·0025
35	·0309	48	·0429	61	·0018
36	·0369	49	·0376	62	·0012
37	·0428	50	·0322		

An inspection of the table shows, what a graph would exhibit still more clearly, how rapidly the chance increases to a maximum and then diminishes. The number of failures, 170, might seem to be unexpectedly outside the range for which the chance is considerable ; but it is of something the same order of magnitude, being only about four times that for which the chance is a maximum ; and the fact that, for various reasons, one failure implies a whole set of failures must affect the result. The failures, except one pair, are grouped in sets of twenty-four. If, in our formulae, we were to suppose M and N both divided by something like twenty, leaving μ unchanged, we might be nearer the mark. This would not affect the value of M/r for the maximum chance ; it would still be in the neighbourhood of $e^{\mu}-1$. But the range for which the chance is considerable would be much wider. Suppose, for instance, that P is the chance of r failures, as hitherto calculated, and P' the chance of r' failures in N' trials, where the number of possible events is M', these numbers being connected with the others by the relations $M = \kappa M'$, $N = \kappa N'$, $r = \kappa r'$; *i.e.* the M events hang together in groups of κ, of which all or none occur, and the number of distinct trials is similarly diminished. Then $\mu = M/N = M'/N'$, and is the same in each case, while we have

$$P = {}_{M}C_{r}e^{-\mu r}(1-e^{-\mu})^{M-r}; \quad P' = {}_{M'}C_{r'}e^{-\mu r'}(1-e^{-\mu})^{M'-r'};$$

and the maxima of the two occur for corresponding values of r, r'. P', however, might be more fairly compared, not with P but with κP (this measuring roughly the chance extending over a range of κ values in the

neighbourhood of r). We have

$$\frac{P'^{\kappa}}{\kappa P} = \frac{1}{\kappa}\left\{\frac{M'(M'-1)\dots(M'-r'+1)}{r'!}\right\}^{\kappa} \div \frac{M(M-1)\dots(M-r+1)}{r!},$$

and, M being very large compared with r and κ, we get, by Stirling's theorem,

$$\frac{P'}{\sqrt[\kappa]{(\kappa P)}} = \frac{1}{\sqrt{(2\pi r')}}\left(2\pi\frac{r'}{\kappa}\right)^{1/(2\kappa)}\left(1-\frac{1}{12r'}\right)$$

nearly, or (roughly) a ratio not very different from $1/\sqrt{(2\pi r')}$. An interesting conclusion follows. If κP is fairly large, say about $\frac{1}{2}$, $\sqrt[\kappa]{(\kappa P)}$ is not very different from unity, and P' approximates to $1/\sqrt{(2\pi r')}$, so that it is considerably less than κP. If, on the other hand, κP is small, $\sqrt[\kappa]{(\kappa P)}$ is much greater, and, even taking account of the fractional multiplier $1/\sqrt{(2\pi r')}$, P' is much greater than κP. This is well illustrated in the case of $F = 14$ if $\kappa = 9$. When $r' = 5$ and $r = 45$, κP is roughly $\frac{1}{2}$ (from the table at the beginning of § 15), and $\sqrt[\kappa]{(\kappa P)} = 1\frac{1}{2}$. Therefore P' is nearly equal to the multiplier $1/\sqrt{(2\pi r')}$, which lies between $\frac{1}{5}$ and $\frac{1}{6}$, and this is not much more than one-third the value of κP. This is what happens in the neighbourhood of the maximum values. When, however, $r' = 10$ and $r = 90$, $\kappa P = 10^{-9}$ nearly; $\sqrt[\kappa]{(\kappa P)} = 10^{-1}$ nearly, so that $P' = \frac{1}{70}$ nearly, while κP has the very small value 10^{-9}. Thus for the "grouped" system, though the maximum chance comes in the corresponding place, it is considerably less than for the "ungrouped" system, but comparatively high chances are much more widely spread. This may help to explain the 170 failures for $F = 14$, for which the chance by our original formula is very small, while it is a maximum for forty-two failures.

For the succeeding numbers $F = 15$, 16, 17, the chances, reckoned as before on the basis of random distribution, are as follows:

Number of failures	Chance		
	$F = 15$	$F = 16$	$F = 17$
0	·02114	·89790	·99937
1	·08153	·09669	·00063
2	·15720	·00521	
3	·20209	·00019	
4	·19485	·000005	

It is remarkable how rapidly, on this basis, the chance of even a few failures diminishes. For $F = 17$, the chance of even one failure is

extremely small, and for $F = 18$ it is much smaller still. It might seem probable that not only the map-theorem system of congruences, but the millions of systems which may be associated with a given normal map of eighteen or more divisions can all be solved. Yet the assumptions, based on the idea of random distribution, which underlie our investigations of probability are, as before noted, somewhat artificial. Moreover, the strange persistence of failures when constants 2 and 1 are associated with two adjacent subdivisions and all the others are zeros cannot be ignored. As noted in § 13, this holds even for fifteen-face and sixteen-face figures, in all the various ways in which the divisions to which 1 and 2 are assigned may be selected. It suggests that, while the proportion of failures will be fewer as the number of divisions increases, there may possibly be certain peculiar arrangements of constants which lead to difficulties even when F is largely increased. Yet when we take account, not only of the above estimate of chances, but of the very small and diminishing proportion of failures in the cases we have been able actually to examine, the possibility of failure for the one particular system connected with the map-colour theorem, for any normal map of more than fourteen divisions, out of the millions of systems which may be associated with it, appears almost inconceivable†.

Note on a series for $\Delta^m 0^n/m^n$ when m, n are very large, n/m^2 small, and n of the order $m \log m$ or more.

We have $$\frac{\Delta^m 0^n}{m^n} = \sum_{s=0}^{m} (-1)^s {}_mC_s \left(1-\frac{s}{m}\right)^n.$$

But $$\left(1-\frac{s}{m}\right)^n = \exp\left(-\frac{ns}{m}\right) \exp\left(-\frac{ns^2}{2m^2}-\frac{ns^3}{3m^3}-\ldots\right)$$

$$= \exp(-ns/m)(1+a_2 s^2+a_3 s^3+\ldots), \quad \text{say,}$$

where the series is the product of the expansions in powers of s of $(1-s/m)^n$ and $e^{ns/m}$, and is therefore absolutely convergent.

Now, writing s for $s(s-1)\ldots(s-q+1)$, we have

$$s^p = \sum_{q=1}^{p} {}_pb_q\, s_q,$$

where ${}_pb_q\, q!$ is the coefficient of z^p in $p!\,(e^z-1)^q$, and therefore vanishes if

† It is known that the failure of such a system cannot occur (see references at beginning of paper) for a map having fewer than 28 divisions, a number largely in excess of those for which the chance of *any* failures has, on our reckoning, become quite inappreciable.

$p < q$. A typical term under the original Σ is therefore

$$a_p\, {}_pb_q \sum_{s=0}^{m} (-1)^s {}_mC_s\, s_q\, e^{-\mu s},$$

if we write μ for n/m. Thus the coefficient of $a_p\, {}_pb_q$ is

$$\begin{aligned}(-1)^q m_q e^{-\mu q} \Sigma (-1)^{s-q} {}_{m-q}C_{s-q}\, e^{-\mu(s-q)} \\ = (-1)^q m_q e^{-\mu q} (1-e^{-\mu})^{m-q} \\ = (-1)^q m_q k^q (1-e^{-\mu})^m\end{aligned}$$

where $k = (e^{\mu}-1)^{-1}$; q not exceeding m.

Therefore $$\frac{\Delta^m 0^n}{m^n} = (1-e^{-\mu})^m \left\{1 + \sum_{q=1}^{m} (-1)^q A_q m_q k^q\right\},$$

where $$A_q = a_q\, {}_qb_q + a_{q+1}\, {}_{q+1}b_q + \ldots.$$

Since ${}_pb_q = 0$ when $p < q$, the a's below a_q do not appear and the A's continually diminish.

Another form may be given to the result if, with Cayley, we put $1+X_s$ for $e^{-ns^2/(2m^2)-ns^3/(3m^3)-\ldots}$.

For $${}_pb_q\, q! = q^p - {}_qC_1(q-1)^p + {}_qC_2(q-2)^p - \ldots.$$

It follows that

$$A_q = \frac{1}{q!}(X_q - {}_qC_1 X_{q-1} + {}_qC_2 X_{q-2} - \ldots \text{ to } q \text{ terms}),$$

$$\frac{\Delta^m 0^n}{m^n} = (1-e^{-\mu})^m \left\{1 - X_1 mk + \frac{X_2 - 2X_1}{2!} m_2 k^2 - \ldots \text{ to } m+1 \text{ terms}\right\}.$$

This formula is exact.

Retaining in X_1, X_2 the terms of lowest degree in n/m^2, we have the value

$$(1-e^{-\mu})^m \left\{1 + \frac{n}{2m^2} mk - \frac{n}{2m^3} m_2 k^2 + \ldots\right\}$$

used in the paper. This may also be derived from the first form above independently of X.

That the A's do continually diminish is most easily seen by observing

(1) that the "leading" term in a_p, *i.e.* the term involving n/m^2 to its lowest power, is, when $p = 2t$ (even),

$$(-1)^t n^t / (2^t\, t!\, m^{2t}).$$

(2) ${}_pb_q$ being the coefficient of z^p in $p!\,(e^z-1)^q/q!$ is certainly less than the coefficient of z^p in $p!\,e^{zq}/q!$, *i.e.* less than $q^p/q!$, generally very much less. Now applying Stirling's formula to the product of the absolute value of the expression in (1) and $q^{2t}/q!$ we get

$$\frac{(e/q)^q}{2\pi\sqrt{q}}\left(\frac{neq^2}{2m^2}\right)^t t^{-t-\frac{1}{2}}.$$

Putting $q=\lambda m^2/n$, this will be found to be a maximum, for a given value of q, when $2t=\lambda^2 m^2/n-1$ nearly. If we put $2t=\lambda q$, which is near enough, the value obtained for the maximum is

$$\frac{(e^{1+\frac{1}{2}\lambda}/q)^q}{\sqrt{(2\lambda)q\pi}}. \tag{1}$$

Since p cannot be less than q, this is available only when $\lambda \geqslant 1$. When $\lambda \leqslant 1$, the greatest value available is when $2t=q$, giving the result

$$\frac{(\lambda^{\frac{1}{2}}e^{\frac{3}{2}}/q)^q}{\sqrt{2\,.\,q\pi}}. \tag{2}$$

In either case the value is extremely small, unless λ is very large; and as λ cannot exceed μ, this does not in general give a higher value. It can do so only when $n>m\log m$, and then $m_q k^q$ is itself exceedingly minute for the higher values of q.

We have been estimating possible values of $|a_{p\,p}b_q|$ when $p=2t$. In summing $\sum_{2t=q}^{\infty} a_{2t\,2t}b_q$ for integral t's, the signs alternate, and for a range of diminishing numerical values the sum will be less than the maximum term. For odd values of p a like result holds. This points to the conclusion that the A_q's rapidly diminish, at least until $m_q k^q$ becomes inappreciable; *i.e.* $A_q m_q k^q$ is negligible after the first few terms, and, under the conditions stated, $(1-e^{-\mu})^m$ may be taken as a good approximation to $\Delta^m 0^n/m^n$.

ON SOME CLASSES OF FOURIER SERIES

By S. Verblunsky.

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1. Consider the following six classes of Fourier series, namely the Fourier series of functions of period 2π which satisfy respectively the condition of being

(i) L integrable,

(ii) bounded,

(iii) R integrable,

(iv) continuous,

(v) of bounded variation,

(vi) absolutely continuous.

We denote any one of these six classes of Fourier series by $\{k\}$ where $k = 1, 2, \ldots, 6$. If

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

denotes any member of $\{k\}$ for some fixed value of k, then we denote the class of series

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \tag{2}$$

by $\{\bar{k}\}$. If (λ_n) $(n = 0, 1, \ldots)$ is a sequence of numbers with the property that if the series (1) be any member of $\{k\}$ for some fixed value of k, then

$$\tfrac{1}{2}a_0\lambda_0 + \sum_{n=1}^{\infty} \lambda_n (a_n \cos nx + b_n \sin nx) \tag{3}$$

is a member of $\{l\}$, where l is a fixed one of the numbers 1, 2, ..., 6, then we

say that (λ_n) transforms $\{k\}$ to $\{l\}$. Similarly, if (μ_n) $(n = 1, 2, \ldots)$ is a sequence of numbers with the property that if the series (2) is any member of $\{\bar{k}\}$ for some fixed value of k, then

$$\sum_{n=1}^{\infty} \mu_n (b_n \cos nx - a_n \sin nx)$$

is a member of $\{l\}$, where l is a fixed one of the numbers $1, 2, \ldots, 6$, then we say that (μ_n) transforms $\{\bar{k}\}$ to $\{\bar{l}\}$.

On the hypothesis that we can enunciate a necessary and sufficient condition [Condition (k, l)] which (λ_n) must satisfy in order that it may transform $\{k\}$ to $\{l\}$, there will exist 36 conditions of this kind. On the hypothesis that we can enunciate a necessary and sufficient condition [Condition $(\bar{k}, l)$] which (μ_n) must satisfy in order that it may transform $\{\bar{k}\}$ to $\{l\}$, there will exist 36 further conditions. The main object of this paper is to obtain these 72 conditions.

We shall in all that follows suppose that $\lambda_0 = 0$. This involves no loss of generality and simplifies the formulae.

Many of the conditions (k, l) can be expressed in the form that one or other of the series

$$\Sigma \lambda_n \cos nx, \quad \Sigma \frac{\lambda_n}{n} \sin nx, \quad \Sigma \frac{\lambda_n}{n^2} \cos nx$$

should belong to one of the six classes. This, however, is not possible in all cases. To express all the 36 conditions it becomes necessary to introduce three new classes of Fourier series, namely the Fourier series of functions of period 2π which satisfy respectively the condition of being

(a) of bounded variation in mean,

(b) absolutely continuous in mean,

(c) of R integrable variation in mean.

These three classes of Fourier series we denote by $\{a\}$, $\{b\}$, $\{c\}$ respectively. The definitions of the terms (a), (b), (c) will be given at the appropriate points.

If a trigonometric series (1) belongs to one of the classes $\{k\}$, we write

$$\tfrac{1}{2}a_0 + \Sigma (a_n \cos nx + b_n \sin nx) \subset \{k\}.$$

We use a similar notation when the series belongs to one of the classes $\{a\}$, $\{b\}$, $\{c\}$, or to one of the classes $\{\bar{k}\}$, $\{\bar{a}\}$, $\{\bar{b}\}$, $\{\bar{c}\}$.

The 36 conditions (k, l) can be expressed as in the following table, where to save space we use the notation

$$[\lambda_n] = \Sigma\lambda_n \cos nx.$$

	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$	$\{6\}$
$\{1\}$	$F, \bar{z}$ $\left[\frac{\lambda_n}{n}\right] \subset \{\bar{5}\}$	$[\lambda_n] \subset \{2\}$	$[\lambda_n] \subset \{2\}$	$[\lambda_n] \subset \{2\}$	$[\lambda_n] \subset \{5\}$	$[\lambda_n] \subset \{5\}$
$\{2\}$	$\left[\frac{\lambda_n}{n}\right] \subset \{\bar{5}\}$	$F, \bar{F}$ $\left[\frac{\lambda_n}{n}\right] \subset \{\bar{5}\}$	$\left[\frac{\lambda_n}{n}\right] \subset \{\bar{c}\}$	$z, \bar{z}$ $[\lambda_n] \subset \{1\}$	$[\lambda_n] \subset \{a\}$	$[\lambda_n] \subset \{b\}$
$\{3\}$	$\left[\frac{\lambda_n}{n}\right] \subset \{\bar{5}\}$	$\left[\frac{\lambda_n}{n}\right] \subset \{\bar{5}\}$	$F, \bar{F}$ $\left[\frac{\lambda_n}{n}\right] \subset \{\bar{5}\}$	$[\lambda_n] \subset \{1\}$	$[\lambda_n] \subset \{a\}$	$[\lambda_n] \subset \{b\}$
$\{4\}$	$\left[\frac{\lambda_n}{n}\right] \subset \{\bar{a}\}$	$\left[\frac{\lambda_n}{n}\right] \subset \{\bar{5}\}$	$\left[\frac{\lambda_n}{n}\right] \subset \{\bar{5}\}$	$F, \bar{F}$ $\left[\frac{\lambda_n}{n}\right] \subset \{\bar{5}\}$	$[\lambda_n] \subset \{a\}$	$[\lambda_n] \subset \{a\}$
$\{5\}$	$\left[\frac{\lambda_n}{n}\right] \subset \{\bar{1}\}$	$\left[\frac{\lambda_n}{n}\right] \subset \{\bar{2}\}$	$\left[\frac{\lambda_n}{n}\right] \subset \{\bar{3}\}$	$\left[\frac{\lambda_n}{n}\right] \subset \{\bar{4}\}$	$F, \bar{F}$ $\left[\frac{\lambda_n}{n}\right] \subset \{\bar{5}\}$	$z, \bar{z}$ $\left[\frac{\lambda_n}{n}\right] \subset \{\bar{6}\}$
$\{6\}$	$\left[\frac{\lambda_n}{n^2}\right] \subset \{5\}$	$\left[\frac{\lambda_n}{n}\right] \subset \{\bar{2}\}$	$\left[\frac{\lambda_n}{n}\right] \subset \{\bar{2}\}$	$\left[\frac{\lambda_n}{n}\right] \subset \{\bar{2}\}$	$\left[\frac{\lambda_n}{n}\right] \subset \{\bar{5}\}$	$F, \bar{z}$ $\left[\frac{\lambda_n}{n}\right] \subset \{\bar{5}\}$

For example, to find condition (4, 2) we take the square in the fourth row and second column. The condition is

$$\Sigma \frac{\lambda_n}{n} \sin nx \subset \{5\}.$$

The conditions are not all distinct. Thus conditions (1, 2), (1, 3), (1, 4) are identical. This means that (λ_n) cannot transform $\{1\}$ to $\{2\}$ without transforming $\{1\}$ to $\{4\}$.

The proofs of the conditions (k, l) are of such a nature that by trivial changes the conditions $(\bar{k}, l)$ may be established. In what follows we shall not therefore give proofs of the conditions $(\bar{k}, l)$. To obtain the enunciation of the condition $(\bar{k}, l)$, change cos to sin and sin to cos in the condition (k, l). Thus condition $(\bar{4}, 2)$ is

$$\Sigma \frac{\lambda_n}{n} \cos nx \subset \{5\}.$$

We call the condition $(\bar{k}, l)$ the conjugate of the condition (k, l). The conditions marked F are due to Fekete*, as are also the conjugates of the conditions marked $\bar{F}$. The conditions marked z are due to Zygmund†, as are also the conjugates of the conditions marked $\bar{z}$.

2. In this section we state as lemmas, and prove, five known results.

LEMMA 1‡. *The necessary and sufficient condition that the series* (1) *should belong respectively to the classes* $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{6\}$ *is that the sequences* $\{S_r(x)\}$ *of Cesàro means of order unity of* (1) *should be respectively*

(ii) *uniformly bounded,*

(iii) *uniformly R integrable,*

(iv) *uniformly continuous,*

(v) *of uniformly bounded variation,*

(vi) *uniformly absolutely continuous.*

Of the terms which enter into the enunciation, (ii) and (iv) require no explanation. Concerning the other three, it is desirable to give definitions which are equivalent to if somewhat different from the usual ones, because the terms (a), (b), (c) of §1 will be seen to be related to them.

I. A periodic function $f(t)$ is said to be R integrable if, given $\epsilon > 0$, there is an $\eta > 0$ such that, if

$$(a_1, b_1), \ \ldots, \ (a_n, b_n)$$

denotes any finite number of non-overlapping intervals in $(0, 2\pi)$ which satisfy

$$b_i - a_i \leqslant \eta,$$

and if γ_i, δ_i be arbitrarily chosen to satisfy

$$a_i \leqslant \gamma_i < \delta_i \leqslant b_i,$$

then

$$\left| \sum_{i=1}^{n} [f(\delta_i) - f(\gamma_i)](b_i - a_i) \right| \leqslant \epsilon.$$

II. A periodic function $f(t)$ is said to be of bounded variation if there is a constant κ such that, if

$$(a_1, b_1), \ \ldots, \ (a_n, b_n)$$

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* Fekete, *Acta Reg. Univ. Hung.*, 1 (1922–23), 148-166.

† Zygmund, *Bulletin Polonaise* (1927), 343–347.

‡ Fekete, *loc. cit.*, § 3.

denote any finite number of non-overlapping intervals in $(0, 2\pi)$, then

$$\left| \sum_{i=1}^{n} f(b_i) - f(a_i) \right| \leqslant \kappa.$$

III. A periodic function $f(t)$ is said to be absolutely continuous if, given ϵ, there is an η such that, if

$$(a_1, b_1), \quad \ldots, \quad (a_n, b_n)$$

denote any finite number of non-overlapping intervals in $(0, 2\pi)$ which satisfy

$$\sum_{i=1}^{n} b_i - a_i \leqslant \eta,$$

then

$$\left| \sum_{i=1}^{n} f(b_i) - f(a_i) \right| \leqslant \epsilon.$$

IV. A sequence $\{f_r(x)\}$ of periodic functions is said to be uniformly R integrable, of uniformly bounded variation, uniformly absolutely continuous, respectively, if each function of the sequence is respectively R integrable, of bounded variation, absolutely continuous, and if the numbers ϵ, η, κ which enter into the definitions are independent of r.

Consider now the proof of Lemma 1. To show that the condition is necessary, we shall consider case (iii), the other cases being similar and simpler. If (1) is the Fourier series of the R integrable function $f(x)$, we have

$$S_r(x) = \frac{1}{2r\pi} \int_0^{2\pi} \left(\frac{\sin \frac{1}{2} rt}{\sin \frac{1}{2} t} \right)^2 f(x+t)\, dt.$$

By definition I, and by the periodicity of $f(t)$,

$$\left| \sum_{i=1}^{n} [f(\delta_i + t) - f(\gamma_i + t)](b_i - a_i) \right| \leqslant \epsilon$$

for all t. Hence

$$\left| \sum_{i=1}^{n} [S_r(\delta_i) - S_r(\gamma_i)](b_i - a_i) \right|$$

$$\leqslant \frac{1}{\frac{1}{2} r\pi} \int_0^{2\pi} \left(\frac{\sin \frac{1}{2} rt}{\sin \frac{1}{2} t} \right)^2 \left| \sum_{i=1}^{n} [f(\delta_i + t) - f(\gamma_i + t)](b_i - a_i) \right| dt$$

$$\leqslant \epsilon,$$

which proves that $\{S_r(x)\}$ is uniformly R integrable.

To show that the condition is sufficient, we again consider case (iii). The condition (iii) implies that there is a constant κ such that

$$|S_r(0)-S_r(x)|<\kappa$$

for all r. But

$$2\pi S_r(0)=\int_0^{2\pi} S_r(x)\,dx+\int_0^{2\pi}[S_r(0)-S_r(x)]\,dx$$

and

$$\int_0^{2\pi} S_r(x)\,dx=\pi a_0.$$

Hence

$$|S_r(0)|<\frac{|a_0|}{2}+\kappa$$

and

$$|S_r(x)|<\frac{|a_0|}{2}+2\kappa.$$

We now wish to prove that (1) is a Fourier series. To do this without using the Riesz-Fischer theorem we observe that, in virtue of the last relation, the sequence of functions

$$\sigma_r(x)=\int_0^x S_r(t)\,dt \quad (0\leqslant x\leqslant 2\pi)$$

is uniformly absolutely continuous and uniformly bounded. Hence there is a subsequence $\{\sigma_{n_r}(x)\}$ which converges uniformly to an absolutely continuous function $F(x)$, say. Hence

$$\lim_{r\to\infty}\int_0^{2\pi}\sigma_{n_r}(x)\begin{matrix}\cos\\ \sin\end{matrix}kx\,dx=\int_0^{2\pi}F(x)\begin{matrix}\cos\\ \sin\end{matrix}kx\,dx.$$

The first integral is, if $n_r>k$,

$$-\pi\frac{b_k}{k}\left(1-\frac{k}{n_r}\right) \quad \text{or} \quad \pi\frac{a_k}{k}\left(1-\frac{k}{n_r}\right),$$

according as cos or sin is taken. Hence

$$\Sigma\frac{a_n\sin nx-b_n\cos nx}{n}$$

is the Fourier series of an absolutely continuous function, and therefore (1) is a Fourier series.

It remains to prove that (1) is the Fourier series of an R integrable function. The sequence $\{S_n(x)\}$ converges in a set E of measure 2π to a

function $f(x)$, say. If γ_i, δ_i in definition I be chosen so as to belong to E, we have, from

$$\left| \sum_{i=1}^{n} [S_r(\delta_i) - S_r(\gamma_i)] (b_i - a_i) \right| \leqslant \epsilon$$

for all r, the limiting relation

$$\left| \sum_{i=1}^{n} [f(\delta_i) - f(\gamma_i)] (b_i - a_i) \right| \leqslant \epsilon.$$

If now ξ be a point of CE we define $f(\xi) = \overline{\lim}_{x \to \xi} f(x)$ $(x \subset E)$, and the preceding relation holds even if γ_i, δ_i belong to CE. Thus $f(x)$ is R integrable.

LEMMA 2. *Let $\{P_n(t)\}$ be a sequence of periodic functions such that*

$$\overline{\lim_{n \to \infty}} \int_0^{2\pi} |P_n(t)| \, dt = \infty.$$

Then there is a continuous periodic function $\psi(t)$ such that

$$\overline{\lim_{n \to \infty}} \left| \int_0^{2\pi} \psi(t) P_n(t) \, dt \right| = \infty.$$

Lemma 2 is a consequence of a theorem of Hahn* in the theory of functionals. The method we use in the proof was suggested by a lemma of Steinhaus†.

If the result of the lemma is false, then, whatever be the continuous periodic function $f(t)$, we must have

$$\overline{\lim_{n \to \infty}} \left| \int_0^{2\pi} f(t) P_n(t) \, dt \right| < \infty. \tag{A}$$

We shall show that (A) implies a contradiction.

Since $P_n(t)$ is periodic, so is sgn $P_n(t)$. Let

$$\int_0^{2\pi} \operatorname{sgn} P_n(t) \, . \, P_n(t) \, dt = \kappa_n,$$

so that

$$\overline{\lim_{n \to \infty}} \kappa_n = \infty.$$

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* Hahn, *Monatshefte für Math. und Phys.*, 32 (1922), 3–88 (59).

† Steinhaus, *Math. Zeitschrift*, 5 (1919), 186–221 (217–219).

We can find a continuous periodic function $f_n(t)$ such that

$$|f_n(t)| \leqslant 1$$

and such that

$$\int_0^{2\pi} f_n(t)\, P_n(t)\, dt > \kappa_n \left(1 - \frac{1}{n}\right).$$

Let (ϵ_i) be a monotone diminishing sequence of positive terms such that

$$\sum_{i=1}^{\infty} \epsilon$$

is convergent. Then, however the ϵ_i be chosen subject to the conditions stated, and however the increasing sequence (n_i) of integers be chosen, the function defined by

$$F(t) = \sum_{i=1}^{\infty} \epsilon_i f_{n_i}(t)$$

is continuous and periodic. We shall show that the ϵ_i and the n_i can be so chosen that

$$\overline{\lim} \left| \int_0^{2\pi} F(t)\, P_n(t)\, dt \right| = \infty,$$

which will contradict (A).

We just take $\epsilon_1 = 1$. We then have

$$\int_0^{2\pi} F(t)\, P_n(t)\, dt = \int_0^{2\pi} f_{n_1}(t)\, P_n(t)\, dt + \sum_{i=2}^{\infty} \epsilon_i \int_0^{2\pi} f_{n_i}(t)\, P_n(t)\, dt$$

$$> \int_0^{2\pi} f_{n_1}(t)\, P_n(t)\, dt - \sum_{i=2}^{\infty} \epsilon_i \int_0^{2\pi} |P_n(t)|\, dt.$$

Choose $n_1 > 1$ such that

$$\kappa_{n_1} \left(1 - \frac{1}{n_1}\right) > 2,$$

and let $\epsilon_2, \epsilon_3, \ldots$ be subjected to the condition

$$\sum_{i=1}^{\infty} \epsilon_i \leqslant 1 \Big/ \int_0^{2\pi} |P_{n_1}(t)|\, dt.$$

Then

$$\int_0^{2\pi} F(t)\, P_{n_1}(t)\, dt > 1.$$

For ϵ_2 choose a number satisfying

$$\epsilon_2 < 1 \Big/ \int_0^{2\pi} |P_{n_1}(t)|\, dt.$$

To find n_2 we write

$$\int_0^{2\pi} F(t)\,P_n(t)\,dt = \int_0^{2\pi} f_{n_1}(t)\,P_n(t)\,dt + \epsilon_2 \int_0^{2\pi} f_{n_2}(t)\,P_n(t)\,dt + \sum_{i=3}^{\infty} \epsilon_i \int_0^{2\pi} f_{n_i}(t)\,P_n(t)\,dt$$

$$> \epsilon_2 \int_0^{2\pi} f_{n_2}(t)\,P_n(t)\,dt - \sum_{i=3}^{\infty} \epsilon_i \int_0^{2\pi} |P_n(t)|\,dt - A_{n_1},$$

where, by (A),
$$\left| \int_0^{2\pi} f_{n_1}(t)\,P_n(t)\,dt \right| < A_{n_1}$$

for all n. Choose $n_2 > n_1$ such that

$$\epsilon_2 \kappa_{n_2} \left(1 - \frac{1}{n_2}\right) > 3 + A_{n_1},$$

and subject ϵ_3, ϵ_4, ... to the condition

$$\sum_{i=3}^{\infty} \epsilon_i \leqslant 1 \Big/ \int_0^{2\pi} |P_{n_2}(t)|\,dt.$$

Then
$$\int_0^{2\pi} F(t)\,P_{n_2}(t)\,dt > 2.$$

For ϵ_3 we choose a number such that

$$\epsilon_3 < 1 \Big/ \int_0^{2\pi} |P_{n_2}(t)|\,dt,$$

$$\epsilon_2 + \epsilon_3 < 1 \Big/ \int_0^{2\pi} |P_{n_1}(t)|\,dt.$$

To find n_3 we write

$$\int_0^{2\pi} F(t)\,P_n(t)\,dt$$

$$= \int_0^{2\pi} \left[\sum_{i=1}^{2} \epsilon_i f_{n_i}(t) \right] P_n(t)\,dt + \epsilon_3 \int_0^{2\pi} f_{n_3}(t)\,P_n(t)\,dt + \sum_{i=4}^{\infty} \epsilon_i \int_0^{2\pi} f_{n_i}(t)\,P_n(t)\,dt$$

$$> \epsilon_3 \int_0^{2\pi} f_{n_3}(t)\,P_n(t)\,dt - \sum_{i=4}^{\infty} \epsilon_i \int_0^{2\pi} |P_n(t)|\,dt - A_{n_2},$$

where, by (A),
$$\left| \int_0^{2\pi} \left[\sum_{i=1}^{2} \epsilon_i f_{n_i}(t) \right] P_n(t)\,dt \right| < A_{n_2}$$

for all n. Choose $n_3 > n_2$ such that

$$\epsilon_3 \kappa_{n_3} \left(1 - \frac{1}{n_3}\right) > 4 + A_{n_2},$$

and subject $\epsilon_4, \epsilon_5, \ldots$ to the condition

$$\sum_{i=4}^{\infty} \epsilon_i \leqslant 1 \Big/ \int_0^{2\pi} |P_{n_3}(t)|\,dt.$$

Then
$$\int_0^{2\pi} F(t)\,P_{n_3}(t)\,dt > 3.$$

It is clear how this process may be continued to give the function $F(t)$ for which

$$\int_0^{2\pi} F(t)\,P_{n_k}(t)\,dt > k.$$

This proves the lemma.

LEMMA 3. *If the bounded functions $\{P_n(t)\}$ are not equivalent to a uniformly bounded sequence, then there exists a periodic function $X(t)$, which is integrable L, such that*

$$\overline{\lim_{n\to\infty}} \left| \int_0^{2\pi} X(t)\,P_n(t)\,dt \right| = \infty.$$

To make the enunciation quite clear, we introduce the following definition:

V. A sequence $\{f_r(t)\}$ of functions is said to be equivalent to a uniformly bounded sequence if there is a uniformly bounded sequence $\{\phi_r(t)\}$ of functions such that, for each r, $f_r(t)$ is equivalent to $\phi_r(t)$.

Lemma 3 is due to Steinhaus*. The proof which we give is precisely similar to the proof of Lemma 2.

If the result of the lemma is false, then, whatever be the L integrable function $f(t)$, we must have

$$\overline{\lim_{n\to\infty}} \left| \int_0^{2\pi} f(t)\,P_n(t)\,dt \right| < \infty. \qquad \text{(B)}$$

We shall show that (B) implies a contradiction.

To each $P_n(t)$ there corresponds a number κ_n such that

$$|P_n(t)| < \kappa_n + \epsilon$$

for all t with the possible exception of a set of measure zero, and for all $\epsilon > 0$, while

$$|P_n(t)| > \kappa_n - \epsilon$$

* Steinhaus, *loc. cit.*, 219–221.

in a set $E_n(\epsilon)$ of positive measure for each $\epsilon > 0$. Let

$$f_n(t) = \frac{\operatorname{sgn} P_n(t)}{m \cdot E_n(\kappa_n/n)} \quad \text{for} \quad t \subset E_n(\kappa_n/n),$$

$$= 0 \qquad\qquad\qquad \text{for} \quad t \subset CE_n(\kappa_n/n).$$

Then

$$\int_0^{2\pi} |f_n(t)|\,dt = 1; \qquad \int_0^{2\pi} f_n(t)\,P_n(t)\,dt > \kappa_n\left(1-\frac{1}{n}\right).$$

Let (ϵ_i) be a diminishing sequence of positive terms such that

$$\sum_{i=1}^{\infty} \epsilon_i$$

is convergent. Then, whatever be the increasing sequence (n_i) of integers,

$$F(t) = \sum_{i=1}^{\infty} \epsilon_i f_{n_i}(t)$$

is integrable L. Just as in the proof of Lemma 2, we can show that

$$\overline{\lim_{n\to\infty}} \left| \int_0^{2\pi} F(t)\,P_n(t)\,dt \right| = \infty$$

for a suitable $F(t)$. This proves the lemma.

LEMMA 4. *If $\{G_n(t)\}$ is a sequence of bounded functions with the property that*

$$\lim_{n\to\infty} \int_0^{2\pi} G_n(t)\,X(t)\,dt$$

exists for every bounded function $X(t)$, then the functions

$$H_n(x) = \int_0^x G_n(t)\,dt$$

converge to an absolutely continuous function.

This lemma is a special case of Theorem I to be proved below.

LEMMA 5. *If $g(x)$ is a periodic function of bounded variation, and $f(x)$ is a periodic function which is integrable L, then*

$$F(x) = \int_0^{2\pi} f(x+t)\,g(t)\,dt$$

is absolutely continuous.

Write $\phi(x) = \int_0^x f(t)\,dt.$

Then
$$F(x) = \int_0^{2\pi} f(t)\, g(t-x)\, dt$$
$$= \int_0^{2\pi} g(t-x)\, d\phi(t)$$
$$= -\int_0^{2\pi} \phi(t)\, dg(t-x)$$
$$= -\int_0^{2\pi} \phi(t+x)\, dg(t).$$

Let now $(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)$ be a finite number of non-overlapping intervals contained in $(0, 2\pi)$. Then

$$\sum_{i=1}^{n} [F(\beta_i) - F(\alpha_i)] = -\int_0^{2\pi} dg(t) \left[\sum_{i=1}^{n} \phi(t+\beta_i) - \phi(t+\alpha_i) \right].$$

Since $\phi(x)$ is absolutely continuous,

$$\sum_{i=1}^{n} \phi(t+\beta_i) - \phi(t+\alpha_i)$$

tends to zero with
$$\sum_{i=1}^{n} \beta_i - \alpha_i$$

uniformly in t. Thus

$$\left| \sum_{i=1}^{n} F(\beta_i) - F(\alpha_i) \right| = o\left(\int_0^{2\pi} |dg(t)| \right)$$
$$= o(1),$$

which proves the lemma.

3. In this section we obtain the seven conditions (1, *l*) (6, 6).

The conditions (1, 2) (1, 3) (1, 4): *these are identical with the condition*

$$\sum_{n=1}^{\infty} \lambda_n \cos nx \subset \{2\}. \tag{4}$$

To prove the necessity of the condition in the three cases we need only show that, if (λ_n) transforms $\{1\}$ to $\{2\}$, then (4) is satisfied.

Let (1) be the Fourier series of any function $f(t)$ which is L integrable. Let $T_n(x)$ denote the n-th Cesàro mean of the series (3) and $\sigma_n(x)$ the corresponding mean of the series in (4). Then

$$T_n(x) = \frac{1}{\pi} \int_0^{2\pi} f(t+x)\, \sigma_n(t)\, dt.$$

By Lemma 1 (ii) it is necessary for $\{T_n(0)\}$ to be a bounded sequence; *i.e.* that

$$\left|\int_0^{2\pi} f(t)\,\sigma_n(t)\,dt\right| < \kappa. \tag{5}$$

If now $\{\sigma_n(t)\}$ is not uniformly bounded, then, since it is a sequence of continuous functions, it cannot be equivalent to a uniformly bounded sequence. By Lemma 3 there exists an L integrable function $X(t)$ such that

$$\overline{\lim} \int_0^{2\pi} X(t)\,\sigma_n(t)\,dt = \infty,$$

which contradicts (5). Hence $\{\sigma_n(t)\}$ is uniformly bounded, and so, by Lemma 1 (ii), the condition (4) is satisfied.

To prove the sufficiency of the condition in the three cases, we need only show that, if (4) holds, then (λ_n) transforms $\{1\}$ to $\{4\}$. Let

$$\sum_{n=1}^{\infty} \lambda_n \cos nx$$

be the Fourier series of the bounded function $g(x)$. Then $\{\sigma_n(x)\}$ is uniformly bounded and converges almost everywhere to $g(x)$. Hence

$$\int_0^{2\pi} f(t+x)\,\sigma_n(t)\,dt \to \int_0^{2\pi} f(t+x)\,g(t)\,dt$$

for all x, and the convergence is bounded. Thus the series (3) is summable $(C, 1)$ to the continuous function

$$\frac{1}{\pi}\int_0^{2\pi} f(t+x)\,g(t)\,dt,$$

and the Cesàro means of (3) are uniformly bounded. Hence, as in the proof of sufficiency in Lemma 1, the series (3) is the Fourier series of a continuous function.

The conditions (1, 5) (1, 6): *these are identical with the condition*

$$\sum_{n=1}^{\infty} \lambda_n \cos nx \subset \{5\}. \tag{6}$$

To prove the necessity of the condition in both cases we need only show that, if (λ_n) transforms $\{1\}$ to $\{5\}$, then (6) is satisfied. By condition (1, 4), the series in (6) is necessarily the Fourier series of the bounded function $g(t)$. If $T(x)$ denotes the sum of the series (3), then

$$T(x) = \frac{1}{\pi}\int_0^{2\pi} f(t)\,g(t-x)\,dt.$$

Consider a sequence (S_n) of sets of points, where S_n consists of ν_n pairs of points $(\alpha_i^{(n)}, \beta_i^{(n)})$ $(i = 1, 2, \ldots, \nu_n)$ which form the end points of ν_n non-overlapping intervals in $(0, 2\pi)$, where ν_n is an integer depending on n in an arbitrarily assigned manner. We shall suppose that the intervals $(\alpha_i^{(n)}, \beta_i^{(n)})$ are so chosen that the end points are points of approximate continuity of the function $g(t)$.

Write
$$T_n = \sum_{i=1}^{\nu_n} T(\beta_i^{(n)}) - T(\alpha_i^{(n)}),$$

$$g^{(n)}(t) = \sum_{i=1}^{\nu_n} g(t-\beta_i^{(n)}) - g(t-\alpha_i^{(n)}).$$

Then
$$T_n = \frac{1}{\pi} \int_0^{2\pi} f(t)\, g^{(n)}(t)\, dt.$$

Since $T(x)$ is of bounded variation, the sequence (T_n) is bounded. On the other hand, if $g(x)$ is not equivalent to a function of bounded variation, we can choose the sets S_n in such a manner that not only is $\{g^{(n)}(0)\}$ an unbounded sequence, but, in virtue of the approximate continuity at the end points of the intervals $(\alpha_i^{(n)}, \beta_i^{(n)})$, the sequence $\{g^{(n)}(t)\}$ is not equivalent to a uniformly bounded sequence. Hence, by Lemma 3, there is an L integrable $f(t)$ such that

$$\overline{\lim}\ T_n = \infty.$$

It follows that $g(x)$ is equivalent to a function of bounded variation.

To prove the sufficiency of the condition in both cases, we need only show that, if (6) is satisfied, then (3) is absolutely continuous. Since $g(t)$ is of bounded variation, $\sigma_n(t)$ tends to $g(t)$ boundedly. Hence

$$\int_0^{2\pi} f(t+x)\, \sigma_n(t)\, dt \to \int_0^{2\pi} f(t+x)\, g(t)\, dt,$$

and the convergence is bounded. Thus the series (3) is summable $(C, 1)$ to the function

$$\frac{1}{\pi} \int_0^{2\pi} f(t+x)\, g(t)\, dt,$$

which, by Lemma 5, is absolutely continuous, and the Cesàro means of (3) are uniformly bounded. It follows that (3) is the Fourier series of an absolutely continuous function.

The conditions (1, 1), (6, 6): *these are identical with the condition*

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n} \sin nx \subset \{5\}. \tag{7}$$

We need consider only the second case. To prove that (7) is necessary, let

$$\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{8}$$

be any member of $\{6\}$. Then

$$\Sigma n(b_n \cos nx - a_n \sin nx)$$

is a member of $\{1\}$ and

$$\Sigma n(a_n \cos nx + b_n \sin nx) \tag{9}$$

is a member of $\{\bar{1}\}$. Conversely, if (9) is a member of $\{\bar{1}\}$ then (8) is a member of $\{6\}$. If then (λ_n) transforms $\{6\}$ to $\{6\}$, (λ_n/n) transforms $\{\bar{1}\}$ to $\{6\}$. By condition $(\bar{1}, 6)$, (7) is satisfied.

On the other hand, if (7) is satisfied, then, by condition $(\bar{1}, 6)$, (λ_n/n) transforms $\{\bar{1}\}$ to $\{6\}$, so that (λ_n) transforms $\{6\}$ to $\{6\}$.

4. In this section we obtain the six conditions $(5, l)$.

The conditions (5, 2), (5, 3), (5, 4), (5, 5), (5, 6) *are respectively that*

$$\sum_{n=1}^{\infty} \frac{\lambda_n \sin nx}{n} \tag{10}$$

should belong respectively to $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{6\}$.

Since $$\sum_{n=1}^{\infty} \frac{\sin nx}{n} \subset \{5\}, \tag{11}$$

the condition is necessary in every case.

To prove the sufficiency, let $T_n(x)$ denote the n-th Cesàro mean of the series (3) and $M_n(x)$ the corresponding mean of (10). If (1) is the Fourier series of the function of bounded variation $f(x)$, we have

$$T_n(x) = \frac{1}{\pi} \int_0^{2\pi} f(x+t)\, M_n'(t)\, dt$$

$$= -\frac{1}{\pi} \int_0^{2\pi} M_n(t-x)\, df(t).$$

If (10) is a member of $\{2\}$, $\{3\}$, ..., $\{6\}$ respectively, then, by Lemma 1, $\{M_n(t)\}$ has the corresponding properties (ii), (iii), ..., (vi). Since

$$\int_0^{2\pi} |df(t)|$$

is finite, it follows that $\{T_n(x)\}$ has the same properties, and thus, by Lemma 1, the series (3) belongs to the corresponding class $\{2\}$, $\{3\}$, ..., $\{6\}$.

The condition (5, 1) *is*

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n} \sin nx \subset \{1\}. \tag{12}$$

The necessity of the condition follows from (11). To prove the sufficiency, let (12) be satisfied and let

$$\Sigma (a_n \cos nx + b_n \sin nx) \subset \{5\}.$$

Then

$$\Sigma (b_n \cos nx - a_n \sin nx) \subset \{\bar{5}\}$$

and

$$\Sigma \frac{\lambda_n}{n} \cdot \frac{\cos nx}{n} \subset \{6\}.$$

Hence, by condition $(\bar{5}, 6)$,

$$\Sigma \frac{\lambda_n}{n} (b_n \cos nx - a_n \sin nx) \subset \{6\},$$

and so

$$\Sigma \lambda_n (a_n \cos nx + b_n \sin nx) \subset \{1\}.$$

5. The five conditions $(6, l)$, where $l = 1, 2, \ldots, 5$, are immediate consequences of the five conditions $(\bar{1}, l)$ and do not require special treatment.

6. In this section we obtain the six conditions (2, 2), (3, 2), (4, 2), (3, 3), (4, 3), (4, 4).

The condition is identical for the six cases with

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n} \sin nx \subset \{5\}. \tag{13}$$

To prove the necessity in all six cases we need only show that, if (λ_n) transforms $\{4\}$ to $\{2\}$, then (13) holds. Let, then, (1) be the Fourier series of a continuous function $f(x)$. Let $T_n(x)$ be the n-th Cesàro mean of (3) and let $M_n(x)$ be the corresponding mean of the series in (13). Then

$$T_n(x) = \frac{1}{\pi} \int_0^{2\pi} f(x+t) M_n'(t)\, dt.$$

If (3) is a member of $\{2\}$, then, by Lemma 1 (ii), we must have

$$|T_n(x)| < \kappa,$$

and, in particular,

$$\left| \int_0^{2\pi} f(t) M_n'(t)\, dt \right| < \kappa. \tag{14}$$

But, if (13) is not satisfied, then, by Lemma 1 (iv),

$$\overline{\lim} \int_0^{2\pi} |M_n'(t)|\, dt = \infty.$$

There then exists, by Lemma 2, a periodic continuous function $\psi(x)$ such that

$$\overline{\lim} \int_0^{2\pi} \psi(t)\, M_n'(t)\, dt = \infty.$$

Since this contradicts (14), it follows that (13) is satisfied.

To prove the sufficiency in the six cases, we need only show that, if (13) holds, then (λ_n) transforms $\{2\}$ to $\{2\}$, $\{3\}$ to $\{3\}$, and $\{4\}$ to $\{4\}$. If

$$\Sigma\,(a_n \cos nx + b_n \sin nx) \subset \{2\},$$

then

$$\Sigma\, nb_n \cdot \frac{\sin nx}{n} \subset \{2\},$$

and so, by condition (5, 2),

$$\Sigma\, \lambda_n b_n \sin nx \subset \{2\}.$$

Further,

$$\Sigma\, \frac{\lambda_n}{n} \cos nx \subset \{\bar{5}\}$$

and

$$\Sigma\, na_n \cdot \frac{\cos nx}{n} \subset \{2\},$$

so that, by condition $(\bar{5}, 2)$,

$$\Sigma\, \lambda_n a_n \cos nx \subset \{2\}.$$

Hence

$$\Sigma\, \lambda_n (a_n \cos nx + b_n \sin nx) \subset \{2\}.$$

The proof for the classes $\{3\}$ and $\{4\}$ is precisely similar.

7. In this section we obtain the condition (2, 4).

The condition (2, 4) *is*

$$\sum_{n=1}^{\infty} \lambda_n \cos nx \subset \{1\}. \tag{15}$$

To prove the necessity, let (1) be the Fourier Series of a bounded function $f(t)$. Let $T_n(x)$ denote the n-th Cesàro mean of the series (3), and let $\sigma_n(x)$ be the corresponding mean of the series in (15). Then

$$T_n(x) = \frac{1}{\pi} \int_0^{2\pi} f(x+t)\, \sigma_n(t)\, dt. \tag{16}$$

If (3) is a member of $\{4\}$ it is necessary that $\lim T_n(0)$ should exist for every bounded $f(t)$, *i.e.* that

$$\lim \int_0^{2\pi} f(t)\,\sigma_n(t)\,dt$$

should exist. By Lemma 4,

$$\lim \int_0^x \sigma_n(t)\,dt$$

is an absolutely continuous function, and further, by Lemma 2,

$$\int_0^{2\pi} |\sigma_n(t)|\,dt < \kappa.$$

Hence

$$\Sigma \frac{\lambda_n}{n} \sin nx$$

is summable $(C, 1)$ to an absolutely continuous function, and its Cesàro means are uniformly bounded. Hence it is the Fourier series of an absolutely continuous function and (15) is satisfied.

To prove that the condition is sufficient, let the series in (15) be the Fourier series of the L integrable function $g(t)$. Then, if $M_n(x)$ denote the n-th Cesàro mean of (1), we have

$$T_n(x) = \frac{1}{\pi} \int_0^{2\pi} M_n(t)\,g(t-x)\,dt.$$

Since

$$|M_n(t)| < \kappa$$

and $M_n(t)$ converges almost everywhere to $f(t)$,

$$\lim T_n(x) = \frac{1}{\pi} \int_0^{2\pi} f(t)\,g(t-x)\,dt$$

for all x, and the convergence is bounded. The last expression is a continuous function of x. Thus (3) is summable $(C, 1)$ to a continuous function and its Cesàro means are uniformly bounded. It is, therefore, the Fourier series of a continuous function.

8. THEOREM I. *If $\{G_n(t)\}$ is a sequence of bounded functions with the property that*

$$\lim \int_0^{2\pi} G_n(t)\,X(t)\,dt \tag{17}$$

exists for every R integrable function $X(t)$, then the functions

$$H_n(x) = \int_0^x G_n(t)\,dt$$

converge to an absolutely continuous function.

The proof consists of four parts.

(i) We first observe that

$$\int_0^{2\pi} |G_n(t)|\,dt < \kappa.$$

For by hypothesis, (17) exists for every continuous $X(t)$, and the result follows by Lemma 2.

(ii) Thus the functions $H_n(x)$ are of uniformly bounded variation. Consider the R integrable function $X(t)$ defined by $X(t) = 1$ for $0 \leqslant t \leqslant x$, $X(t) = 0$ for $x < t \leqslant 2\pi$. Then

$$H(x) = \lim \int_0^x G_n(t)\,dt \tag{18}$$

exists for all x. Thus $H(x)$ is of bounded variation.

(iii) We next prove that $H(x)$ is continuous. If not, it has a point of ordinary discontinuity, say ξ. We may without loss of generality suppose that

$$H(\xi+0) - H(\xi) > \delta,$$

where δ is a positive number. We can find a diminishing sequence (ϵ_n) tending to zero such that

$$\int_\xi^{\xi+\epsilon_n} G_{n_r}(t)\,dt > \delta,$$

where n_r is an integer depending on n. By selecting this subsequence $G_{n_r}(t)$ and altering the notation, we may suppose that

$$\int_\xi^{\xi+\epsilon_n} G_n(t)\,dt > \delta. \tag{19}$$

We shall construct an R integrable function $X(t)$ satisfying

$$|X(t)| \leqslant 1, \tag{20}$$

such that (17) does not exist. This will prove that $H(x)$ is continuous.

In $(0, \xi)$, and in $(\xi+\epsilon_1, 2\pi)$, put $X(t)=0$. Let

$$\int_{\xi}^{\xi+\epsilon_1} G_1(t)\,dt = l_1,$$

where $l_1 > \delta$. We can find an n_1 such that

$$\int_{\xi}^{\xi+\epsilon_{n_1}} |G_1(t)|\,dt < \epsilon,$$

where $\epsilon < \frac{1}{2}\delta$. In $(\xi+\epsilon_{n_1}, \xi+\epsilon_1)$ put $X(t)=1$. Then, whatever be the definition of $X(t)$ in $(\xi, \xi+\epsilon_{n_1})$ subject to (20), we have

$$\left|\int_0^{2\pi} G_1(t)\,X(t)\,dt - l_1\right| < \epsilon. \tag{21}$$

Now
$$\lim \int_{\xi+\epsilon_{n_1}}^{\xi+\epsilon_1} G_n(t)\,X(t)\,dt = l_2, \tag{22}$$

say. We can find an $N_1 > n_1$ such that

$$\left|\int_{\xi+\epsilon_{n_1}}^{\xi+\epsilon_1} G_{N_1}(t)\,X(t)\,dt - l_2\right| < \tfrac{1}{2}\epsilon. \tag{23}$$

Further
$$\int_{\xi}^{\xi+\epsilon_{N_1}} G_{N_1}(t)\,dt > \delta.$$

We can find an $n_2 > N_1$ such that

$$\int_{\xi}^{\xi+\epsilon_{n_2}} |G_{N_1}(t)|\,dt < \tfrac{1}{2}\epsilon.$$

Put $X(t)=0$ in $(\xi+\epsilon_{N_1}, \xi+\epsilon_{n_1})$. In $(\xi+\epsilon_{n_2}, \xi+\epsilon_{N_1})$ we can, by choosing $X(t)=+1$ or $X(t)=-1$, alter the value of

$$\int_{\xi+\epsilon_{n_1}}^{\xi+\epsilon_{N_1}} G_{N_1}(t)\,X(t)\,dt$$

by more than $2\delta-\epsilon$. Hence we can alter the value of

$$\int_{\xi}^{\xi+\epsilon_{N_1}} G_{N_1}(t)\,X(t)\,dt$$

by more than $2\delta-2\epsilon$ whatever be the definition of $X(t)$ in $(\xi, \xi+\epsilon_{n_2})$. Since now $X(t)$ is given in $(\xi+\epsilon_{n_1}, \xi+\epsilon_1)$, we can alter the value of

$$\int_{\xi}^{\xi+\epsilon_1} G_{N_1}(t)\,X(t)\,dt \tag{24}$$

by more than $2\delta-2\epsilon$. Hence we can make (24) differ from any assigned number, l_1 say, by more than $\delta-\epsilon$; *i.e.* by (21) we can ensure that

$$\left|\int_0^{2\pi} G_1(t)\,X(t)\,dt-\int_0^{2\pi} G_{N_1}(t)\,X(t)\,dt\right|>\delta-2\epsilon,$$

whatever be the definition of $X(t)$ in $(\xi,\ \xi+\epsilon_{n_2})$. This process can clearly be repeated so as to obtain a sequence (N_r) such that

$$\left|\int_2^{2\pi} G_{N_r}(t)\,X(t)\,dt-\int_0^{2\pi} G_{N_{r+1}}(t)\,X(t)\,dt\right|>\delta-2\epsilon.$$

Further $X(t)$ is integrable R, for it has only an enumerable set of discontinuities. Thus (17) does not exist, and hence the result follows.

(iv) We now prove that $H(x)$ is absolutely continuous.

Let $P(x)$, $N(x)$ denote the positive and negative variations of $H(x)$ in $(0,\ 2\pi)$. Then, if $H(x)$ is not absolutely continuous, at least one of $P(x)$, $N(x)$ is not absolutely continuous.

If $P(x)$ is not absolutely continuous, there exists a set E_1 of measure zero with the following properties. If $\{(\alpha_n,\ \beta_n)\}$ denotes any enumerable set of non-overlapping intervals which enclose E_1, then there is a fixed positive number η such that

$$\sum_n P(\beta_n)-P(\alpha_n)>\eta.$$

Further, if $\{(\alpha_n,\ \beta_n)\}$ denotes any set of non-overlapping intervals which enclose any set E of measure zero which has no point in common with E_1, then

$$\sum_n P(\beta_n)-P(\alpha_n)$$

tends to zero with $\qquad \sum_n \beta_n-\alpha_n.$

If $N(x)$ is not absolutely continuous, there is a set E_2 of measure zero with the following properties. The set E_2 has no point in common with E_1. If $\{(\alpha_n,\ \beta_n)\}$ denotes any set of non-overlapping intervals which enclose E_2, then there is a fixed positive number ζ such that

$$\sum_{(n)} N(\beta_n)-N(\alpha_n)>\zeta.$$

Further, if $\{(\alpha_n, \beta_n)\}$ denotes any set of non-overlapping intervals which enclose a set E of measure zero which has no point in common with E_2, then

$$\sum_{(n)} N(\beta_n)-N(\alpha_n)$$

tends to zero with $\sum_{(n)} (\beta_n - \alpha_n)$.

If $H(x)$ is not absolutely continuous, we may without loss of generality suppose that $P(x)$ is not absolutely continuous. Then, since

$$H(\beta) - H(\alpha) = [P(\beta) - P(\alpha)] - [N(\beta) - N(\alpha)],$$

it follows that there exists a set E of measure zero with the following properties. If $\{(\alpha_n, \beta_n)\}$ denotes any set of non-overlapping intervals which enclose E, then

$$\sum_n H(\beta_n) - H(\alpha_n) > \eta$$

for all sufficiently small $\sum_n (\beta_n - \alpha_n)$

where η is a fixed positive number.

Let $\{(\alpha_n^{(1)}, \beta_n^{(1)})\}$ be such a set of non-overlapping intervals. Then, if we denote by $V_n^{(1)}$ the total variation of $H(x)$ in $(\alpha_n^{(1)}, \beta_n^{(1)})$, since

$$\sum_{n=1}^{\infty} V_n^{(1)}$$

is convergent, we can find an integer r_1 such that

$$\sum_{n=r_1+1}^{\infty} V_n^{(1)} < \tfrac{1}{4}\eta.$$

If we write $e_1 = \sum_{n=1}^{r_1} (\alpha_n^{(1)}, \beta_n^{(1)})$,

then the total variation of $P(x)$ on Ee_1 will exceed $\eta - \frac{1}{4}\eta$. Thus, if $\{(\alpha_n^{(2)}, \beta_n^{(2)})\}$ is a set of non-overlapping intervals contained in e_1 and containing Ee_1, we shall have

$$\sum_{(n)} H(\beta_n^{(2)}) - H(\alpha_n^{(2)}) > \eta - \tfrac{1}{4}\eta$$

for all sufficiently small $\sum_{(n)} (\beta_n^{(2)} - \alpha_n^{(2)})$.

If we denote by $V_n^{(2)}$ the total variation of $H(x)$ in $(\alpha_n^{(2)}, \beta_n^{(2)})$, then, since

$$\sum_{n=1}^{\infty} V_n^{(2)}$$

is convergent, we can find an integer r_2 such that

$$\sum_{n=r_2+1}^{\infty} V_n^{(2)} < \tfrac{1}{8}\eta.$$

If we write $$e_2 = \sum_{n=1}^{r_2} (\alpha_n^{(2)}, \beta_n^{(2)}),$$

then the total variation of $P(x)$ on Ee_2 will exceed $\eta - \frac{1}{4}\eta - \frac{1}{8}\eta$.

Proceeding in this manner, we construct a sequence (e_n) of sets with the following properties:

(a) e_n consists of a finite number of non-overlapping intervals.

(b) $$e_n \supset e_{n+1}.$$

(c) $$\lim me_n = 0.$$

(d) If the constituent intervals of e_n are

$$(\alpha_1^{(n)}, \beta_1^{(n)}), \quad \ldots, \quad (\alpha_{n_r}^{(n)}, \beta_{n_r}^{(n)}),$$

then $$\sum_{i=1}^{n_r} H(\beta_i^{(n)}) - H(\alpha_i^{(n)}) > \tfrac{1}{2}\eta.$$

Let $\gamma(t) = 1$ in e_n and 0 in ce_n for any particular n. Then $\gamma(t)$ is R integrable. Hence

$$\lim_{i \to \infty} \int_{e_n} G_i(t)\, dt > \tfrac{1}{2}\eta.$$

To each n we can find a number $\nu(n)$ such that

$$\int_{e_n} G_{\nu(n)}(t)\, dt > \tfrac{1}{2}\eta.$$

By selecting the subsequence $\{G_{\nu(n)}(t)\}$ and altering the notation, we may suppose that

$$\int_{e_n} G_n(t)\, dt > \eta. \tag{25}$$

We shall construct a function $X(t)$ which is R integrable, which satisfies

$$|X(t)| \leqslant 1, \tag{26}$$

and for which (17) does not exist. This will prove that $H(x)$ is absolutely integrable.

The method is precisely similar to that used in (iii). In Ce_1 put $X(t) = 0$. Let

$$\int_{e_1} G_1(t)\, dt = l_1,$$

where $l_1 > \eta$. We can find an integer n_1 such that

$$\int_{e_{n_1}} |G_1(t)|\,dt < \epsilon,$$

where $\epsilon < \frac{1}{2}\eta$. In $e_1 - e_{n_1}$ put $X(t) = 1$. Then

$$\left| \int_0^{2\pi} G_1(t)\,X(t)\,dt - l_1 \right| < \epsilon, \tag{27}$$

whatever be the definition of $X(t)$ in e_{n_1} subject to (26). Now

$$\lim \int_{e_1 - e_{n_1}} G_n(t)\,X(t)\,dt = l_2,$$

say. We can find an $N_1 > n_1$ such that

$$\left| \int_{e_1 - e_{n_1}} G_{N_1}(t)\,X(t)\,dt - l_2 \right| < \tfrac{1}{2}\epsilon.$$

Further

$$\int_{e_{N_1}} G_{N_1}(t)\,dt > \eta.$$

We can find an $n_2 > N_1$ such that

$$\int_{e_{n_2}} |G_{N_1}(t)|\,dt < \tfrac{1}{2}\epsilon.$$

Put $X(t) = 0$ in $e_{n_1} - e_{N_1}$. In $e_{N_1} - e_{n_2}$, we can, by choosing $X(t) = 1$ or $X(t) = -1$, make

$$\int_{e_{N_1} - e_{n_2}} G_{N_1}(t)\,X(t)\,dt$$

have one of two values which differ by more than $2\eta - \epsilon$. Hence, whatever be the definition of $X(t)$ in e_{n_2} subject to (26), we can make

$$\int_{e_{n_1}} G_{N_1}(t)\,X(t)\,dt$$

change by more than $2\eta - \epsilon$. Since $X(t)$ is given in $e_1 - e_{n_1}$, we can make the value of

$$\int_0^{2\pi} G_{N_1}(t)\,X(t)\,dt$$

differ from an assigned number, l_2 say, by more than $\eta - \epsilon$. Hence, by (27),

$$\left| \int_0^{2\pi} G_1(t)\,X(t)\,dt - \int_0^{2\pi} G_{N_1}(t)\,X(t)\,dt \right| > \eta - 2\epsilon.$$

This process may clearly be continued so as to give a sequence (N_r) such that

$$\left| \int_0^{2\pi} G_{N_r}(t) X(t)\, dt - \int_0^{2\pi} G_{N_{r+1}}(t) X(t)\, dt \right| > \eta - 2\epsilon.$$

The function $X(t)$ thus defined can only be discontinuous at the enumerable set of end points of the intervals constituting e_n for all n, and at the points of $\lim e_n$ which is of measure zero. Hence $X(t)$ is R integrable. This completes the proof.

9. In this section we obtain the condition (3, 4).

The condition (3, 4) *is*

$$\sum_{n=1}^{\infty} \lambda_n \cos nx \subset \{1\}. \tag{28}$$

The sufficiency of the condition follows *a fortiori* from condition (2, 4). To prove that the condition is necessary, let (1) be the Fourier series of the R integrable function $f(x)$, and let (3) be the Fourier series of a continuous function. Then, if $T_n(x)$ denotes the n-th Cesàro mean of (3) and $\sigma_n(x)$ the corresponding mean of the series in (28), we have

$$T_n(x) = \frac{1}{\pi} \int_0^{2\pi} f(x+t)\, \sigma_n(t)\, dt.$$

Now $\lim T_n(x)$

exists for all x, so that, in particular,

$$\lim \int_0^{2\pi} f(t)\, \sigma_n(t)\, dt$$

must exist for every R integrable function $f(t)$. By Theorem 1 this requires that

$$\lim \int_0^x \sigma_n(t)\, dt$$

should be absolutely continuous, and further, by Lemma 2, that

$$\int_0^{2\pi} |\sigma_n(t)|\, dt < \kappa.$$

Hence $\Sigma \frac{\lambda_n}{n} \sin nx$

is summable $(C, 1)$ to an absolutely continuous function, and the Cesàro

means of this series are uniformly bounded. It is, therefore, the Fourier series of an absolutely continuous function. Hence (28) is satisfied.

10. In this section we obtain the five conditions (2, 5), (3, 5), (4, 5), (4, 6), (4, 1).

We first prove that the conditions (2, 5), (3, 5), (4, 5), (4, 6) are identical. To this end we must show that if any one of the four events is produced by (λ_n), then the other three events will occur.

(i) Let (λ_n) transform $\{4\}$ to $\{6\}$. Then (λ_n) transforms $\{4\}$ to $\{5\}$. Let (μ_n) be chosen in any manner so that

$$\Sigma \mu_n \cos nx \subset \{2\}.$$

Then, by condition (1, 4), (μ_n) transforms $\{1\}$ to $\{4\}$. Hence $(\lambda_n \mu_n)$ transforms $\{1\}$ to $\{6\}$. By condition (1, 6),

$$\Sigma \lambda_n \mu_n \cos nx \subset \{5\}.$$

Thus (λ_n) transforms every even member of $\{2\}$ into a member of $\{5\}$.

Next, let (μ_n) be chosen in any manner so that

$$\Sigma \mu_n \sin nx \subset \{2\}.$$

Then (μ_n) transforms $\{\bar{1}\}$ to $\{4\}$, by condition $(\bar{1}, 4)$. Hence $(\lambda_n \mu_n)$ transforms $\{\bar{1}\}$ to $\{6\}$. By condition $(\bar{1}, 6)$,

$$\Sigma \lambda_n \mu_n \sin nx \subset \{5\}.$$

Thus (λ_n) transforms every odd member of $\{2\}$ into a member of $\{5\}$, Hence (λ_n) transforms $\{2\}$ to $\{5\}$, and *a fortiori* it transforms $\{3\}$ to $\{5\}$.

(ii) Let (λ_n) transform $\{2\}$ to $\{5\}$. Then (λ_n) transforms $\{3\}$ to $\{5\}$ and $\{4\}$ to $\{5\}$. Let (μ_n/n) be chosen in any manner so that

$$\Sigma \frac{\mu_n \sin nx}{n} \subset \{4\}.$$

Then, by condition (5, 4), (μ_n) transforms $\{5\}$ to $\{4\}$. Hence $(\lambda_n \mu_n)$ transforms $\{2\}$ to $\{4\}$. By condition (2, 4),

$$\Sigma \lambda_n \mu_n \cos nx \subset \{1\}.$$

But

$$\Sigma \frac{\mu_n}{n} \cos nx$$

can represent any even member of $\{\bar{4}\}$. Hence $(n\lambda_n)$ transforms every even member of $\{\bar{4}\}$ into a member of $\{1\}$.

Next, let (μ_n/n) be chosen in any manner so that

$$\Sigma \frac{\mu_n}{n} \cos nx \subset \{4\}.$$

Then, by condition $(\bar{5}, 4)$, (μ_n) transforms $\{\bar{5}\}$ to $\{4\}$. But (λ_n) transforms $\{\bar{2}\}$ to $\{\bar{5}\}$. Hence $(\lambda_n \mu_n)$ transforms $\{\bar{2}\}$ to $\{4\}$. By condition $(\bar{2}, 4)$,

$$\Sigma \lambda_n \mu_n \sin nx \subset \{1\}.$$

But

$$\Sigma \frac{\mu_n}{n} \sin nx$$

can represent any odd member of $\{\bar{4}\}$. Hence $(n\lambda_n)$ transforms every odd member of $\{\bar{4}\}$ into a member of $\{1\}$. Thus $(n\lambda_n)$ transforms $\{\bar{4}\}$ to $\{1\}$.

Writing

$$A_n(x) = a_n \cos nx + b_n \sin nx,$$

$$B_n(x) = b_n \cos nx - a_n \sin nx,$$

then, if

$$\Sigma A_n(x) \subset \{4\},$$

we have

$$\Sigma B_n(x) \subset \{\bar{4}\}$$

and

$$\Sigma n\lambda_n B_n(x) \subset \{1\},$$

so that

$$\Sigma \lambda_n A_n(x) \subset \{6\}.$$

Thus (λ_n) transforms $\{4\}$ to $\{6\}$.

(iii) Let (λ_n) transform $\{4\}$ to $\{5\}$. Then, if (μ_n/n) is chosen in any manner so that

$$\Sigma \frac{\mu_n}{n} \sin nx \subset \{2\},$$

the sequence (μ_n) transforms $\{6\}$ to $\{4\}$ by condition $(6, 4)$. Hence $(\lambda_n \mu_n)$ transforms $\{6\}$ to $\{5\}$. By condition $(6, 5)$,

$$\Sigma \lambda_n \frac{\mu_n \sin nx}{n} \subset \{5\}.$$

But

$$\Sigma \frac{\mu_n}{n} \sin nx$$

can represent any odd member of $\{2\}$. Hence (λ_n) transforms every odd member of $\{2\}$ into a member of $\{5\}$. In a similar manner we can show that (λ_n) transforms every even member of $\{2\}$ into a member of $\{5\}$. Thus (λ_n) transforms $\{2\}$ to $\{5\}$. By case (ii), all four events happen.

(iv) Let (λ_n) transform $\{3\}$ to $\{5\}$. Then (λ_n) transforms $\{4\}$ to $\{5\}$, so that, by case (iii), all four events happen.

This proves that the four conditions are identical. To enunciate the condition we require the following definition.

A periodic function $g(t)$ is of bounded variation in mean, if there is a constant κ such that, if (α_1, β_1), ..., (α_n, β_n) is any finite set of non-overlapping intervals in $(0, 2\pi)$, then

$$\int_0^{2\pi} \left| \sum_{i=1}^{n} \{g(t-\beta_i)-g(t-\alpha_i)\} \right| dt < \kappa.$$

The conditions (2, 5), (3, 5), (4, 5), (4, 6) *are identical with the condition that*

$$\sum_{n=1}^{\infty} \lambda_n \cos nx \tag{29}$$

is the Fourier series of the integrable function $g(t)$ which is of bounded variation in mean.

In virtue of what precedes, we need only show that the condition is necessary and sufficient in case (2, 5).

Suppose first that (λ_n) transforms $\{2\}$ to $\{5\}$. Let (1) be the Fourier series of a bounded function $f(t)$, and let $T_n(x)$, $\sigma_n(x)$ denote the n-th Cesàro means of (3) and (29). Then

$$T_n(x) = \frac{1}{\pi} \int_0^{2\pi} f(t+x)\, \sigma_n(t)\, dt.$$

Since (3) is of bounded variation, $\lim T_n(0)$ exists, so that

$$\lim \int_0^{2\pi} f(t)\, \sigma_n(t)\, dt$$

exists for every bounded $f(t)$. Hence, by Lemma 4,

$$\lim \int_0^{x} \sigma_n(t)\, dt$$

is absolutely continuous. Thus

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n} \sin nx$$

is summable $(C, 1)$ to an absolutely continuous function. Further, by Lemma 2,

$$\int_0^{2\pi} |\sigma_n(t)|\, dt < \kappa,$$

so that the Cesàro means of the series in question are uniformly bounded. Hence (25) is the Fourier series of an integrable function $g(t)$. If now we denote by $\tau_n(x)$ the n-th Cesàro mean of (1), we have

$$T_n(x) = \frac{1}{\pi}\int_0^{2\pi} \tau_n(t+x)\, g(t)\, dt.$$

Since $\{\tau_n(x)\}$ is uniformly bounded and converges almost everywhere to $f(x)$, we have

$$\begin{aligned} T(x) &= \lim T_n(x) \\ &= \frac{1}{\pi}\int_0^{2\pi} f(x+t)\, g(t)\, dt \\ &= \int_0^{2\pi} f(t)\, g(t-x)\, dt. \end{aligned} \tag{30}$$

If now $g(t)$ is not of bounded variation in mean, we can find a sequence of sets S_n with the following properties. S_n consists of ν_n pairs of points, where ν_n is a number depending on n, which are the end points of ν_n non-overlapping intervals $(\alpha_1^{(n)}, \beta_1^{(n)})$, ..., $(\alpha_{\nu_n}^{(n)}, \beta_{\nu_n}^{(n)})$ in $(0, 2\pi)$, and

$$\overline{\lim_{n\to\infty}} \int_0^{2\pi} \left| \sum_{i=1}^{\nu_n} \{g(t-\beta_i^{(n)}) - g(t-\alpha_i^{(n)})\} \right| dt = \infty.$$

If we write

$$g_n(t) = \sum_{i=1}^{\nu_n} \{g(t-\beta_i^{(n)}) - g(t-\alpha_i^{(n)})\},$$

then

$$\overline{\lim} \int_0^{2\pi} |g_n(t)|\, dt = \infty.$$

Hence, by Lemma 2, there is a continuous periodic function $f(t)$ such that

$$\overline{\lim} \left| \int_0^{2\pi} f(t)\, g_n(t)\, dt \right| = \infty.$$

But

$$\int_0^{2\pi} f(t)\, g_n(t)\, dt = \sum_{i=1}^{\nu_n} T(\beta_i^{(n)}) - T(\alpha_i^{(n)}),$$

so that

$$\overline{\lim} \left| \sum_{i=1}^{\nu_n} T(\beta_i^{(n)}) - T(\alpha_i^{(n)}) \right| = \infty,$$

which contradicts the fact that $T(x)$ is of bounded variation. This proves that the condition is necessary.

To prove that the condition is sufficient, we observe that, since (29) is the Fourier series of the L integrable function $g(t)$, the relation (30)

holds. We then have, if (α_1, β_1), ..., (α_n, β_n) denote any finite set of non-overlapping intervals in $(0, 2\pi)$.

$$\left| \sum_{i=1}^{n} T(\beta_i) - T(\alpha_i) \right| = \frac{1}{\pi} \left| \int_0^{2\pi} f(t) \sum_{i=1}^{n} \{g(t-\beta_i) - g(t-\alpha_i)\}\, dt \right|$$

$$\leqslant \frac{\kappa}{\pi} \operatorname{Max} |f(t)|,$$

so that $T(x)$ is of bounded variation.

The condition (4, 1) *is that*

$$\Sigma \frac{\lambda_n}{n} \sin nx$$

is the Fourier series of the integrable function $g(t)$ *which is of bounded variation in mean.*

For if (λ_n) transforms $\{4\}$ to $\{1\}$, then (λ_n/n) transforms $\{\overline{4}\}$ to $\{6\}$ and conversely. The result therefore follows from condition $(\overline{4}, 6)$.

If a finite measurable periodic function $g(t)$ is of bounded variation in mean, then it is necessarily L integrable. For if in the definition we take for the finite set of non-overlapping intervals the single interval $(0, x)$, where $0 < x < 2\pi$, we have

$$\int_0^{2\pi} |g(t-x) - g(t)|\, dt < \kappa.$$

Hence
$$\int_0^{2\pi} dx \int_0^{2\pi} |g(t-x) - g(t)|\, dt < 2\pi\kappa,$$

and so
$$\int_0^{2\pi} dt \int_0^{2\pi} |g(t-x) - g(t)|\, dx < 2\pi\kappa.$$

This means that the function of x,

$$g(t-x) - g(t),$$

is absolutely integrable for almost all t. Let t_0 be a value of t for which it is integrable. Then

$$\int_0^{2\pi} |g(t_0-x) - g(t_0)|\, dx < \infty,$$

and so
$$\int_0^{2\pi} |g(t_0-x)|\, dx < \infty,$$

or, by the periodicity of $g(x)$,

$$\int_0^{2\pi} |g(x)|\,dx < \infty.$$

We say that a sequence $\{g_r(t)\}$ of periodic functions is of uniformly bounded variation in mean if each function is of bounded variation in mean and if there is a single constant κ satisfying the required conditions for every $g_r(t)$.

We can prove the following result.

LEMMA 6. *The condition* (2, 5) *can be expressed as follows*:

$$\sum_{n=1}^{\infty} \lambda_n \cos nx$$

has its Cesàro means of uniformly bounded variation in mean.

To prove that this condition is sufficient, let $T_r(x)$, $\sigma_r(x)$ be the r-th Cesàro means of the series (3) and (29). Let (1) be the Fourier series of a bounded function $f(x)$. Then

$$T_r(x) = \frac{1}{\pi} \int_0^{2\pi} f(t)\,\sigma_r(t-x)\,dt.$$

If now $(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)$ denote any finite set of non-overlapping intervals in $(0, 2\pi)$, we have

$$\left| \sum_{i=1}^{n} T_r(\beta_i) - T_r(\alpha_i) \right| \leqslant \frac{1}{\pi} \int_0^{2\pi} |f(t)| \left| \sum_{i=1}^{n} \{\sigma_r(t-\beta_i) - \sigma_r(t-\alpha_i)\} \right| dt$$

$$\leqslant \frac{\kappa}{\pi} \operatorname{Max} |f(t)|.$$

It follows that the sequence $\{T_r(x)\}$ is of uniformly bounded variation, and hence, by Lemma 1 (v), that (3) is the Fourier series of a function of bounded variation.

To prove that the condition of the lemma is necessary, suppose that (λ_n) transforms $\{2\}$ to $\{5\}$. Then $\{T_r(x)\}$ is of uniformly bounded variation. If now $\{\sigma_r(t)\}$ is not of uniformly bounded variation in mean, we can select a subsequence $\{\sigma_n(t)\}$ with the property that there exists a sequence (S_n) of sets, where S_n consists of ν_n pairs of points $(\alpha_i^{(n)}, \beta_i^{(n)})$ $(i = 1, 2, \ldots, \nu_n)$ forming the end points of ν_n non-overlapping intervals in $(0, 2\pi)$, such that

$$\overline{\lim} \int_0^{2\pi} \left| \sum_{i=1}^{\nu_n} \{\sigma_n(t-\beta_i^{(n)}) - \sigma_n(t-\alpha_i^{(n)})\} \right| dt = \infty.$$

Writing $$g_n(t) = \sum_{i=1}^{\nu_n} \{\sigma_n(t-\beta_i^{(n)}) - \sigma_n(t-\alpha_i^{(n)})\},$$

we have $$\overline{\lim} \int_0^{2\pi} |g_n(t)|\, dt = \infty,$$

so that, by Lemma 2, there is a continuous periodic function $f(t)$ such that

$$\overline{\lim} \left| \int_0^{2\pi} f(t)\, g_n(t)\, dt \right| = \infty;$$

i.e. such that

$$\overline{\lim} \left| \sum_{i=1}^{\nu_n} T_n(\beta_i^{(n)}) - T_n(\alpha_i^{(n)}) \right| = \infty.$$

This contradicts the fact that $\{T_r(x)\}$ is of uniformly bounded variation.

By Lemma 6 and the original condition (2, 5) we can enunciate

THEOREM II. *The necessary and sufficient condition for*

$$\Sigma\, (a_n \cos nx + b_n \sin nx)$$

to be the Fourier series of a function which is of bounded variation in mean is that the Cesàro means of the series are uniformly of bounded variation in mean.

11. In this section we obtain the four conditions (2, 6), (3, 6), (2, 1), (3, 1).

We first prove that the conditions (2, 6), (3, 6) are identical. If (λ_n) transforms $\{2\}$ to $\{6\}$, then (λ_n) transforms $\{3\}$ to $\{6\}$. Suppose, conversely, that (λ_n) transforms $\{3\}$ to $\{6\}$. Let (μ_n/n) be chosen in any manner so that

$$\Sigma \frac{\mu_n}{n} \sin nx \subset \{2\}.$$

Then, by condition (6, 4), (μ_n) transforms $\{6\}$ to $\{4\}$. Hence $(\lambda_n \mu_n)$ transforms $\{3\}$ to $\{4\}$. By condition (3, 4),

$$\Sigma \lambda_n \mu_n \cos nx \subset \{1\},$$

and so

$$\Sigma \lambda_n \frac{\mu_n}{n} \sin nx \subset \{6\}.$$

Since $$\Sigma \frac{\mu_n}{n} \sin nx$$

may represent any odd member of $\{2\}$, it follows that (λ_n) transforms every odd member of $\{2\}$ into a member of $\{6\}$. In a similar manner we can show that (λ_n) transforms every even member of $\{2\}$ into a member of $\{6\}$. Hence (λ_n) transforms $\{2\}$ to $\{6\}$. This proves that the two conditions are identical.

We now introduce a definition. *If $g(t)$ is a periodic function such that, given $\eta > 0$, there is an $\epsilon > 0$ with the property that, given any finite set of non-overlapping intervals (α_1, β_1), ..., (α_n, β_n) in $(0, 2\pi)$, such that*

$$\sum_{i=1}^{n} \beta_i - \alpha_i \leqslant \epsilon,$$

then

$$\int_0^{2\pi} \left| \sum_{i=1}^{n} \{g(t-\beta_i) - g(t-\alpha_i)\} \right| dt \leqslant \eta,$$

we say that $g(t)$ is absolutely continuous in mean.

The conditions (2, 6), (3, 6) *are identical with the condition that*

$$\sum_{n=1}^{\infty} \lambda_n \cos nx \tag{31}$$

is the Fourier series of an integrable function $g(t)$ which is absolutely continuous in mean.

We need only consider the case (2, 6). To show that the condition is necessary, let (λ_n) transform $\{2\}$ to $\{6\}$. Then, by condition (2, 4), it is necessary for the series (31) to be the Fourier series of an integrable function $g(t)$. If $f(t)$ denotes any bounded function whose Fourier series is (1), then, by the same argument as for (30), we infer that the sum $T(x)$ of the series (3) is

$$T(x) = \frac{1}{\pi} \int_0^{2\pi} f(t)\, g(t-x)\, dt. \tag{32}$$

If now $g(t)$ is not absolutely continuous in mean, there is a sequence (S_n) of sets of points with the following property. S_n consists of ν_n pairs of points, where ν_n is a number depending on n, which are the end points of ν_n non-overlapping intervals $(\alpha_1^{(n)}, \beta_1^{(n)})$, ..., $(\alpha_{\nu_n}^{(n)}, \beta_{\nu_n}^{(n)})$ in $(0, 2\pi)$, such that

$$\lim_{n \to \infty} \sum_{i=1}^{\nu_n} \beta_i^{(n)} - \alpha_i^{(n)} = 0$$

and

$$\int_0^{2\pi} \left| \sum_{i=1}^{\nu_n} g(t-\beta_i^{(n)}) - g(t-\alpha_i^{(n)}) \right| dt > \eta.$$

We write $$g_n(t) = \sum_{i=1}^{\nu_n} \{g(t-\beta_i^{(n)}) - g(t-\alpha_i^{(n)})\}.$$

Then, since

$$\frac{1}{\pi}\int_0^{2\pi} f(t)\, g_n(t)\, dt = \sum_{i=1}^{\nu_n} T(\beta_i^{(n)}) - T(\alpha_i^{(n)}),$$

and, since $T(x)$ is absolutely continuous for every bounded $f(t)$, we have

$$\lim_{n\to\infty} \int_0^{2\pi} f(t)\, g_n(t)\, dt = 0 \tag{33}$$

for every bounded $f(t)$.

If we select a fixed value of n which we call ν, then, taking

$$f(t) = \operatorname{sgn} g_\nu(t),$$

we have, by (33),

$$\lim_{n\to\infty} \int_0^{2\pi} \{\operatorname{sgn} g_\nu(t)\}\, g_n(t)\, dt = 0. \tag{34}$$

Further, $$\int_0^{2\pi} \{\operatorname{sgn} g_\nu(t)\}\, g_\nu(t)\, dt > \eta. \tag{35}$$

If we denote by E_n the set in which

$$|g_n(t)| > \eta/(16\pi),$$

then (34) and (35) imply that

$$\lim_{n\to\infty} m(E_\nu . E_n) = 0. \tag{36}$$

Since $g_n(t)$ is integrable, there corresponds to each n an ϵ_n such that, if $\mathfrak{E}$ denote any set of measure not exceeding ϵ_n, then

$$\int_{\mathfrak{E}} |g_n(t)|\, dt \leqslant \tfrac{1}{8}\eta. \tag{37}$$

We can find an $n_1 > 1$ such that

$$m(E_1 . E_{n_1}) < \frac{\epsilon_1}{2}.$$

We can find an $n_2 > n_1$ such that

$$m(E_1 . E_{n_2}) < \frac{\epsilon_1}{2^2}; \quad m(E_{n_1} . E_{n_2}) < \frac{\epsilon_{n_1}}{2}.$$

We can find an $n_3 > n_2$ such that

$$m(E_1 . E_{n_3}) < \frac{\epsilon_1}{2^3}; \quad m(E_{n_1} . E_{n_3}) < \frac{\epsilon_{n_1}}{2}; \quad m(E_{n_2} . E_{n_3}) < \frac{\epsilon_{n_2}}{2};$$

and so on.

Then, writing $$\mathfrak{E}_1 = E_1 - E_1\left(\sum_{i=1}^{\infty} E_{n_i}\right),$$

we have $$m\mathfrak{E}_1 \geqslant mE_1 - \epsilon_1.$$

Writing $$\mathfrak{E}_{n_1} = E_{n_1} - E_{n_1}\left(\sum_{i=2}^{\infty} E_{n_i}\right),$$

we have $$m\mathfrak{E}_{n_1} \geqslant mE_{n_1} - \epsilon_{n_1},$$

and $$\mathfrak{E}_1 . \mathfrak{E}_{n_1} = 0.$$

Generally, writing

$$\mathfrak{E}_{n_r} = E_{n_r} - E_{n_r}\left(\sum_{i=r+1}^{\infty} E_{n_i}\right),$$

we have $$m\mathfrak{E}_{n_r} \geqslant mE_{n_r} - \epsilon_{n_r},$$

and no two of the sets $\mathfrak{E}_{n_r}$ have a point in common. Further, $\mathfrak{E}_{n_r}$ is contained in E_{n_r}. Since we have, in E_{n_r},

$$|g_{n_r}(t)| > \eta/16\pi,$$

it follows that $$\int_{CE_{n_r}} |g_{n_r}(t)|\,dt \leqslant \tfrac{1}{8}\eta. \tag{38}$$

Hence $$\int_{E_{n_r}} |g_n(t)|\,dt > \frac{7\eta}{8},$$

and so, by (37), $$\int_{\mathfrak{E}_{n_r}} |g_{n_r}(t)|\,dt > \frac{3\eta}{4}. \tag{39}$$

Further, by (38) and (37),

$$\int_{C\mathfrak{E}_{n_r}} |g_{n_r}(t)|\,dt \leqslant \tfrac{1}{4}\eta.$$

If now we put $$f(t) = \operatorname{sgn} g_n(t)$$

in $\mathfrak{E}_{n_r}$ for $r = 1, 2, \ldots$, and equal to zero elsewhere, we have

$$\left|\int_0^{2\pi} f(t)\, g_{n_r}(t)\,dt\right| \geqslant \int_{\mathfrak{E}_{n_r}} |g_{n_r}(t)|\,dt - \int_{C\mathfrak{E}_{n_r}} |g_{n_r}(t)|\,dt$$
$$> \tfrac{1}{2}\eta,$$

which contradicts (33). The necessity of the condition is established.

To prove the sufficiency of the conditions, we observe that, if the condition is satisfied, then (32) holds. Given $\eta > 0$, there is an $\epsilon > 0$ such that, if (α_1, β_1), ..., (α_n, β_n) denote a non-overlapping set of intervals in $(0, 2\pi)$, with

$$\sum_{i=1}^{n} (\beta_i - \alpha_i) \leqslant \epsilon,$$

then

$$\int_0^{2\pi} \left| \sum_{i=1}^{n} \{g(t-\beta_i) - g(t-\alpha_i)\} \right| dt \leqslant \eta.$$

Hence

$$\left| \sum_{i=1}^{n} T(\beta_i) - T(\alpha_i) \right| \leqslant \frac{\eta}{\pi} \operatorname{Max} |f(x)|,$$

and so $T(x)$ is absolutely continuous.

The conditions (2, 1), (3, 1) *are identical with the condition that*

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n} \sin nx$$

is the Fourier series of the integrable function $g(x)$ which is absolutely continuous in mean.

This is an immediate consequence from the conditions $(\bar{2}, 6)$, $(\bar{3}, 6)$.

We say that a sequence $\{g_r(t)\}$ of periodic functions is uniformly absolutely continuous in mean if each function is absolutely continuous in mean, and if, given η, there is an ϵ which satisfies the required conditions for every $g_r(t)$.

We can prove the following result.

LEMMA 7. *The condition* (2, 6) *can be expressed as follows*:

$$\sum_{n=1}^{\infty} \lambda_n \cos nx$$

has its Cesàro mean uniformly absolutely continuous in mean.

To prove that the condition is sufficient, let $T_r(x)$, $\sigma_r(x)$ denote the r-th Cesàro means of the series (3) and (31) respectively. Let (1) be the Fourier series of a bounded function $f(x)$. Then

$$T_r(x) = \frac{1}{\pi} \int_0^{2\pi} f(t)\, \sigma_r(t-x)\, dt.$$

If now (α_1, β_1), ..., (α_n, β_n) denote any finite set of non-overlapping intervals in $(0, 2\pi)$ such that

$$\sum_{i=1}^{n} (\beta_i - \alpha_i) \leqslant \epsilon,$$

we have

$$\left| \sum_{i=1}^{\infty} \{T_r(\beta_i) - T_r(\alpha_i)\} \right| \leqslant \frac{1}{\pi} \int_0^{2\pi} |f(t)| \left| \sum_{i=1}^{n} \{\sigma_r(t-\beta_i) - \sigma_r(t-\alpha_i)\} \right| dt$$

$$\leqslant \frac{\eta}{\pi} \operatorname{Max} |f(t)|.$$

It follows that the sequence $\{T_r(x)\}$ is uniformly absolutely continuous, and hence, by Lemma 1 (vi), that (3) belongs to $\{6\}$.

To prove that the condition is necessary, suppose that (λ_n) transforms $\{2\}$ to $\{6\}$. Then $\{T_r(x)\}$ is uniformly absolutely continuous. If now $\{\sigma_r(t)\}$ is not uniformly absolutely continuous in mean, we can select a subsequence $\{\sigma_n(t)\}$ with the property that there exists a sequence (S_n) of sets such that, with the notation previously used,

$$\lim_{n \to \infty} \sum_{i=1}^{\nu_n} (\beta_i^{(n)} - \alpha_i^{(n)}) = 0$$

and

$$\int_0^{2\pi} \left| \sum_{i=1}^{\nu_n} \{\sigma_n(t-\beta_i^{(n)}) - \sigma_n(t-\alpha_i^{(n)})\} \right| dt > \eta.$$

Writing

$$g_n(t) = \sum_{i=1}^{\nu_n} \{\sigma_n(t-\beta_i^{(n)}) - \sigma_n(t-\alpha_i^{(n)})\},$$

we have, for every bounded $f(t)$,

$$\lim \int_0^{2\pi} f(t)\, g_n(t)\, dt = 0$$

and

$$\int_0^{2\pi} |g_n(t)|\, dt > \eta.$$

These two relations are formally the same as (33) and (35) which have been shown to imply a contradiction. This proves the lemma.

From Lemma 7 and the original condition (2, 6) we infer

THEOREM IV. *The necessary and sufficient condition that*

$$\Sigma (a_n \cos nx + b_n \sin nx)$$

should be the Fourier series of a function which is absolutely continuous in mean is that the Cesàro means of the series should be uniformly absolutely continuous in mean.

12. In this section we obtain the condition (2, 3).

We first introduce some definitions. *A periodic function $g(t)$ is said to be R integrable in mean if, given $\eta > 0$, there is an $\epsilon > 0$ such that, if $(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)$ are any finite set of non-overlapping intervals in* $(0, 2\pi)$

with $$\beta_i - \alpha_i \leqslant \epsilon,$$

and if γ_i, δ_i *are any two numbers which satisfy*

$$\alpha_i \leqslant \gamma_i < \delta_i \leqslant \beta_i,$$

then $$\int_0^{2\pi} \left| \sum_{i=1}^{n} \{g(t-\delta_i) - g(t-\gamma_i)\}(\beta_i - \alpha_i) \right| dt \leqslant \eta.$$

A sequence $\{g_r(t)\}$ of periodic functions is said to be uniformly R integrable in mean if each function is R integrable in mean and if, given η, there is an ϵ which satisfies the requirements of the preceding definition for every $g_r(t)$.

We require the following lemma.

LEMMA 8. *The condition* (2, 3) *may be expressed as follows*:

$$\sum_{n=1}^{\infty} \lambda_n \cos nx \tag{40}$$

has its Cesàro means uniformly R integrable in mean.

To prove that the condition is sufficient, let $T'_r(x)$, $\sigma_r(x)$ denote respectively the r-th Cesàro means of the series (3) and (40). Then, if (1) is the Fourier series of the bounded function $f(t)$, we have

$$T_r(x) = \frac{1}{\pi} \int_0^{2\pi} f(t)\, \sigma_r(t-x)\, dt.$$

Using the notation given above, we have

$$\left| \sum_{i=1}^{n} \{T_r(\delta_i) - T_r(\gamma_i)\}(\beta_i - \alpha_i) \right|$$

$$\leqslant \frac{1}{\pi} \int_0^{2\pi} |f(t)| \left| \sum_{i=1}^{n} \{\sigma_r(t-\delta_i) - \sigma_r(t-\gamma_i)\}(\beta_i - \alpha_i) \right| dt$$

$$\leqslant \frac{\eta}{\pi} \operatorname{Max} |f(t)|.$$

Hence the sequence $\{T_r(x)\}$ is uniformly R integrable. By Lemma 1 (iii) the series (3) is the Fourier series of an R integrable function.

To prove that the condition is necessary, suppose that (λ_n) transforms $\{2\}$ to $\{3\}$. Then $\{T_r(x)\}$ is uniformly R integrable. Hence, given $\eta > 0$, there is an $\epsilon > 0$ such that

$$\left| \sum_{i=1}^{n} \{T_r(\delta_i) - T_r(\gamma_i)\}(\beta_i - \alpha_i) \right| \leqslant \eta,$$

where the α's, β's, γ's, δ's have the meanings given above.

If now $\{\sigma_r(t)\}$ is not uniformly R integrable in mean, there is an $\eta > 0$ and a subsequence $\{\sigma_\nu(t)\}$ with the following property. To each ν there corresponds an ϵ_ν such that $\lim \epsilon_\nu = 0$. Further, to each ν there correspond n_ν non-overlapping intervals in $(0, 2\pi)$,

$$(\alpha_1^{(\nu)}, \beta_1^{(\nu)}), \quad \ldots, \quad (\alpha_{n_\nu}^{(\nu)}, \beta_{n_\nu}^{(\nu)}),$$

where n_ν is an integer depending on ν, such that

$$\beta_i^{(\nu)} - \alpha_i^{(\nu)} \leqslant \epsilon_\nu,$$

and a set of numbers $\gamma_i^{(\nu)}$, $\delta_i^{(\nu)}$ satisfying

$$\alpha_i^{(\nu)} \leqslant \gamma_i^{(\nu)} < \delta_i^{(\nu)} \leqslant \beta_i^{(\nu)}$$

such that

$$\int_0^{2\pi} \left| \sum_{i=1}^{n_\nu} \{\sigma_\nu(t-\delta_i^{(\nu)}) - \sigma_\nu(t-\gamma_i^{(\nu)})\} (\beta_i^{(\nu)} - \alpha_i^{(\nu)}) \right| dt \geqslant \eta.$$

We write

$$g_\nu(t) = \sum_{i=1}^{n_\nu} \{\sigma_\nu(t-\delta_i^{(\nu)}) - \sigma_\nu(t-\gamma_i^{(\nu)})\} (\beta_i^{(\nu)} - \alpha_i^{(\nu)}).$$

We then have

$$\int_0^{2\pi} |g_\nu(t)|\, dt \geqslant \eta$$

and, by the hypothesis that (λ_n) transforms $\{2\}$ to $\{3\}$, we have, for every bounded $f(t)$,

$$\lim_{\nu \to \infty} \int_0^{2\pi} f(t)\, g_\nu(t)\, dt = 0.$$

These two relations are formally the same as (35) and (33) respectively, and these latter have been shown to imply a contradiction. Hence $\{\sigma_r(t)\}$ is uniformly R integrable in mean. This proves the lemma.

Let $g(t)$ be a periodic function. We say that the variation of $g(t)$ is R integrable in mean [or that $g(t)$ is of R integrable variation in mean] if, given $\eta > 0$, there is an $\epsilon > 0$ such that, if (α_1, β_1), ..., (α_n, β_n) denote any finite set of non-overlapping intervals in $(0, 2\pi)$ with

$$\beta_i - \alpha_i \leqslant \epsilon,$$

and γ_i, δ_i denote any two numbers satisfying

$$\alpha_i \leqslant \gamma_i < \delta_i \leqslant \beta_i,$$

then

$$\int_0^{2\pi} \left| \sum_{i=1}^{n} \{dg(t-\delta_i) - dg(t-\gamma_i)\} (\beta_i - \alpha_i) \right| \leqslant \eta.$$

We now have the following result.

The condition (2, 3) *is that*

$$\sum_{n=1}^{\infty} \frac{\lambda_n \sin nx}{n} \tag{41}$$

is the Fourier series of a function $g(t)$ *whose variation is* R *integrable in mean.*

We first prove that the condition is necessary. By condition (2, 2) it is necessary that (41) should be the Fourier series of a function $g(t)$ which is of bounded variation. It is further necessary, by Lemma 8, that, given η, there is an ϵ such that

$$\int_0^{2\pi} \left| \sum_{i=1}^{n} \{\sigma_r(t-\delta_i)-\sigma_r(t-\gamma_i)\}(\beta_i-\alpha_i) \right| \leqslant \eta, \tag{42}$$

where $\sigma_r(x)$ denotes the r-th Cesàro mean of

$$\Sigma \lambda_n \cos nx.$$

We keep ϵ, η, and the α's, β's, γ's, δ's the same in the subsequent argument. Then, writing

$$G(t) = \sum_{i=1}^{n} \{g(t-\delta_i)-g(t-\gamma_i)\}(\beta_i-\alpha_i)$$

and

$$\kappa_r(t) = \frac{1}{2r\pi} \left(\frac{\sin \frac{1}{2}rt}{\sin \frac{1}{2}t} \right)^2,$$

we have

$$\sum_{n=1}^{n} \{\sigma_r(t-\delta_i)-\sigma_r(t-\gamma_i)\}(\beta_i-\alpha_i) = \int_0^{2\pi} \kappa_r(\tau)\, dG(\tau-t). \tag{43}$$

Since the integrand in (42) is a trigonometric polynomial, the set of points in which it is of one sign consists of a finite number of intervals. If E denotes any set contained in $(0, 2\pi)$ which consists of a finite number of intervals, then (42) implies that

$$\int_E \left| \sum_{i=1}^{n} \{\sigma_r(t-\delta_i)-\sigma_r(t-\gamma_i)\}(\beta_i-\alpha_i) \right| dt \leqslant \eta, \tag{44}$$

and conversely (44) implies (42) with 2η in place of η.

By (43) we have

$$\int_E dt \int_0^{2\pi} \kappa_r(\tau)\, dG(\tau-t) \leqslant \eta. \tag{45}$$

Now

$$\int_0^x dt \int_0^{2\pi} \kappa_r(\tau)\, dG(\tau-t)$$

is the r-th Cesàro mean of the Fourier series of $G(x)$. If then E consists of the ν intervals (a_i, b_i) $(i = 1, 2, \ldots, \nu)$, we have

$$\int_E dt \int_0^{2\pi} \kappa_r(\tau)\, dG(\tau - t) = \int_0^{2\pi} \kappa_r(t) \sum_{i=1}^{\nu} \{G(b_i + t) - G(a_i + t)\}\, dt.$$

It follows that

$$\int_0^{2\pi} \kappa_r(t) \sum_{i=1}^{\nu} \{G(b_i + t) - G(a_i + t)\}\, dt \leqslant \eta. \tag{46}$$

Now $g(t)$, which is of bounded variation, may be supposed to be everywhere continuous on the right. So then is $G(t)$. Thus

$$\lim_{t \to 0} \sum_{i=1}^{\nu} \{G(b_i + t) - G(a_i + t)\} = \sum_{i-1}^{\nu} \{G(b_i) - G(a_i)\}. \tag{47}$$

The relation (46) implies that

$$\lim_{r \to \infty} \int_0^{2\pi} \kappa_r(t) \sum_{i=1}^{\nu} \{G(b_i + t) - G(a_i + t)\}\, dt \leqslant \eta. \tag{48}$$

But the first member of (48) is equal to the first member of (47). Hence

$$\sum_{i=1}^{\nu} \{G(b_i) - G(a_i)\} \leqslant \eta. \tag{49}$$

This, however, means that the total variation of $G(t)$ does not exceed 2η; *i.e.* that

$$\int_0^{2\pi} \left| \sum_{i=1}^{n} \{dg(t - \delta_i) - dg(t - \gamma_i)\} (\beta_i - \alpha_i) \right| dt \leqslant 2\eta. \tag{50}$$

Thus $g(t)$ is of R integrable variation in mean.

To prove that the condition is sufficient we must show that, keeping the α's, β's, γ's, δ's fixed, the relation (50) implies that $\{\sigma_r(t)\}$ is uniformly R integrable in mean.

Now (50), in which 2η is replaced by η, implies (49). Since the finite set of intervals (a_i, b_i) $(i = 1, 2, \ldots, \nu)$ may be arbitrarily chosen, (49) implies that

$$\sum_{i=1}^{\nu} \{G(b_i + t) - G(a_i + t)\} \leqslant \eta$$

for all t. This in its turn implies (46) and so (44). This establishes the result.

SOME IMPORTANT DOUBLY INFINITE LINE SYSTEMS IN HIGHER SPACE

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Introduction.

The principal object of the present paper is the study of the properties of certain elementary doubly infinite line systems in higher space. The method employed is to represent the lines of $[n]$ by their Grassmann coordinates, and to take these as point coordinates in higher space, so that to the lines of $[n]$ correspond the points of a certain locus in this space. To the lines of the system under consideration will then correspond the points of a surface on this locus. We examine the relations between the line system and this representative surface, and in particular the relations between the representative surfaces of two line systems, one of which is a projection of the other. This leads to the consideration of normal and supernormal loci, from which all the others may be derived. The systems considered are those whose representative surfaces are the non-ruled surfaces whose prime sections are rational or elliptic curves; it is shown that the line systems generate threefold loci which are closely related to each other and which can be derived as projections of known loci.

In § 1 the necessary properties of the locus representing the lines of $[n]$ are briefly developed. The general properties of doubly infinite line systems are considered in § 2, in § 3 the non-ruled surfaces whose prime sections are rational or elliptic curves are enumerated, and also the threefolds with elliptic curve sections which are not the locus of a single infinity of planes. The line systems which correspond to these surfaces are examined in the next three sections. The later sections deal with the particular case in which the line system is supposed to lie in [4], and the relations of the representative loci to the V_6^5 which represents all the lines of [4] are considered. The paper concludes with an investigation of the representative locus for the ∞^3 lines lying on the general quadric primal in [4].

1. *The representation of the lines of* $[n]$ *by points.*

1.1. *Line coordinates.*

It is well known that, if $X = (x_0, \ldots, x_n)$ and $Y = (y_0, \ldots, y_n)$ are two points of $[n]$, the ratios of the $\frac{1}{2}n(n+1)$ quantities

$$p_{ij} = x_i y_j - x_j y_i$$

uniquely determine the line and are uniquely determined by it. They are, in fact, the extension to n dimensional space of the Plücker line coordinates in ordinary space. They are connected by a number of linearly independent quadratic relations of the form

$$p_{\alpha\beta}p_{\gamma\delta} + p_{\alpha\gamma}p_{\delta\beta} + p_{\alpha\delta}p_{\beta\gamma} = 0, \tag{1.11}$$

where $\alpha, \beta, \gamma, \delta$ represent any four of the suffixes $0, \ldots, n$. We may regard the p_{ij} as homogeneous coordinates in a space $[N]$, where $N = \frac{1}{2}n(n+1) - 1$; the points of $[N]$ whose coordinates satisfy (1.11) are then the points of a certain locus G_n which is in unexceptional correspondence with the lines of $[n]$.

1.2. *Elementary properties of the locus* G_n.

The general properties of the locus have been considered by Severi†. The points of G_n which correspond to lines through a point A of the $[n]$ lie in a $[n-1]$ a, so that G_n contains ∞^n $[n-1]$. Any two of these, a, a' say, have a common point which maps the line joining the corresponding points A, A' of the $[n]$. The points of G_n which correspond to lines lying in a $[n-k]$ of the $[n]$ lie on a G_{n-k}; in particular the map of the ∞^2 lines of a plane ϖ of $[n]$ is a plane π on G_n. If two lines of the $[n]$ intersect, the corresponding points of G_n both lie in the $[n-1]$ which corresponds to the point of intersection, so that the line joining these points lies entirely on G_n. Conversely, the points of any line lying on G_n map the lines of a plane pencil in the $[n]$, and the line is the intersection of a definite $[n-1]$ a and a definite plane π. The lines of G_n which pass through a point L map all the plane pencils of which the corresponding line l is a member, so that the points of these lines represent all the lines of the $[n]$ which meet l. The lines on G_n lie in the tangent space to G_n at L and generate a cone whose vertex is L. This cone contains ∞^1 $[n-1]$ a, corresponding to the points of l, and ∞^{n-2} planes π, corresponding to planes through l, and any one of the $[n-1]$ meets any of the planes in a line through L. The prime section of the cone is thus a locus generated by ∞^1 $[n-2]$ and ∞^{n-2} lines,

† Severi, **1**.

any two spaces of different systems intersecting ; it is accordingly the rational normal V_{n-1}^{n-1} of $[2n-3]$, defined by the joining lines of corresponding points of two related $[n-2]$ which do not intersect†. This accords with the fact that the dimension of the tangent space to G_n is $2(n-1)$, which is the freedom of the lines of $[n]$.

The points of a prime section of G_n correspond to the lines of $[n]$ whose coordinates satisfy a linear relation, that is, to the lines of a linear complex. In particular, the lines which meet a secundum, or $[n-2]$, of the $[n]$ form such a system and are mapped on a prime section of G_n; this is a special section and contains the tangent $[2n-2]$ to G_n at all the points which map lines lying in the given secundum; that is, the prime touches G_n at all points of a G_{n-2}. It follows that a ruled surface of order m is mapped on G_n by a curve of order m ; the genus of the curve will be equal to that of the surface since the points of the former are in (1, 1) correspondence with the generators of the latter. In particular, a rational scroll of order $n-1-k$ is represented by a rational curve of this order; and we can show that if the surface lies in its normal space, namely $[n-k]$‡, the same is true of the representing curve. We shall lose no generality in supposing k to be zero, since we could otherwise consider the G_{n-k} corresponding to the $[n-k]$ in which the surface lies. Now we have seen that the tangent space at a point of G_n cuts G_n in a cone projecting the normal V_{n-1}^{n-1} of $[2n-3]$; the section of this by a general $[n-1]$ is a normal rational C^{n-1}, and this maps the generators of a ruled surface of order $n-1$ special in the fact that it possesses a linear directrix. The general surface, not possessing this peculiarity, will *a fortiori* be mapped by a normal curve.

It follows that a regulus is mapped by the points of a conic, a cubic surface in [4] by a twisted cubic, and so on.

It is important to notice that, if a G_r $(r<n)$ lies on G_n, representing the lines of some $[r]$ of the $[n]$, the space $[R]$ which contains G_r cannot have any further intersection with G_n. To prove this we observe, first, that the theorem will follow by induction if we can prove it for the case in which $r=n-1$, for we can consider a succession of loci G_n, G_{n-1}, ..., G_{r+1}, G_r, each of which contains all the ones succeeding it in the series, and apply the theorem to each consecutive pair. Now, if the theorem is not true for the G_{n-1} of $[n]$, every linear complex which contains all the lines of some $[n-1]$ of the $[n]$ necessarily contains other fixed lines. In particular this will be true of the n special linear complexes which consist of the lines meeting n linearly independent $[n-2]$ of the $[n-1]$. These have thus

† Veronese, **1**. Segre, **1**.

‡ Veronese, **1**.

common lines not lying in the $[n-1]$. But this is impossible, for, since the $[n-2]$ are independent, they have no common point, and any transversal line of them all must lie in the containing $[n-1]$. This proves the result stated, when $r = n-1$, and the general theorem follows.

1.3. *Relations between projected line systems.*

It is clear from what has been said above that, if any line system Σ of the $[n]$ is mapped on G_n by the points of a locus S, and if we consider the $[n]$ as being immersed in some space of higher dimension $n+h$, then Σ will be represented upon G_{n+h} by a locus identical with S projectively, which lies on a particular G_n of G_{n+h}, namely that corresponding to the particular $[n]$ in which Σ lies. It is thus appropriate to consider the representation of a system of lines by points quite apart from the consideration of any particular Grassmannian locus, and to consider merely the correspondence between the points of the representative locus S and the lines of the system Σ. The geometrical properties of S, such as the nature of the curves, surfaces, ..., which it contains, and the relations of these loci to one another, can all be translated directly into properties of subsystems of lines belonging to Σ. Such properties are independent of any G_n on which S may be supposed to lie. There are, however, properties of Σ whose interpretation involves reference to the appropriate G_n; such a property is that of two lines of Σ intersecting, which is represented by the fact that a certain chord of S lies on G_n. We thus are able to distinguish among the properties of the system Σ those which are essentially independent of the space in which it lies, and correspond to properties of the representative locus S, and those which essentially belong to the fact that the system Σ lies in a definite $[n]$, whose interpretation involves reference to G_n.

We now suppose that the dimension of Σ does not exceed $n-2$, n being the dimension of the containing space. Then through a general point of the space no lines of Σ pass. If from such a point we project the system Σ on to a $[n-1]$, we obtain, in this space, a system of lines Σ' which will be mapped on G_{n-1} by some locus S'. Now, since the projection is from a point which is not on any line of Σ, it follows that those properties of Σ which simply concern its submanifolds are equally properties of Σ', and that the only differences between Σ and Σ' are those whose interpretation would involve reference to the Grassmannian loci. It follows that the relation between the loci S and S' is such that in general a curve on the one corresponds to a curve on the other having precisely the same order and genus, that every point of S gives rise to a unique point of S', but that possibly, for particular points, a point of S' may arise from two or more

points of S; this would happen, for example, if the vertex of projection lay in a plane with two of the lines of Σ. It follows that the only difference between S and S' projectively is that possibly S' may lie in a lower space than S and be capable of derivation from S by projection from some space not meeting S. It follows that *the representative locus S' of the projection of a system Σ from a point not lying on any of its lines is either projectively identical with the representative locus of Σ or can be obtained from this locus by a general projection.*

That both these cases can arise follows by the following examples. Consider, first, the ∞^2 lines of the rational normal V_3^3 of [5], which join corresponding points of two projectively related planes. On the locus G_5, these lines correspond to the points of a surface, and since the lines are clearly in rational correspondence with the points of one of the planes of V_3^3, it follows that this surface is rational. The lines of V_3^3 which meet a line in this plane form a regulus, and thus correspond to the points of a conic on G_5, whence it follows that the surface S is a Veronese surface. Now consider the projection of the system into [3] from two arbitrary points. The ∞^2 lines become a congruence in space which is easily seen to be of order three and class one, that is to say, the lines are the axes of a cubic developable, namely that formed by the projection of the planes of V_3^3. But it is known that on the quadric in five dimensions, which is G_3, the axes of a cubic developable are represented by the points of a Veronese surface. In this case, then, the loci S, S' are identical. In the second case, we may consider the ∞^2 chords of a normal quartic in four dimensions. These are mapped on G_4, which lies in [9], by the points of a surface which, as we shall show later, is in fact of order nine. It is only of importance to notice here that the lines in question do not lie in a linear complex, so that the surface is actually in the [9] and not in any prime thereof. The projection of this system on to [3] from a general point is a congruence whose representative surface must still be of order nine, but since G_3 is only in [5] this surface must be a projection of the former from a [3] which does not meet it.

The above holds when the vertex of projection O lies on no line of Σ. When the vertex lies on one general line of Σ, every ruled surface lying in Σ and not containing this line l projects into a surface of Σ' of the same order, while a surface containing l as a simple generator projects into a surface in Σ' whose order is diminished by one. Thus every curve on S which does not pass through the point L of S corresponding to l corresponds to a similar curve on S', while curves passing simply through L give rise to curves on S' of order less by one than the curve on S. Thus, to within a projective transformation, we may identify S' with the projection of S from some

space which meets S in the point L†. An example of this case will now be given. Consider the normal V_4^4 of lines in [7], obtained by joining the corresponding points of two projectively related solids. On G_7 the ∞^3 lines of this are represented by the points of a rational threefold, which contains a conic passing through any pair of its points, corresponding to the regulus determined by the lines of V_4^4 meeting a line of one of its generating [3]. This locus is thus the V_3^8 of [9] represented on [3] by all the quadrics of space. Consider now the projection of the system into [3] from four points of the locus. The ∞^3 lines of V_4^4 then project into the lines of a tetrahedral complex‡. This is represented on G_3 by a particular V_3^4 which can be represented on [3] by quadrics passing through four points; it is thus the projection of the V_3^8 of [9] from four general points of itself.

The properties of this section thus suggest the utility of the consideration of *normal* and *supernormal* loci, defined as follows. A locus of $[k]$ will be called *normal* if it cannot be obtained by projection of any locus in higher space from points which do not lie on the locus; it will be called *supernormal* if it cannot be obtained as a projection of some locus in higher space from a vertex which meets the latter locus only in a finite number of general points. The reasoning of the last section shows that if a line system Σ is represented by a surface S on the Grassmannian variety G_n appropriate to the space in which Σ lies, that S cannot be supernormal unless Σ is, and that if Σ is supernormal then S is supernormal§. It is clear that, by considering the representation of a supernormal locus, we can get by projection the properties of the representations of loci derived from the given one by various methods.

The definition just given does not involve the normality of a supernormal locus, but we shall suppose in what follows that all supernormal loci are considered to be in their normal spaces. If Σ is supernormal in this generalized sense it is easily seen that the same is true of S.

2. *Doubly infinite line systems.*

2.1. *The order and class of a doubly infinite line system.*

We now consider more particularly the characteristic properties of doubly infinite line systems in space of dimension n which is greater than

† The theory may easily be generalized to the case when O lies on an infinite number of lines of Σ.

‡ Todd, **1**, 443.

§ For the supernormal locus from which S is derived will lie on some G_n, and will map a line system Σ' of which Σ is the projection: since Σ is supernormal this must coincide with Σ.

three. The lines of the system Σ in this case generate a threefold locus M, through a general point of which will pass a finite number of lines of Σ. This number may be as great as six, this occurring when M is the cubic primal in [4]; but in general will be unity. We shall confine our attention to cases in which this happens; the theorems stated below will require modification if this is not true. The primal M may possess other line systems, but we shall be little concerned with them in general.

We can define for the system Σ two projective characters, namely the *order* and the *class*, which are defined respectively as the order and class of the congruence of lines in ordinary space into which Σ is projected from an arbitrary $[n-4]$. Thus the order r of Σ is the number of its lines incident with a given general $[n-3]$, and the class s is the number of its lines lying in a general prime. The locus M is therefore of order r, and its prime section is a surface which contains s lines of the system Σ. Now consider the representative surface S. Its order is the number of its points which lie on a secundum, that is it is equal to the number of lines of Σ which belong to two linear complexes. Now if we consider two linear complexes which consist of the lines incident with two $[n-2]$ of the $[n]$, these being chosen so as to lie in a prime, it is easily seen that the lines common to the two complexes consist (a) of all the lines of the prime, and (b) of all the lines incident with the $[n-3]$ in which the two $[n-2]$ intersect. There are s lines of Σ in system (a) and r in system (b), by the definition of the numbers r and s; consequently the order of S must be $r+s$. Again, the points of S on a section by a prime touching G_n along a G_{n-2} represent the lines of Σ meeting an $[n-2]$, this meets M in a curve through the general point of which passes one line of Σ. It follows that the curve sections of S and of M have the same genus. Consequently *a line system Σ, of which the order and class are respectively r, s, is represented by a surface of order $r+s$, and the curve sections of this surface and of the threefold generated by the lines have the same genus.*

2.2. *Relations between projected line systems.*

We now apply to the particular case of doubly infinite line systems the results which we have obtained above in 1.3. If we project Σ from a general point of M, we have seen that the projected system Σ' is mapped on G_{n-1} by a surface S' which can be regarded as the projection of S from a space which meets it in a single point. The order of S' is thus $r+s-1$. It is clear, however, that the order and class of Σ' are given by $r' = r-1$, $s' = s$, so that *projection does not alter the class of a system.* It follows that the systems Σ for which s is constant and r varies are derivable by projection from the particular supernormal system in which r has its maximum value,

and that the corresponding surfaces S are related in the same way to the surface which represents this supernormal system, this surface being itself supernormal.

In the present paper we consider the supernormal systems Σ for which the representative surface S is a non-ruled surface of rational or elliptic prime section, and discuss the line systems obtained by projection of these. These surfaces, together with certain important threefold loci which arise later, are now considered.

3. *Surfaces and threefold loci whose curve sections have genus* 0, 1.

3.1. *The Veronese surface and the Del Pezzo surfaces.*

It is well known that any surface which is not ruled, and which has rational curves for its prime section, is either the Veronese surface in [5] or one of its general projections†. This surface is the quartic surface mapped on the plane by all the conics; and its chief property is that it contains a double infinity of conics, mapped by the lines of the plane, one passing through any two points of the surface.

The surfaces whose prime sections are elliptic and which are not ruled are of order n and lie in n dimensions, where n does not exceed nine‡. The novenic surface in [9] is that mapped on the plane by the aggregate of cubic curves which lie in it, and the projections of this surface from points of itself give surfaces of lower order represented by systems of cubic curves with certain base points. We shall denote the surfaces so obtained by the term "proper" Del Pezzo surfaces; they exist for all values of n from 3 to 9 inclusive, and when $n = 2$ the corresponding surface can be regarded as a plane counted doubly, and branching along a quartic curve of genus three§. Apart from the surfaces of this series there is a surface in eight dimensions, of order eight, which can be represented on the plane by means of quartic curves with two fixed nodes, or on a quadric by the totality of its quadric sections. The projection of this from a point of itself is the septimic surface of the proper series. The relation between the surfaces can be expressed diagrammatically as follows, an arrow denoting general projection from a point of the surface:

$$V_2^9 \to V_2^8 \to V_2^7 \to V_2^6 \to V_2^5 \to V_2^4 \to V_2^3$$

$$V_2^{8*} \nearrow$$

† Picard, 1, 2.
‡ Del Pezzo, 1.
§ Geiser, 1.

The novenic surface, and the octavic surface last mentioned, are *supernormal*. All other non-ruled surfaces of elliptic section which are normal are particular cases of those described. The star in the table above denotes the particular octavic surface not belonging to the "proper" series. The surfaces of the proper series contain a finite number of lines; for $n = 9, 8, 7, 6, 5, 4, 3$ the numbers are respectively $0, 1, 3, 6, 10, 16, 27$; the surface V_2^8* on the other hand contains no lines.

3.2. *Threefolds with elliptic curve sections.*

3.21. Any locus of dimension three and order greater than two whose curve sections are rational contains a single infinity of planes or else is a cone projecting a Veronese surface. The threefolds with elliptic curve section which are not the locus of a single infinity of planes have been considered by Enriques† and Scorza‡. They are of order $n-1$ in $[n]$, where n is less than or equal to nine. They form several classes, according to the nature of the supernormal loci from which they may be derived. These supernormal loci are as follows: (*a*) the V_3^8 in $[9]$ represented by all the quadrics of space, (*b*) a V_3^6 of $[7]$ mapped by cubic surfaces passing through three skew lines, (*c*) a V_3^6 of $[7]$ mapped by cubic surfaces passing through a twisted cubic curve and having a node at a fixed point of the curve, (*d*) a V_3^5 of $[6]$ which is mapped on space by cubic surfaces through a rational quartic curve, (*e*) the general intersection of two quadrics in $[5]$, (*f*) the general cubic primal in $[4]$. The loci of the same order as these which are not supernormal, but which are obtained as the projections of other supernormal loci of the series from certain points of themselves, can sometimes be regarded as special cases of the supernormal loci, but for present purposes it is convenient to consider them separately.

Now it is clear that the line systems which are mapped by the surfaces with elliptic prime sections generate threefold loci whose curve sections are themselves elliptic, and which have one line of the generating system passing through each general point. We have, therefore, to consider all the loci of this character which can exist, and examine the supernormal loci mentioned above, particular cases of these, and their projections, with this end in view.

If any threefold locus is projected from a point of itself, the neighbourhood of this point will in general give rise to a *plane* on the projected locus,

† Enriques, **1**.

‡ Scorza, **1**. See also Semple, **1**, for an account of certain of these loci.

while a plane will also arise if there is a quadric surface on the locus which passes through the point of projection. Any line on the locus through the projecting point gives on the projected locus a multiple point, whose multiplicity is equal to the number of intersections of the line with a residual secundum section; if the threefold has elliptic sections this number is two, and the multiple point is a node. Any conic through the point of projection gives rise to a line. We shall see that the loci which we investigate contain in general a finite number of nodes and planes, and, in general, several line systems.

The *class* of a line system lying on such a locus can be found immediately, since the prime section of the locus is a Del Pezzo surface containing a known number of lines, which must arise either as the section of planes lying on the threefold locus, or from the line systems which lie on this.

3.22. Now consider the loci in class (α) above, of which the parent is the "Veronese threefold", the V_3^8 of [9] represented on space by all the quadrics. This locus contains no lines. Its projection from a general point is a V_3^7 of [8], containing one plane, the projection of the neighbourhood of the projecting point, and a single system of lines, mapped by lines through the base point which is added in the course of projection to the system of quadrics. Since two of the lines of this system lie in a prime, as follows at once from the representation, it follows that the line system on this locus, as well as on all its projections, is of class two. The further projections of this locus are mapped by the systems of quadrics which possess further base points; they contain line systems mapped by lines through these points or by cubic curves through five such points; planes corresponding to the base points or to the planes which join threes of them, and nodes corresponding to the lines joining pairs of the points. We have in fact the scheme

	[9]	[8]	[7]	[6]	[5]	[4]
	V_3^8	V_3^7	V_3^6	V_3^5	V_3^4	V_3^3
Planes	—	1	2	4	8	15
Line systems	—	1	2	3	4	6
Nodes	—	—	1	3	6	10

describing the singularities of these loci; all obtained by successive projection of the V_3^8. The V_3^3 of [4] is the well known Segre cubic primal, the V_3^4 in [5] is the particular intersection of two quadrics which maps from [3] the lines of a tetrahedral complex. The V_3^5 has been studied by

the author† as the locus which represents the lines of [4] which meet three planes.

The prime section of V_3^8 is, of course, V_2^8*. It is easy to see that on those of the above loci, which contain several distinct line systems, all these are equivalent.

3.23. Next consider the loci of class (b), which are derived from the V_3^6 of [7] which is mapped on [3] by cubic surfaces passing through three skew lines. This locus clearly contains three precisely equivalent line systems, each generating the locus simply; these are mapped by the transversals of pairs of the three lines. Since its prime section is the sextic Del Pezzo surface, containing six lines, each of these systems is of class two. The projection of this locus from a general point of itself is a quintic threefold in [6] possessing three nodes, the projections of the lines through the point of projection, and the author has shown† that such a locus must be of the same type as that which occurred above in class (a). The system of cubic surfaces passing through three lines and a fixed point may in fact be transformed into quadrics with three base points by means of a Cremona transformation determined by cubic surfaces which have the fixed point as a node and contain the three lines‡. Hence the projections of the V_3^6 give rise to no new loci.

The relation between the loci of classes (a) and (b) is therefore expressible diagrammatically by the scheme

(a) $\qquad V_3^8 \to V_3^7 \to V_3^6$

(b) $\qquad\qquad\qquad\qquad V_3^6 \nearrow V_3^5 \to V_3^4 \to V_3^3,$

and all the line systems concerned are of class two.

3.24. Now take the loci (c), starting with the V_3^6 of [7] mapped on [3] by cubic surfaces which pass through a twisted cubic curve γ, and which have a node at a fixed point O of γ. Such a V_3^6 contains two line systems, mapped respectively by chords of γ and by lines through O; both these line systems generate the locus simply, and are of class three, and it follows from Scorza's results that they are precisely equivalent. The projection of this locus from a general point of itself is a V_3^5 in [6] containing two nodes arising from two lines of V_3^6 which pass through the point of projection, and is mapped by the cubic surfaces of the system just described which pass through an additional point P. The binodal V_3^5 has been briefly

† Todd, **2**, 539 *seq.*

‡ See Todd, **3**, 290, and also Babbage, **1**, 413.

described by the author†; it arises as the section of the Grassmannian locus G_4 by a [6] which meets one of the generating [3] of the locus in a plane. This plane, of course, is that which arises from the neighbourhood of the point of projection. It contains three line systems, two of which are the projections of those on V_3^6; the third is mapped by conics which pass through the points O, P, and have two further intersections with γ; on V_3^5 this line system is distinguished from the other two by the fact that all the lines of the system meet the plane on V_3^5. This system, like the other two, is of class three, since the quintic surface which is the prime section of the locus contains ten lines, one lying in the plane on V_3^5 and three belonging to each of the three systems.

The projection of V_3^5 into [5] from a point of itself is a V_3^4 containing five nodes, four planes, and four line systems; in this case all the line systems are equivalent; the four planes are mapped by the two isolated base points P, P' which the system of cubics now possesses, by the plane OPP', and by the quadric surface containing γ, P, P'. Each line system now meets one of the planes. The projection on to [4] is a cubic primal with nine nodes, nine planes, and six equivalent systems of lines, each meeting three of the planes; this is one of the loci described by Segre‡.

3.25. The V_3^5 in [6], which is the parent of the loci in system (d), is mapped on [3] by cubic surfaces through a rational quartic curve, and it contains in general an irreducible line system, mapped by chords of this curve, three lines passing through each point of the surface§. In order to obtain a simple system out of this we must allow the quartic to break up.

Now the general V_3^5 in [6], as shown by Scorza, is the section of the locus G_4 by a [6] bearing no particular relation to the locus. In order to obtain a V_3^5 containing a reducible line system we have to consider particular cases of this. The author has given a table of such cases‖. It is, in fact, the case that the cubic surfaces which pass through a twisted cubic and a line meeting it represent such a V_3^5, one simply generating line system then corresponding to the chords of the cubic curve, and another doubly generating system corresponding to the transversal lines of the cubic and the straight line. The corresponding V_3^5 possesses a double point, being the second case on the list mentioned above, but has no planes lying on it. The node in this case corresponds to the point in which the

† Todd, **2**, 539, §§ 3.441, 3.442.
‡ Segre, **2**.
§ Scorza, **1**; Todd, **2**, 538, 543.
‖ Todd, **2**, 537.

line and cubic meet, and the planes through the line correspond to an infinity of quadric surfaces on V_3^5 through the node. The projection of V_3^5 from the node is the familiar V_3^3 of planes in [5], and is mapped by the cubic surfaces of the system in question which have a double point at the point corresponding to the node on V_3^5. (This representation of V_3^3 clearly equally arises if the V_3^6 of [7], in class (*c*) above, is projected from one of its lines.) It is not difficult to see that four chords of the cubic lie on a general surface of the system, so that the line system in question has class four. The projections of this locus on to [5] and [4] now give other line systems of class four; in [5] we get a V_3^4 which contains two planes, four nodes, and two simple generating systems of lines; if we map this locus by adding a base point to the system of cubics described above, the two line systems correspond respectively to the chords of the cubic curve and to the conics passing through the isolated base point, which meet the line once and the cubic three times; one such conic passes through an arbitrary point, lying on the quadric joining this point and the isolated base point to the cubic. The two planes are mapped respectively by the neighbourhood of the isolated base point, and by the plane joining this to the line; it follows that the former line system, which is the projection of that on V_3^5, meets the latter of these planes while the latter meets the first. Projecting again on to [4] we get a cubic primal with eight nodes, five planes, and four simply generating systems of lines; if we consider the mapping obtained by adding a second base point, the five planes are mapped by the two base points, the joins of these to the line, and the quadric containing the cubic and the two points; the line systems are mapped by chords of the cubic, conics through one of the base points meeting the line once and the cubic three times, and cubic curves passing through the two points and meeting the line twice and the cubic four times. We can then verify that each of the four line systems in question has two of the five planes for directrices†.

It is easy to see that the V_3^5 of [6] mapped on [3] by cubic surfaces passing through a quartic curve which possesses a node without degenerating, and which have a node at this point, is precisely similar to the one just described.

3.26. The general V_3^5, as we have seen, contains an irreducible system of lines generating the locus triply. Its projection on to [5] from a point of itself, however, is a V_3^4 containing a plane and three double points, and this contains, in addition to the triply generating system arising from that on V_3^5, a simple system of lines of class five, all of whose lines meet the plane. The projection of this is a seven nodal cubic primal containing two planes,

† Segre, 2.

intersecting in a node and each containing three of the remaining six nodes; this contains three simply generating line systems of class five†, two of which have a single directrix plane and the other having two such planes. The former are the systems similar to the projection of the system on V_3^4.

3.27. Next, considering the loci of class (e), the general quadric intersection in [5] is a V_3^4 mapped on [3] by cubic surfaces passing through a quintic curve of genus two, containing an irreducible line system which generates the locus quadruply. In order to get a simple line system we must make the system of chords of this curve reducible. This can be done as follows. (i) The quintic curve may split up into a rational quartic and one of its trisecants. In this case the locus is the projection of the V_3^5 of [6] represented by the cubic surfaces passing through this quartic curve, and has already been considered. (ii) The curve may split up into a cubic curve and a trisecant conic. In this case the plane of the conic represents a plane on the V_3^4, and it is not difficult to see that the locus is the same as the one previously mentioned (in 3.26). Finally, we have the possibility (iii) that the curve may consist of a cubic and two of its chords. In this case we clearly have a V_3^4 containing four equivalent line systems of the fourth class and four nodes, but not containing any planes. The projection of this into [4] gives an eight nodal cubic primal with five planes, which is identical with that obtained by projection of the V_3^4 obtained from the nodal V_3^5 of [6]. This result can be expressed in the scheme

$$\begin{array}{ll} (d)' & V_3^5 \to V_3^4 \searrow \\ (e) & \qquad\qquad V_3^4 \nearrow \end{array} \quad V_3^3,$$

the accent denoting the first of the two cases which arose under (d) above.

3.28. Finally, we have to consider cubic primals whose line system degenerates, and it is easy to see that those which are not comprehended under the above classes are in general six nodal and possess two simple generating line systems of class six. These primals are the general determinantal cubics in [4]†.

There are certain particular cases of some of the above cubic primals which should be noted. In the first place, if the cubic just mentioned has for equation a *symmetrical* determinant the equations which determine its double points reduce to three independent ones, and instead of six isolated nodes we have a double normal quartic curve. There is still a simply generating line system of class six, composed of the chords of this curve.

† Segre, 2

Again, it is shown by Segre† that there exists a cubic-primal which contains a double line and a double point not lying on this; it then necessarily contains the plane joining the line to the point. This contains two simply generating systems of lines, one having the plane as directrix, and the other consisting of lines meeting the line. These systems both have class five, and thus are particular cases of those already considered.

We shall now examine the association of the line systems lying on the threefolds described above with the corresponding surfaces on the Grassmannian locus.

4. *Line systems represented by Veronese surfaces.*

We have seen in 2.1 that the order of the representative surface of a line system of order r and class s is $r+s$. Now, if we exclude the degenerate line systems whose lines either all pass through a point or all lie in a plane. the order and class of any doubly infinite line system must be at least unity. Hence, if the representative surface is a Veronese surface we must have

(i) $r = 1, \quad s = 3;$

(ii) $r = 2, \quad s = 2;$

(iii) $r = 3, \quad s = 1.$

Now the first of these systems cannot exist in more than three dimensions. For suppose that there were such systems in [4], then they would be mapped on G_4 by Veronese surfaces. If a Veronese surface exist on G_4, not lying on one of the quadric V_4^2 which this contains, any [5] containing it must either meet the locus again in a plane or else meet it in a V_3^3 of planes‡. In the latter case it lies in the tangent [6] to G_4 at one of its points, so that the Veronese surface represents lines all meeting some line; it is clear that the only such system which has order and class mentioned above consists of the join of points of the line to points of a conic. This system has $r = s = 2$, and lies on a quadric line cone. The other case corresponds to a line system for which $r = 3$, $s = 1$, and generates a cubic primal with rational linear sections, thus having a double plane. This is the general projection of the V_3^3 of planes in [5], so that the system (3, 1) generates this locus or one of its projections, as observed previously. The

† Segre, **2.**

‡ Todd, **2**, 538, 542.

system (1, 3) thus cannot exist in [4]; in [3] it is simply the aggregate of chords of a twisted cubic.

The three line systems represented by Veronese surfaces are therefore

(i) (1, 3) normal in [3], the chords of a twisted cubic curve,

(ii) (2, 2) normal in [4], joining points of a line and a conic,

(iii) (3, 1) normal in [5], and lying on a V_3^3 of lines and planes.

5. *Line systems represented by Del Pezzo surfaces: general results.*

We now consider the line systems which are represented by the surfaces of elliptic prime section. The condition on the surface that the prime section should be elliptic imposes restrictions on the possible values for r and s. For the V_3 generated by the lines has elliptic curve sections. Hence either the V_3 lies in space of four dimensions at least (when normal) and has order three or more, or else V_3 has order two, and consists of a doubly counted solid with some quartic surface as branch locus; this satisfies the necessary conditions, since the general double line with four branch points possesses the properties of an elliptic curve. Hence we must have $r \geqslant 3$ if the line system is normal in four or more dimensions. Leaving aside for the present the case in which the system is normal in [3], we can suppose that $r \geqslant 3$. Again, if $s = 1$ one line will lie in an arbitrary prime. Considering then the pencil of primes passing through a secundum, each of the primes through this secundum will contain a line of V_3 meeting the secundum in a point; as the prime varies this point will describe the curve in which the secundum meets V_3; this curve is accordingly rational, contrary to hypothesis. The only possibility is, therefore, that the point of intersection of the line with the secundum should be fixed for all positions of the prime. This cannot happen, however, since through the general point of the secundum passes a line of the system which, if it does not lie in the secundum, determines a prime through it in which the line does not pass through the fixed point ; and if every line meeting the secundum lay in it we should have the V_3 generated by an infinity of planes, a case which we have excluded. (A similar statement holds if every line meeting the general secundum lies in a fixed prime.)

We can, therefore, suppose that we have $r \geqslant 3$, $s \geqslant 2$. We then wish to obtain the correlation between the surfaces with elliptic prime sections and the various line systems just described.

If the V_3 generated by a line system contains a plane which is not a directrix plane for the system in question, there will be in general a pencil of lines belonging to the system which lie in the plane, and the only line

of the system which passes through the general point of the plane will be a line of this pencil. This will manifest itself in the existence of a line on the representative surface S, and every line on S, representing a plane pencil of lines of the system Σ, must correspond to such a plane on the V_3. This remark will enable us to choose between certain line systems of the same characteristic, as will be evident later.

The order of the representative surface of a line system of order r and class s has been seen to be $r+s$. In view of the inequalities for r and s which we have supposed above to be satisfied, this gives, for the various surfaces of elliptic section, the following possibilities.

Order of surface.	Line system (r, s) of order r and class s.
9	(7, 2), (6, 3), (5, 4), (4, 5), (3, 6)
8	(6, 2), (5, 3), (4, 4), (3, 5)
7	(5, 2), (4, 3), (3, 4)
6	(4, 2), (3, 3)
5	(3, 2)

In the case of the octavic surface there are two possibilities, corresponding to the two surfaces V_2^8 and V_2^{8*}.

6. *Enumeration of the line systems mapped by Del Pezzo surfaces.*

6.1. *Systems of class two.*

From the table just given it follows that the line system of order seven and class two is mapped by a novenic Del Pezzo surface. We have seen that the lines on the V_3^7 of 3.22 are of this class; so that we have the result that *the lines which generate the* V_3^7 *of* [8] *are mapped on* G_8 *by a novenic surface of elliptic prime section.* This V_3^7 contains a plane, mapped by the base point of the system of quadrics; it is a directrix plane for the line system. The projections of this locus from points of itself give other line systems of class two. In [7] we have a V_3^6 on which there are two line systems represented by the ordinary octavic Del Pezzo surface; the V_3^6 contains two planes, one a directrix of each system of lines, and hence, by what was said above, each of the line systems contains a plane pencil, and the representative surface contains a line. The lines on the trinodal V_3^5 of [6], of either of the three systems, correspond to the septimic Del Pezzo surface; each system has one of the four planes of V_3^5 as directrix, and the corresponding surface contains three lines. Similarly each line system on the V_3^4 in [5], or on the V_3^3 in [4], belonging to this system, is mapped by a

Del Pezzo surface of the sixth or fifth order. In particular, the representation of the lines of one system of the Segre cubic primal by points of a quintic surface accords with the fact† that these lines are common to four linear complexes in [4], and that the quintic surface is the intersection of G_4 with four primes of general position.

Each system of lines on the V_3^6 of [7] described in 3.23 is of order six and class two, and hence corresponds to an octavic surface; since, however, the V_3^6 contains no planes at all this surface contains no lines, and is therefore the supernormal surface V_2^{8*}. The fact that the projection of this surface from a general point is the V_2^7 of the proper series confirms the fact that the projection of this V_3^6 from a point of itself is the V_3^5 just mentioned.

6.2. *Systems of class three.*

For systems of class three the greatest order is six. The line systems on the V_3^6 of [7] mapped by nodal cubics containing a cubic curve and having a fixed node thereon (3.24) are of order six and class three, and thus are mapped by novenic surfaces; by projection we get the line systems on the binodal V_3^5, the five nodal V_3^4, and the nine nodal V_3^3, mapped respectively by octavic, septimic, and sextic surfaces. The octavic surface is naturally the projection of the novenic surface, according with the fact that there is a plane on the binodal V_3^5 which is not a directrix of the line system derived as the projection of one on V_3^6. The other line system on V_3^5, equally of the fifth order and third class, has the plane as directrix, and is mapped by the supernormal octavic surface. The projections of the two systems on to [5] are equivalent, as we have seen.

6.3. *Systems of class four, five, or six.*

The lines on the nodal V_3^5 of [6], which generate it simply, form a system of order five and class four, and are thus mapped by the novenic surface; the projections of this locus, namely the four nodal V_3^4 of [5] and the eight nodal V_3^3 of [4] (3.25), correspond to the octavic and septimic surfaces. The other four nodal V_3^4 mentioned in 3.27, which contains no planes, contains four systems of lines of order four and class four; the corresponding surface cannot contain any lines, and must therefore be V_2^{8*}.

The lines of the simple system on the three nodal V_3^4 which is the projection of the general V_3^5 of [6] form a system of order four and

† Castelnuovo, **1**; Todd, **2**, 518, 542.

class five (3.26); these are thus represented by the novenic surface, and the projection of the system, lying on the seven nodal cubic primal of [4], is mapped by the octavic surface of the proper series. This primal contains three simple systems, one having each of the two planes as directrices, the one obtained by projection from V_3^4 being of this type; and the third, having both planes as directrices, is also of class five and corresponds to the supernormal surface V_2^8*.

Finally, the line system which generates the six nodal V_3^3 simply is of class six, and corresponds to the novenic surface.

The two particular cases mentioned above may also be considered; the chords of a normal quartic curve generate a cubic primal on which the curve is double, and six of the chords lie in a prime; the corresponding surface is thus the novenic surface; and similarly the lines on the cubic primal with a double line and a double point are each of class five, and correspond respectively to the two octavic surfaces, since one of the systems has the plane on the locus as directrix and the other has not.

This completes the relation of the threefold loci to the corresponding surfaces; the results may be summarised in the table annexed.

Representative surface	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$
V_2^9	(V_3^8) ↓ $V_3^7(P)$ $\underline{(1,2)_1}$	V_3^6 $\underline{2(1,3)}$	$V_3^5(n)$ $\underline{(1,4)}$ $(2,6)$	(V_3^5) ↓ $V_3^4(3n,P)$ $\underline{(1,5)_1}$ $(3,10)$	$V_3^3(6n)$ $\underline{2(1,6)}$ $(4,15)$
↓	↙	↓	↓	↘	
V_2^8 $V_2^{8}*$	$V_3^6(n,2P)$ V_3^6 $\underline{2(1,2)_1}$ $\underline{3(1,2)}$	$V_3^5(2n,P)$ $\underline{2(1,3)}$ $\underline{(1,3)_1}$	$V_3^4(4n,2P)$ $V_3^4(4n)$ $\underline{2(1,4)_1}$ $(2,6)$ $\underline{4(1,4)}$	$V_3^3(7n,2P)$ $\underline{2(1,5)_1}$ $\underline{(1,5)_2}$ $(3,10)$	
↘ ↙	↘ ↙	↘ ↙	↘ ↙		
V_2^7	$V_3^5(3n,4P)$ $\underline{3(1,2)_1}$	$V_3^4(5n,4P)$ $\underline{4(1,3)_1}$	$V_3^3(8n,5P)$ $\underline{4(1,4)_2}$ $(2,6)$		
↓	↓	↓			
V_2^6	$V_3^4(6n,8P)$ $\underline{4(1,2)_2}$	$V_3^3(9n,9P)$ $\underline{6(1,3)_3}$			
↓	↓				
V_2^5	$V_3^3(10n,15P)$ $\underline{6(1,2)_5}$				

Explanation of the table.

The table shows the various threefold loci which contain the line systems of different class s whose representative surface is the Del Pezzo surface indicated in the first column of the table. The symbols n, P refer to the number of nodes and planes that the locus contains. The line

systems on the locus are represented by symbols (a, b); a being the number of lines which pass through a general point of the locus, and b the class of the system. A suffix outside this bracket indicates the existence of a certain number of *directrix planes* for the system in question. In the case of loci with several non-equivalent systems of lines, the particular system which corresponds to the surface V_2^n is underlined. In the second row of the table, corresponding to octavic surfaces, we have two line systems for each class, corresponding to V_2^8 and V_2^{8*}; of the two given in the table the first refers to V_2^8 and is the projection of the system mapped by V_2^9; the two line systems may generate different loci, as in the cases $s = 2, 4$, or they may belong to different systems on the same locus, as in the cases $s = 3, 5$. An arrow in all cases denotes projection from a general point of the locus. The loci V_3^8, V_3^5 in brackets at the head of two of the columns are loci which do not contain simple generating systems, but which project into the loci occupying lower places in the column.

7. *The projections on to* [4] *of these loci.*

7.1. *Properties of the locus* G_4.

We now suppose that the loci which have been described in the previous sections are projected on to [4] from points whose joining space does not meet any line of the locus. We shall then get, in [4], line systems of order r and class s, and the representative surfaces of such line systems will by what has been said above (1.3) be either the Del Pezzo surfaces themselves or else the projections of these from points not lying on them. We shall now investigate the question of the existence of such surfaces on the locus G_4, which maps the lines of [4], and we shall deduce for each corresponding system in [4] the number of linear complexes which must in general contain it.

For this purpose we make use of the representation of the points of G_4 and its sections on linear spaces. These have been studied in detail by Semple† and by the author‡. The principal results established in these papers are the following:

(i) The locus G_4 contains a system of ∞^6 planes from any one of which it projects on to [6] uniquely; this gives a representation of the locus on [6] in which prime sections of G_4 are mapped by quadrics passing through a V_3^3 of planes which lies in a prime.

† Semple, **2**, § 6.
‡ Todd, **2**, § 4.1.

(ii) The general prime section of G_4 can be represented on [5] by quadrics passing through a cubic scroll. If the prime is a *tangent prime* to G_4 and represents a special linear complex, the cubic scroll breaks up into a quadric surface and a plane, not lying in the solid containing this, which passes through one of the generators.

(iii) The general V_4^5 which is the section of G_4 by a [7] is mapped on [4] by quadrics which pass through a twisted cubic curve; if this lies in a tangent prime the cubic curve breaks up into a conic and a line meeting it; if it lies in two tangent primes the cubic degenerates into three lines, one of which meets the other two.

Now all these representations are obtained by projecting the locus in question from one of the planes of the system mentioned above which it is known to contain. In particular, on G_4 itself, we can certainly find a plane which does not meet a given surface on this locus, since G_4 is of dimension six. If then G_4 is projected from such a plane the surface V_2^n will be projected into a surface in [6] of the same order, and this will be met by V_3^3 in a general prime section, since the points of V_3^3 arise from the points of G_4 which lie in the tangent prime along the plane of projection, and by the choice of this plane which we have made, this cuts the surface in question in a general prime section. Hence *any* V_2^n *on* G_4 *is mapped on* [6] *by a* V_2^n *meeting* V_3^3 *in a* C^n *which is the general prime section of* V_2^n; *and conversely.*

7.2. *Surfaces on* G_4 *with elliptic prime section.*

By the last theorem, any surface on G_4 with elliptic prime sections can be mapped on [6] by a surface of the same order n which meets the base V_3^3 of the representation in an elliptic curve of order n. If this surface necessarily lies on r linearly independent quadrics of the [6] which pass through the V_3^3, then the corresponding V_2^n will lie in r linearly independent primes, and hence the dimension of its containing space will be $9-r$. We shall now consider the cases $n = 6, 7, 8, 9$; the case $n = 5$ need not be considered, since the V_2^5 of [5] is the general section of G_4 by a [5].

We evidently require, in the first place, the number of linearly independent quadrics which pass through the projections into [6] of the various Del Pezzo surfaces. This may be obtained as follows. Considering, first, the surfaces of the proper series, the surface V_2^n of $[n]$ is mapped on the plane by means of cubic curves with $9-n$ base points. Its quadric sections are, therefore, mapped by sextic curves in the plane with $9-n$ fixed nodes; the freedom of these curves is clearly

$$27-3(9-n) = 3n,$$

so that the surface has $3n$ linearly independent quadric sections. The surface V_2^{8*}, mapped on the plane by quartics with two fixed nodes, has for quadric sections curves mapped by octavics with two fixed quadruple points, the freedom of these is $24 = 3.8$, so the formula holds equally well for this surface. Thus the number of linearly independent quadrics of $[n]$ passing through the surface is

$$\tfrac{1}{2}n(n+3)-3n = \tfrac{1}{2}n(n-3).$$

Now if we project the surface from an $[n-k-1]$ on to $[k]$, every quadric passing through the projected surface will arise from a quadric through the original surface which is a cone having the $[n-k-1]$ as vertex; and conversely every such cone gives rise to a quadric through the projected surface. We shall assume the result, easily seen on analytical grounds to be true, that the extra conditions imposed on the quadrics passing through a surface in $[n]$ to have a fixed $[n-k-1]$ of general position as double locus are independent. The number of these conditions is therefore

$$\tfrac{1}{2}n(n+3)-\tfrac{1}{2}k(k+3) = \tfrac{1}{2}(n-k)(n+k+3).$$

When $k = 6$ this reduces to $\frac{1}{2}(n-6)(n+9)$. It follows that the number of linearly independent quadrics which pass through the projection of the V_2^n of $[n]$ of elliptic sections on to $[6]$ is

$$\tfrac{1}{2}n(n-3)-\tfrac{1}{2}(n-6)(n+9) = 27-3n = 3(9-n).$$

We must now consider quadrics which contain the base V_3^3 of planes in addition to such a surface. It is easily seen from the equations of the locus that there are ten linearly independent quadrics passing through V_3^3, so that the postulation of this locus for quadrics is 18. On the other hand, the postulation of an elliptic curve of order n is $2n$. That this is true in the case of the *normal* curve in $[n-1]$ follows from a result of Castelnuovo†. This result is as follows. For a curve in $[r]$ of order n and genus p let χ be defined as the integer satisfying the inequality

$$\frac{n-1}{r-1}-1 \leqslant \chi < \frac{n-1}{r-1}.$$

Then, if π denotes the maximum possible genus for a curve of order n in $[r]$, the postulation of the curve for primals of sufficiently high order m is

$$mn-p+1-d,$$

† Castelnuovo, **2.**

where d depends only on the curve, and satisfies

$$0 \leqslant d \leqslant \pi - p;$$

and this certainly happens if the order m exceeds

$$\chi + \pi - p.$$

For a normal elliptic curve C^n in $[n-1]$ we have $\pi = 1$ and $\chi = 1$, whence $d = 0$ and the theorem applies for $m - 2$; thus the postulation for quadrics is given by $2n-1+1 = 2n$, as stated.

For the general elliptic curve in space of dimension less than $n-1$ we project the normal curve from a space of general position; then, by the same reasoning as in the case of the projected surfaces, the quadrics through the projected curve correspond to cones through the normal curve; and it is easy to see that the postulations of the two curves are the same.

If we now suppose that the further conditions imposed upon quadric primals which contain an elliptic curve of order n on V_3^3 to contain this locus entirely are independent, it follows that the postulation of V_3^3 for quadrics which already contain a ${}_1C^n$ of it is $18-2n$.

Combining the results obtained above, we infer that the freedom of quadric primals in [6] which pass through V_3^3 and through a V_2^n of elliptic section, meeting V_3^3 in a ${}_1C^n$, is

$$3(9-n)-(18-2n) = 9-n.$$

Thus the surfaces of order nine, eight, seven, and six lie respectively on none, one, two, and three independent such quadrics, and the corresponding surfaces on G_4 lie in their normal spaces.

It is not, however, the case that such surfaces lie in the *most general* linear sections of G_4 of appropriate dimension. The general [6] section of G_4, for example, is a V_3^5 which can easily be shown to contain no surfaces other than its complete intersections with primals of various orders. In particular, it contains only surfaces whose order is a multiple of five, and consequently cannot contain a sextic surface. Further, it is possible to show that the general section of G_4 by a [7] contains no septimic Del Pezzo surfaces, which lie in their normal space. For, if the V_4^5, which is the section in question, be projected from the unique plane of the system previously mentioned, which it can be shown to contain†, the V_4^5 is mapped on [4] by quadrics passing through a twisted cubic curve. A V_2^7 on the locus will project into some surface of order not exceeding seven, whose curve sections have genus 0 or 1, and which is met by any quadric through the

† Todd, **2**, § 3 . 31.

base cubic in an elliptic curve. The only such surface which maps a septimic surface on V_4^5 may be verified to be a quadric surface meeting the cubic in a point. This, however, maps a surface which lies in a prime through the plane of projection, and hence is not normal.

These two results could have been deduced from the table given in the previous section, for it is seen that every line system mapped by a sextic or septimic surface has at least one directrix plane, so that, in [4], there is at least one *special* linear complex containing the line system, and a tangent prime to G_4 containing the representative surface.

By projecting the section of G_4 by a general prime on to [5], and obtaining the representation of octavic surfaces lying in this prime, it is easy to verify, by methods analogous to the above, that the *general* prime section of G_4 contains octavic Del Pezzo surfaces which are in their normal spaces. In fact, the table given above shows the existence of line systems represented by such surfaces which contain no directrix planes.

We may remark that, if the line systems considered above are further projected into [3], either from general points or from points of the locus generated by the lines, we shall get in [3] various *congruences* of lines, which are mapped on the quadric in [5], which is, of course, the locus G_3, by surfaces with elliptic prime sections. From this point of view the loci have, with few exceptions, been considered by Fano†, in a paper in which are enumerated the congruences in space for which the sum of the order and class does not exceed eight. It seems, however, that the properties of such congruences are more simply considered as derived from the normal loci with which they are associated. In addition to the congruences whose order and class correspond to those of the systems considered above, we shall get, by projection of the various cubic primals in the table given previously from points of themselves, a number of congruences of the second order and of various classes which are bitangents of certain surfaces, namely those which are the apparent contours of the cubic primals from the point of projection; such surfaces are considered by Segre‡. These surfaces are all of order four, and in particular, for the projection of the Segre primal, we get one system of bitangents of the Kummer surface. The cubic primals will project into double three-dimensional spaces, which branch along the surface in question; these V_3^2 so obtained are rational and may be mapped by the pairs of points in an involution obtained by adding an additional base-point to the system of surfaces representing the V_3^3. Some of these involutions are considered by Babbage§.

† Fano, **1**.
‡ Segre, **2**.
§ Babbage, **1**.

8. *The quadric primal in* [4].

The quadric primal in [4] is peculiar in being generated by an irreducible *triple* infinity of lines, no three of which are coplanar. It is interesting to note the representative locus of this system on G_4. It is easily seen, in the first place, that no linear complex contains the lines in question; so that the representative locus does, in fact, lie in [9] and in no lower space. Through any pair of non-intersecting lines of the quadric passes a solid which meets the quadric in a quadric surface, on which the two lines determine a regulus, the lines of which must be mapped by the points of a conic; if two lines on the quadric meet there is a quadric cone of lines passing through their intersection, and the lines of this are equally mapped by the points of a conic. Thus the locus which maps the lines has a conic through every pair of points. Its order is the number of its points lying in [6], or the number of lines of the quadric common to three linear complexes. Taking two of these to be special complexes whose base planes meet in a line, and the third to be a special complex whose base plane does not meet this line, it is easy to see that the number in question is eight. The locus required is then evidently the V_3^8 of [9] of elliptic section mapped on [3] by all the quadrics of space. Hence we have the result:

The ∞^3 lines on the general quadric primal in [4] *are mapped on G_4 by the points of the Veronese threefold locus V_3^8, which is represented on* [3] *by the system of ∞^9 quadrics of space.*

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ON THE SECOND THEOREM OF CONSISTENCY IN THE THEORY OF SUMMATION BY TYPICAL MEANS

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1. It is known that if Σc_n is summable (l, κ), where $l_n = e^{\lambda_n}$, then it is summable (λ, κ) to the same sum; this is the second theorem of consistency*. This theorem has been extended by Hardy who has shown that summability (λ, κ) implies summability (μ, κ), where μ is an L-function of λ, and $\mu = O(\lambda^\Delta)$, where Δ is any constant however large†.

In this paper we propose to generalise Hardy's result by removing the restriction that μ should be an L-function of λ, and showing that under less stringent conditions summability (λ, κ) still implies summability (μ, κ). We shall prove first (§§ 2–4)

THEOREM 1. *If Σc_n is summable (λ, κ) to sum C, then it is summable (μ, κ) to the same sum, where $\mu = \phi(\lambda)$, and $\phi(t)$ is a function which increases steadily to infinity with t and satisfies the following conditions*:

(i) *κ an integer.*

$$(a) \quad \int_0^w t^r \left| \phi^{(r+1)}(t) \right| dt = O\{\phi(w)\} \quad (r = 1, 2, 3, \ldots, \kappa).$$

(ii) *κ not an integer.*

$$(a) \quad \int_0^w t^r \left| \phi^{(r+1)}(t) \right| dt = O\{\phi(w)\} \quad (r = 1, 2, 3, \ldots, k+1,\ k = [\kappa]),$$

* G. H. Hardy and M. Riesz, "General theory of Dirichlet's series", *Cambridge Mathematical Tract*, No. 18 (1915), 30–32.

† G. H. Hardy, "The second theorem of consistency for summable series", *Proc. London Math. Soc.* (2), 15 (1916), 72–88. For the properties of L-functions (or logarithmico-exponential functions), see G. H. Hardy, "Orders of infinity", *Cambridge Mathematical Tract*, No. 12.

and either (b) $\phi'(t)$ *is a monotonic increasing function,*

or (c_1) $\phi'(t)$ *is a monotonic decreasing function,*

and (c_2) $t\phi''(t) = O\{\phi'(t)\}$.

In §§ 5, 6 we discuss the most interesting special cases.

The notation adopted will, as far as possible, be the same as that used by Hardy and Riesz in the tract referred to above.

Without loss of generality we may suppose that $C = 0$, $\lambda_1 > 0$, $\mu_1 > 0$. We may also suppose $\phi(0) = 0$, since otherwise we can replace $\phi(t)$ by $\phi(t) - \phi(0)$. We are given that

$$C_\lambda^\kappa(w) = \kappa \int_0^w C_\lambda(t)(w-t)^{\kappa-1}\,dt = o(w^\kappa), \tag{1.1}$$

and we have to show that

$$C_\mu^\kappa(\omega) = \kappa \int_0^\omega C_\mu(\tau)(\omega-\tau)^{\kappa-1}\,d\tau = o(\omega^\kappa). \tag{1.21}$$

Putting $\omega = \phi(w)$, $\tau = \phi(t)$, and observing that $C_\mu\{\phi(t)\} = C_\lambda(t)$, we may write (1.21) in the form

$$\kappa \int_0^w C_\lambda(t)\{\phi(w) - \phi(t)\}^{\kappa-1}\phi'(t)\,dt = o\{\phi(w)^\kappa\},$$

or

$$\int_0^w C_\lambda(t)\frac{d}{dt}\{\phi(w) - \phi(t)\}^\kappa dt = o\{\phi(w)^\kappa\}. \tag{1.22}$$

2. We shall require the following lemmas; it is assumed that all necessary derivatives exist.

LEMMA A. *The n-th derivative of* $\{f(x)\}^m$ *is the sum of a number of terms of the form*

$$A\{f(x)\}^{m-r}\{f^{(1)}(x)\}^{a_1}\{f^{(2)}(x)\}^{a_2}\dots\{f^{(n)}(x)\}^{a_n},$$

where $r \leqslant n$, *and the* a*'s are positive integers or zero, such that*

$$\sum_{\nu=1}^{n} a_\nu = r, \quad \sum_{\nu=1}^{n} \nu a_\nu = n.$$

Further, if m *is a positive integer, then* $r \leqslant m$.

This is a particular case of a theorem due to Faa di Bruno on the n-th derivative of a function of a function; the factor $\{f(x)\}^{m-r}$ arises from

the differentiation of $\{f(x)\}^m$ with respect to $f(x)$ and is multiplied by a zero factor if m is a positive integer and $r > m$*.

LEMMA B. *If*

$$\int_0^w t^r |\phi^{(r+1)}(t)|\,dt = O\{\phi(w)\} \quad (r = 0, 1, 2, \ldots, n),$$

then

$$w^r |\phi^{(r)}(w)| = O\{\phi(w)\} \qquad (r = 1, 2, 3, \ldots, n).$$

For

$$\frac{d}{dt}\{t^r \phi^{(r)}(t)\} = rt^{r-1}\phi^{(r)}(t) + t^r \phi^{(r+1)}(t);$$

hence

$$w^r \phi^{(r)}(w) = r\int_0^w t^{r-1}\phi^{(r)}(t)\,dt + \int_0^w t^r \phi^{(r+1)}(t)\,dt;$$

therefore

$$|w^r \phi^{(r)}(w)| \leqslant r\int_0^w t^{r-1}|\phi^{(r)}(t)|\,dt + \int_0^w t^r |\phi^{(r+1)}(t)|\,dt = O\{\phi(w)\}.$$

LEMMA C. *If $\phi(t)$ is a monotonic increasing function, and $\phi'(t)$ is monotonic, then $\{\phi(w)-\phi(t)\}/(w-t)$ (where w is fixed and $t \leqslant w$) is a monotonic function of t which increases or decreases according as $\phi'(t)$ increases or decreases.*

For

$$\frac{d}{dt}\left\{\frac{\phi(w)-\phi(t)}{w-t}\right\} = \frac{\phi(w)-\phi(t)-(w-t)\phi'(t)}{(w-t)^2}$$

$$= \frac{(w-t)\phi'(\eta)-(w-t)\phi'(t)}{(w-t)^2} \qquad (t \leqslant \eta \leqslant w)$$

$$= \frac{\phi'(\eta)-\phi'(t)}{w-t}.$$

This last expression is greater than or equal to 0, or less than or equal to 0, according as $\phi'(t)$ is an increasing or a decreasing function. This proves the lemma.

LEMMA D. *If c_n is real, $0 \leqslant \xi \leqslant \omega$, $k \geqslant 0$, and $0 < \kappa - k \leqslant 1$, then*

$$\frac{\Gamma(\kappa+1)}{\Gamma(k+1)\,\Gamma(\kappa-k)}\left|\int_0^\xi C^k(\tau)(\omega-\tau)^{\kappa-k-1}\,d\tau\right| \leqslant \operatorname*{Max}_{0 \leqslant \tau \leqslant \xi} |C^\kappa(\tau)|.$$

This is a known result†.

* De la Vallée-Poussin, *Cours d'analyse infinitésimale*, 1, 5 ed., 89.

† G. H. Hardy and M. Riesz, "General theory of Dirichlet's series", 29.

LEMMA E. *If $f(x)$ is a positive decreasing function and $xf'(x) = O\{f(x)\}$, then there exists a constant $c > 1$ such that*

$$\frac{f(cx)}{f(x)} > \tfrac{1}{2} \quad \textit{for all } x > x_0.$$

Since $xf'(x) = O\{f(x)\}$ there exists an x_0 such that

$$\left|\frac{xf'(x)}{f(x)}\right| < K \quad \text{for all } x > x_0.$$

Clearly $f(cx)/f(x) \leqslant 1$. Now

$$\begin{aligned}\left|\frac{f(cx)}{f(x)} - 1\right| &= \left|\frac{f(cx)-f(x)}{f(x)}\right| \\ &= \frac{(c-1)\,x\,|f'(\xi)|}{f(x)} \quad (x < \xi < cx) \\ &\leqslant \frac{(c-1)\,\xi\,|f'(\xi)|}{f(\xi)} < K(c-1) \quad \text{for all } x > x_0, \\ &< \tfrac{1}{2} \text{ by appropriate choice of } c.\end{aligned}$$

With c so chosen, $f(cx)/f(x) > \frac{1}{2}$.

3. (i) *Proof when κ is an integer.* In this case

$$C_\lambda(t) = \frac{1}{\kappa!}\left(\frac{d}{dt}\right)^\kappa C_\lambda^\kappa(t).$$

We substitute this expression for $C_\lambda(t)$ in (1.22) and integrate κ times by parts. All the integrated terms vanish, except the one arising from the last integration; this term is a constant multiple of

$$\begin{aligned}C_\lambda^\kappa(w)\left[\left(\frac{d}{dt}\right)^\kappa \{\phi(w)-\phi(t)\}^\kappa\right]_{t=w} &= \kappa!\,\{-\phi'(w)\}^\kappa\, C_\lambda^\kappa(w) \\ &= o\{w^\kappa \phi'(w)^\kappa\} = o\{\phi(w)^\kappa\} \qquad (3.1)\end{aligned}$$

by Lemma B; for the condition of this lemma is satisfied for $r = 0$, since $\phi'(t) \geqslant 0$, and for $r = 1$ by the condition (a).

The integral with which we are left is a constant multiple of

$$\int_0^w C_\lambda^\kappa(t)\left(\frac{d}{dt}\right)^{\kappa+1} \{\phi(w)-\phi(t)\}^\kappa dt, \qquad (3.2)$$

which, by Lemma A, can be expressed as the sum of constant multiples of integrals of the type

$$\int_0^w C_\lambda^\kappa(t)\{\phi(w)-\phi(t)\}^{\kappa-r}\{\phi^{(1)}(t)\}^{\alpha_1}\{\phi^{(2)}(t)\}^{\alpha_2}\dots\{\phi^{(\kappa+1)}(t)\}^{\alpha_{\kappa+1}}dt,$$

where $r\leqslant\kappa$, $\sum_{n=1}^{\kappa+1} n\alpha_n=\kappa+1$, and $\sum_{n=1}^{\kappa+1}\alpha_n=r$; the α's being positive integers or zero. We write this integral, for short, as

$$\int_0^w C_\lambda^\kappa(t)\{\phi(w)-\phi(t)\}^{\kappa-r}\prod_{n=1}^{\kappa+1}\{\phi^{(n)}(t)\}^{\alpha_n}dt. \tag{3.3}$$

Now, by (1.1), $|C_\lambda^\kappa(t)|<\epsilon t^\kappa$ for all $t\geqslant t_0=t_0(\epsilon)$. Therefore, if $w>t_0$, this last integral is less, in absolute value, than

$$\left|\int_0^{t_0} C_\lambda^\kappa(t)\{\phi(w)-\phi(t)\}^{\kappa-r}\prod\{\phi^{(n)}(t)\}^{\alpha_n}dt\right|+\epsilon\int_{t_0}^w t^\kappa\{\phi(w)-\phi(t)\}^{\kappa-r}\prod|\{\phi^{(n)}(t)\}^{\alpha_n}|\,dt.$$

The first integral is less than $K_\epsilon\{\phi(w)\}^{\kappa-r}$, where K is a constant depending on ϵ only. Hence (3.3) is, in absolute value, less than

$$\epsilon\int_0^w t^\kappa\{\phi(w)-\phi(t)\}^{\kappa-r}\prod_{n=1}^{\kappa+1}|\{\phi^{(n)}(t)\}^{\alpha_n}|\,dt+K_\epsilon\{\phi(w)\}^{\kappa-r}$$

$$=\epsilon\int_0^w t^{m-1}|\phi^{(m)}(t)|\{\phi(w)-\phi(t)\}^{\kappa-r}\prod_{n=1}^{\kappa}|\{t^n\phi^{(n)}(t)\}^{\beta_n}|\,dt+K_\epsilon\{\phi(w)\}^{\kappa-r},$$

where $\beta_n=\alpha_n$ $(n\neq m)$, and $\beta_m=\alpha_m-1$ (m being such that $\alpha_m\geqslant 1$); therefore $\Sigma\beta_n=r-1$. The product in this last integral need not be taken beyond $n=\kappa$. For, if $\alpha_{\kappa+1}=0$, the factor in which it occurs can be omitted; while, if $\alpha_{\kappa+1}>0$, then, by the first of the two relations connecting the α's, we must have $\alpha_1=\alpha_2=\alpha_3=\dots=\alpha_\kappa=0$ and $\alpha_{\kappa+1}=1$. We must, therefore, take $\alpha_m=\alpha_{\kappa+1}$; thus $\beta_{\kappa+1}=0$ and the factor in which it occurs need not be included. Hence Lemma B applies to each factor of the product. Applying Lemma B, we see that the integral (3.3) is, in absolute value, less than

$$\epsilon K\{\phi(w)\}^{\kappa-r}\{\phi(w)\}^{r-1}\int_0^w t^{m-1}|\phi^{(m)}(t)|\,dt+K_\epsilon\{\phi(w)\}^{\kappa-r}$$

$$<\epsilon K\{\phi(w)\}^\kappa+K_\epsilon\{\phi(w)\}^{\kappa-r}$$

by condition (a), where K is a constant (not necessarily the same at different occurrences) independent of ϵ. Hence the integral is $o\{\phi(w)^\kappa\}$, since the two equations satisfied by the α's ensure that $r\geqslant 1$. Thus (3.2) is $o\{\phi(w)^\kappa\}$ and this, along with (3.1), establishes (1.22) for κ an integer.

[*Note.*—It would appear from Hardy's paper, mentioned previously, that the conditions (a) which we have imposed on $\phi(t)$ might be replaced by the less exacting conditions $t^r\phi^{(r)}(t) = O\{\phi(t)\}$. But in dealing with the integral corresponding to (3.3) above, Hardy apparently assumes that the exponent of $\phi'(t)$ (s_1 in his notation, α_1 in ours) is never zero. His arguments do not cover the case in which s_1 is zero, but it is easy to rectify this by applying the conditions $t^r\phi^{(r+1)}(t) = O\{\phi'(t)\}$, which he uses in another connection on pages 83–84, *loc. cit.* For general functions these conditions are more stringent than ours, which, in turn, are more stringent than $t^r\phi^{(r)}(t) = O\{\phi(t)\}$. For L-functions the three sets of conditions are equivalent.

With regard to the completion of Hardy's argument, we may insert, just before the last line of § 5 in his paper:

"If $s_1 = 0$, then, by 5.332, there is at least one index, s_n say, which is not zero. Using the relation $\mu^{(n)} = O(\mu'/\sigma^{n-1})$, we can replace $\{\mu^{(n)}\}^{s_n}$ by $O(\sigma^{-n+1}\mu'\{\mu^{(n)}\}^{s_n-1})$. Now, replacing all derivatives of order higher than the first, by means of the relations $\mu^{(r)} = O(\mu\sigma^{-r})$, we see that 5.34 is of the form 5.35 when $s_1 = 0$".

A similar amendment is required on page 84, in dealing with the integral 7.33.]

4. (ii) *Proof when κ is not an integer.* As before we have to establish (1.22). Let k be the integral part of κ.

Integrating (1.22) by parts k times, the integrated terms all vanish and we are left with an integral which is a constant multiple of

$$\int_0^w C_\lambda^k(t)\left(\frac{d}{dt}\right)^{k+1}\{\phi(w)-\phi(t)\}^\kappa\,dt.$$

By Lemma A this can be expressed as the sum of

$$\int_0^w C_\lambda^k(t)\{\phi(w)-\phi(t)\}^{\kappa-k-1}\{\phi'(t)\}^{k+1}\,dt \tag{4.1}$$

together with other integrals of the type

$$\int_0^w C_\lambda^k(t)\{\phi(w)-\phi(t)\}^{\kappa-r}\prod_{n=1}^{k+1}\{\phi^{(n)}(t)\}^{\alpha_n}\,dt, \tag{4.2}$$

where $r<\kappa$, $\sum_{n=1}^{k+1} n\alpha_n = k+1$, $\sum_{n=1}^{k+1}\alpha_n = r$, and certain constants have been

omitted. The integral (4.1) is the only one in which the exponent of $\phi(w)-\phi(t)$ is negative, and requires special treatment. We consider first the integrals of type (4.2).

If we integrate (4.2) once again by parts, the integrated term vanishes since $\kappa-r>0$, and we get a number of integrals which again have to be considered in two groups.

First, we have the integral which is a constant multiple of

$$\int_0^w C_\lambda^{k+1}(t)\{\phi(w)-\phi(t)\}^{\kappa-r-1}\phi'(t)\prod_{n-1}^{k+1}\{\phi^{(n)}(t)\}^{\alpha_n}dt.$$

Since the series is summable (λ, κ) to sum 0, it is summable $(\lambda, k+1)$ to the same sum, by the first theorem of consistency; therefore $C_\lambda^{k+1}(t)=o(t^{k+1})$. Remembering that $k+1=\Sigma n\alpha_n$, and that $\phi'(t)$ is positive, we deduce, as before, that, if $w>t_0=t_0(\epsilon)$, the absolute value of the integral last written is less than

$$\epsilon\int_0^w\{\phi(w)-\phi(t)\}^{\kappa-r-1}\phi'(t)\prod_{n=1}^{k+1}|\{t^n\phi^{(n)}(t)\}^{\alpha_n}|\,dt+K_\epsilon\int_0^{t_0}\{\phi(w)-\phi(t)\}^{\kappa-r-1}\phi'(t)\,dt$$

$$<\epsilon K\{\phi(w)\}^r\int_0^w\{\phi(w)-\phi(t)\}^{\kappa-r-1}\phi'(t)\,dt+K_\epsilon\{\phi(w)\}^{\kappa-r}$$

by Lemma B, since $\Sigma\alpha_n=r$. Hence

$$\int_0^w C_\lambda^{k+1}(t)\{\phi(w)-\phi(t)\}^{\kappa-r-1}\phi'(t)\prod_{n=1}^{k+1}\{\phi^{(n)}(t)\}^{\alpha_n}dt=o\{\phi(w)^\kappa\}. \quad (4.31)$$

A typical member of the second group of integrals is

$$\int_0^w C_\lambda^{k+1}(t)\{\phi(w)-\phi(t)\}^{\kappa-r}\prod_{n=1}^{k+2}\{\phi^{(n)}(t)\}^{\beta_n}dt,$$

where $\Sigma n\beta_n=k+2$ and $\Sigma\beta_n=r$. Treating this integral in the same way as we treated (3.3), we show that it is

$$o\{\phi(w)^\kappa\}. \quad (4.32)$$

By (4.31) and (4.32) each of the integrals (4.2) is

$$o\{\phi(w)^\kappa\}. \quad (4.4)$$

We have now to consider the integral (4.1), and it is here that we assume $\phi'(t)$ monotonic. The two cases which thus arise require different treatment, and they must therefore be considered separately. In both cases we shall assume c_n real, but this involves no actual loss of generality.

We first take the case of $\phi'(t)$ being a monotonic increasing function. We can write (4.1) as

$$\int_0^w C_\lambda^k(t)\left\{\frac{\phi(w)-\phi(t)}{w-t}\right\}^{\kappa-k-1}(w-t)^{\kappa-k-1}\{\phi'(t)\}^{k+1}dt.$$

Applying the second mean value theorem, and observing that $\kappa-k-1<0$, we see that this, by Lemma C, is equal to

$$\left(\frac{\phi(w)}{w}\right)^{\kappa-k-1}\{\phi'(\xi_1)\}^{k+1}\int_{\xi_2}^{\xi_1} C_\lambda^k(t)(w-t)^{\kappa-k-1}dt \quad (0\leqslant\xi_2<\xi_1\leqslant w),$$

and this, by Lemma D, in absolute value, does not exceed

$$\left(\frac{\phi(w)}{w}\right)^{\kappa-k-1}\{\phi'(w)\}^{k+1}.2.\frac{\Gamma(k+1)\,\Gamma(\kappa-k)}{\Gamma(\kappa+1)}\operatorname*{Max}_{0\leqslant t\leqslant\xi_1}|C_\lambda^\kappa(t)|$$

$$=o\left(\frac{\phi(w)}{w}\right)^{\kappa-k-1}\{\phi'(w)\}^{k+1}w^\kappa$$

$$=o[\{\phi(w)\}^{\kappa-k-1}\{w\phi'(w)\}^{k+1}]=o\{\phi(w)^\kappa\}. \qquad (4.5)$$

(4.4) and (4.5) establish (1.22) for $\phi'(t)$ a monotonic increasing function.

Now suppose that $\phi'(t)$ is a monotonic decreasing function. By (c_2) we can choose a constant $c>1$, as in Lemma E, so that $\phi'(ct)/\phi'(t)>\frac{1}{2}$ for all $t>t_0$. Let w be so large that $w/c>t_0$; then (4.1) can be written

$$\left(\int_0^{w/c}+\int_{w/c}^{w}\right)C_\lambda^k(t)\{\phi(w)-\phi(t)\}^{\kappa-k-1}\{\phi'(t)\}^{k+1}dt=I_1+I_2 \quad \text{(say)}.$$

Integrating by parts,

$$I_1=\left[\frac{1}{k+1}C_\lambda^{k+1}(t)\{\phi(w)-\phi(t)\}^{\kappa-k-1}\{\phi'(t)\}^{k+1}\right]_0^{w/c}$$

$$-\frac{1}{k+1}\int_0^{w/c}C_\lambda^{k+1}(t)\,\frac{d}{dt}\,[\{\phi(w)-\phi(t)\}^{\kappa-k-1}\{\phi'(t)\}^{k+1}]dt.$$

The integrated term vanishes at the lower limit, and at the upper limit is

$$o\left(\frac{w}{c}\right)^{k+1}\left\{\phi(w)-\phi\left(\frac{w}{c}\right)\right\}^{\kappa-k-1}\left\{\phi'\left(\frac{w}{c}\right)\right\}^{k+1}$$

$$=o\,w^{k+1}\left\{\left(w-\frac{w}{c}\right)\phi'(\eta)\right\}^{\kappa-k-1}\left\{\phi'\left(\frac{w}{c}\right)\right\}^{k+1}\quad\left(\frac{w}{c}<\eta<w\right)$$

$$=o\,w^{k+1}w^{\kappa-k-1}\{\phi'(w)\}^{\kappa-k-1}\{\phi'(w/c)\}^{k+1}\quad\text{since }\phi'(w)\leqslant\phi'(\eta),$$

$$=o\,w^\kappa\{\phi'(w)\}^{\kappa-k-1}\{\phi'(w)\}^{k+1}\qquad\text{by lemma E,}$$

$$=o\{\phi(w)^\kappa\}. \qquad (4.61)$$

The remaining integral, expressed as two integrals, is

$$(k+1-\kappa)\int_0^{w/c} C_\lambda^{k+1}(t)\{\phi(w)-\phi(t)\}^{\kappa-k-2}\{\phi'(t)\}^{k+2}\,dt$$

$$+(k+1)\int_0^{w/c} C_\lambda^{k+1}(t)\{\phi(w)-\phi(t)\}^{\kappa-k-1}\{\phi'(t)\}^{k}\phi''(t)\,dt$$

$$=J_1+J_2 \quad \text{(say)}.$$

In each of these integrals we split the range of integration as in dealing with integral (3.3), and we obtain for J_1

$$|J_1|<(k+1-\kappa)\,\epsilon\int_{t_1}^{w/c} t^{k+1}\{\phi'(t)\}^{k+1}\{\phi(w)-\phi(t)\}^{\kappa-k-2}\phi'(t)\,dt$$

$$+(k+1+\kappa)\left|\int_0^{t_1} C_\lambda^{k+1}(t)\{\phi(w)-\phi(t)\}^{\kappa-k-2}\{\phi'(t)\}^{k+2}\,dt\right|.$$

The second integral is (for $w>w_2$) less than $K_\epsilon\{\phi(w)\}^{\kappa-k-2}$, where K_ϵ is a constant depending on ϵ. To deal with the first integral we integrate back by parts.

$$|J_1|<\epsilon[t^{k+1}\{\phi(w)-\phi(t)\}^{\kappa-k-1}\{\phi'(t)\}^{k+1}]_0^{w/c}$$

$$+\epsilon(k+1)\int_0^{w/c}\{\phi(w)-\phi(t)\}^{\kappa-k-1}t^k\{\phi'(t)\}^{k+1}dt$$

$$+\epsilon(k+1)\int_0^{w/c}\{\phi(w)-\phi(t)\}^{\kappa-k-1}t^{k+1}\{\phi'(t)\}^{k}\phi''(t)\,dt+K_\epsilon\{\phi(w)\}^{\kappa-k-2}.$$

The integrated term is less than $\epsilon K\phi(w)^\kappa$, as shown above. The first integral is less than

$$\epsilon K\{\phi(w)\}^k\int_0^{w/c}\{\phi(w)-\phi(t)\}^{\kappa-k-1}\phi'(t)\,dt$$

$$=\epsilon K\{\phi(w)\}^k\left[-\frac{1}{\kappa-k}\{\phi(w)-\phi(t)\}^{\kappa-k}\right]_0^{w/c}<\epsilon K\{\phi(w)^\kappa\},$$

$\kappa-k$ being positive. Since the integrand is of constant sign the second integral is less than

$$\epsilon K\{\phi(w)\}^k\left|\int_0^{w/c}\{\phi(w)-\phi(t)\}^{\kappa-k-1}t\phi''(t)\,dt\right|$$

$$<\epsilon K\{\phi(w)\}^k\int_0^{w/c}\{\phi(w)-\phi(t)\}^{\kappa-k-1}\phi'(t)\,dt \quad \text{by condition } (c_2)$$

$$<\epsilon K\{\phi(w)^\kappa\}.$$

Thus $$|J_1| < \epsilon K\{\phi(w)^\kappa\} + K_\epsilon\{\phi(w)\}^{\kappa-k-2},$$

where K is a constant independent of ϵ; therefore

$$J_1 = o\{\phi(w)^\kappa\}. \tag{4.62}$$

Similarly we obtain for J_2, since $\phi''(t)$ is of constant sign,

$$|J_2| < \epsilon K\{\phi(w)\}^k \left| \int_0^{w/c} \{\phi(w)-\phi(t)\}^{\kappa-k-1} t\phi''(t)\,dt \right| + K_\epsilon\{\phi(w)\}^{\kappa-k-1}$$

$$< \epsilon K\{\phi(w)\}^k \int_0^{w/c} \{\phi(w)-\phi(t)\}^{\kappa-k-1}\phi'(t)\,dt + K_\epsilon\{\phi(w)\}^{\kappa-k-1}$$

by condition (c_2). Hence

$$J_2 = o\{\phi(w)^\kappa\}. \tag{4.63}$$

By (4.61), (4.62), and (4.63) it follows that

$$I_1 = o\{\phi(w)^\kappa\}. \tag{4.7}$$

Finally I_2 can be written

$$\int_{w/c}^{w} C_\lambda^k(t) \left\{\frac{\phi(w)-\phi(t)}{w-t}\right\}^{\kappa-k-1} (w-t)^{\kappa-k-1}\{\phi'(t)\}^{k+1}\,dt.$$

Hence, by the second mean value theorem and Lemma C,

$$I_2 = \{\phi'(w)\}^{\kappa-k-1}\{\phi'(\xi_1)\}^{k+1} \int_{\xi_1}^{\xi_2} C_\lambda^k(t)(w-t)^{\kappa-k-1}\,dt \quad (w/c \leqslant \xi_1 < \xi_2 \leqslant w).$$

Therefore, by Lemma D,

$$|I_2| \leqslant \{\phi'(w)\}^{\kappa-k-1}\left\{\phi'\left(\frac{w}{c}\right)\right\}^{k+1} 2\,\frac{\Gamma(k+1)\,\Gamma(\kappa-k)}{\Gamma(\kappa+1)} \operatorname*{Max}_{0\leqslant t\leqslant w} |C_\lambda^\kappa(t)|$$

$$= o[\{w\phi'(w)\}^\kappa] \quad \text{by Lemma E,}$$

$$= o\{\phi(w)^\kappa\}. \tag{4.8}$$

By (4.4), (4.7), and (4.8), (1.22) holds for $\phi'(t)$ a monotonic decreasing function subject to condition (c_2).

This completes the proof of the theorem.

5. The most important cases of the above are embodied in the following theorems.

THEOREM 2A. *If* Σc_n *is summable* (λ, κ), *then it is summable* (μ, κ), *where* $\mu = \phi(\lambda)$, *provided that*

(*a*) *all the derivatives of* $\phi(t)$ *are monotonic,*

(*b*) $\phi'(t)$ *is an increasing function,*

and (*c*) *at least one of the derivatives is a decreasing function.*

The general assumption that $\phi(t)$ increases steadily to infinity with t is, of course, retained throughout.

Suppose, in accordance with conditions (*b*) and (*c*), that

$$\phi(t), \quad \phi'(t), \quad \ldots, \quad \phi^{(s-1)}(t)$$

are increasing functions, and that $\phi^{(s)}(t)$ is a decreasing function, where $s \geqslant 2$. Then, since these functions are positive and $\phi^{(s)}(t)$ is decreasing,

$$0 \leqslant t\phi^{(s)}(t) \leqslant \int_0^t \phi^{(s)}(u)\,du = \phi^{(s-1)}(t) - \phi^{(s-1)}(0) \leqslant \phi^{(s-1)}(t).$$

Hence
$$t\phi^{(s)}(t) = O\{\phi^{(s-1)}(t)\}.$$

It follows that

$$t\phi^{(s)}(t) + \phi^{(s-1)}(t) = O\{\phi^{(s-1)}(t)\},$$

and therefore, by integration,

$$t\phi^{(s-1)}(t) = O\{\phi^{(s-2)}(t)\}.$$

Repeating this process we ultimately obtain

$$t\phi'(t) = O\{\phi(t)\}. \tag{5.1}$$

From (5.1) and condition (*a*) we can deduce that

$$t^r \phi^{(r)}(t) = O\{\phi(t)\} \quad (r = 1, 2, 3, \ldots). \tag{5.2}$$

For suppose (5.2) to be true for $r = 1, 2, \ldots, n$; then, in particular,

$$t^n \phi^{(n)}(t) = O\{\phi(t)\},$$

and therefore
$$|t^n \phi^{(n)}(t)| < K\phi(t) \tag{5.31}$$

for all $t > t_0$.

Write $\psi(t) = \phi^{(n)}(t)$; then, if $\psi'(t)$ is an increasing function,

$$\frac{\psi(t-h)-\psi(t)}{-h} \leqslant \psi'(t) \leqslant \frac{\psi(t+h)-\psi(t)}{h} \quad (0<h<t),$$

while, if $\psi'(t)$ is a decreasing function, these inequalities are reversed. Thus in either case

$$|\psi'(t)| \leqslant \frac{1}{h}\{|\psi(t+h)|+|\psi(t)|+|\psi(t-h)|\}$$

$$< \frac{K}{h}\left\{\frac{\phi(t+h)}{(t+h)^n}+\frac{\phi(t)}{t^n}+\frac{\phi(t-h)}{(t-h)^n}\right\} \tag{5.32}$$

by the inequality (5.31), if $t > t_0+h$.

Now, since $\phi(t)$ is a positive increasing function, $f(t) = 1/\phi(t)$ is a positive decreasing function; and

$$tf'(t) = \frac{-t\phi'(t)}{\{\phi(t)\}^2} = O\left(\frac{1}{\phi(t)}\right) = O\{f(t)\}.$$

Thus the conditions of Lemma E are satisfied, and we can therefore find a constant $c > 1$ such that $f(ct)/f(t) > \frac{1}{2}$ for all sufficiently large t. That is $\phi(ct) < 2\phi(t)$ for all $t > t_1$. We may clearly suppose that $c < \frac{3}{2}$. Then, if in the inequality (5.32) we take $h = (c-1)t$, we have

$$\phi(t+h) = \phi(ct) < 2\phi(t).$$

Further $\phi(t-h) \leqslant \phi(t)$, and $t-h = (2-c)t > \frac{1}{2}t$. Hence, substituting in (5.32),

$$|\psi'(t)| < \frac{K}{t}\left\{\frac{2\phi(t)}{t^n}+\frac{\phi(t)}{t^n}+\frac{2^n\phi(t)}{t^n}\right\} < K_n\frac{\phi(t)}{t^{n+1}}$$

for all $t > \mathrm{Max}(2t_0, t_1)$. Therefore

$$t^{n+1}\phi^{(n+1)}(t) = O\{\phi(t)\},$$

and the truth of (5.2) follows immediately.

To complete the proof we have only to show that the conditions (5.2) imply the conditions (a) of Theorem 1. Since $\phi^{(r)}(t)$ is monotonic, so that $\phi^{(r+1)}(t)$ is of constant sign,

$$\int_0^w t^r|\phi^{(r+1)}(t)|\,dt = \left|\int_0^w t^r\phi^{(r+1)}(t)\,dt\right| = \left|w^r\phi^{(r)}(w) - r\int_0^w t^{r-1}\phi^{(r)}(t)\,dt\right|.$$

Putting $r = 1, 2, 3, \ldots$ in turn, we obtain the desired result.

The case in which $\phi'(t)$ is a decreasing function is excluded from Theorem 2A, but we can include it in

THEOREM 2B. *If* Σc_n *is summable* (λ, κ), *then it is summable* (μ, κ), *where* $\mu = \phi(\lambda)$, *provided that*

(*a*) *all the derivatives of* $\phi(t)$ *are monotonic,*

and (*b*) *there exists a positive constant* c, *such that* $t^c \phi'(t)$ *is an increasing function, and* $t^{-c}\phi'(t)$ *a decreasing function.*

For a given function satisfying condition (*a*), one of the conditions (*b*) is clearly redundant. The other condition implies that

$$t\phi''(t) = O\{\phi'(t)\},$$

as can be seen by differentiating $t^c\phi'(t)$ or $t^{-c}\phi'(t)$, as the case may be. For example, suppose that $\phi'(t)$ is a decreasing function, then $\phi''(t)$ is negative. Differentiating $t^c\phi'(t)$, the condition is that there exists a positive constant c such that

$$t^c\phi''(t) + ct^{c-1}\phi'(t) \geqslant 0,$$

i.e.

$$-t\phi''(t) \leqslant c\phi'(t).$$

Therefore, since $\phi''(t)$ is negative,

$$t\phi''(t) = O\{\phi'(t)\}.$$

This is condition (c_2) of Theorem 1, and the fulfilment of the other conditions is established as in Theorem 2A.

6. In Theorems 2A and 2B we assumed the derivatives of $\phi(t)$ to be monotonic over the range $t \geqslant 0$. But it is clearly sufficient to suppose each derivative *ultimately* monotonic. This may be seen by considering summability (λ', κ) of $\Sigma c'_n$, where $\lambda'_n = \lambda_{h+n}$, $c'_n = c_{h+n}$, and h is a fixed positive integer chosen so large that $\phi^{(r)}(t)$ is monotonic for $t \geqslant \lambda_h$ $(r = 1, 2, \ldots, k+1)$. We thus obtain in particular

THEOREM 3. *If* Σc_n *is summable* (λ, κ) *it is summable* (μ, κ), *where* $\mu = \phi(\lambda)$, *provided that*

(*a*) $\phi(t)$ *is an L-function of* t,

(*b*) $\phi(t) = O(t^\Delta)$, *where* Δ *is some positive constant.*

For, since $\phi(t) \to \infty$, the O condition implies that $\log \phi(t) = O(\log t)$; therefore, differentiating, we get

$$\frac{\phi'(t)}{\phi(t)} = O\left(\frac{1}{t}\right).$$

That is $$t\phi'(t) = O\{\phi(t)\}.$$

Differentiating* again,

$$t\phi''(t) = O\{\phi'(t)\}.$$

This is equivalent to condition (b) of Theorem 2_B. The other condition, that all derivatives of $\phi(t)$ must be ultimately monotonic, is a consequence of the fact that all are L-functions†. The result now follows by Theorem 2_B.

I should like to acknowledge my indebtedness to Mr. A. E. Ingham, to whose helpful advice and encouragement this paper is largely due.

* For the justification of the differentiation, see Hardy, *Orders of infinity*, 2nd ed., 34_4 (§ 5.22). Since $\log t$ and $\phi(t)$ tend to infinity with t, neither of the exceptional cases can arise here.

† *Loc. cit.*, 18.

CONFIGURATIONS OF SPACES WHICH OSCULATE NORMAL RATIONAL CURVES*

By T. G. ROOM.

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In a recent paper Mr. Grace proved the theorem: "In $[n]$ there is one and only one normal rational curve osculating $r+4$ $[r]$'s, where $n = r^2+4r+1$†." A proof, which also covers the general case, was given in detail for the case $r = 1$, and its application was made to the case $r = 2$. At the same time a related problem was discussed: what is the condition that six lines in [8] shall be tangents to a normal rational curve?

Dealing first with the case $r = 1$, Mr. Grace proves the existence of a normal sextic touching five arbitrary lines in [6]. It is shown that each set of four lines determines a plane which meets them all, and further, that, calling the lines $h_1, \ldots, h_5$, and denoting by k_ρ the plane which meets all of them but h_ρ, the five [4]'s $[h_\rho, k_\rho]$ have a common point. If through this point the line is drawn which meets h_ρ and k_ρ, then the intersection of this line with h_ρ is the point where h_ρ touches the sextic. (§§ 9, 10, and 11 of G.) In the related problem of six lines in [8], it is found that each set of five lines determines a [3], giving a double-six of lines and [3]'s; but in this case the necessary and sufficient condition that there shall be a normal rational curve touching the lines is that the [5]'s, which join the lines to the [3]'s they do not meet, should have a common point. After this the points of contact are constructible as before (G, § 14).

* Osculate is used throughout this paper to mean contact of the highest possible order, *i.e.*, a $[r]$ osculates a curve if it has $r+1$ "consecutive" points common with it.

† *Proc. Camb. Phil. Soc.*, 25 (1929), 421. This paper will be referred to as G. Other papers to which reference is made in the course of the present paper are two by the writer, *Proc. London Math. Soc.* (2), 28 (1927), 312, and *ibid.*, 31 (1930), 455. These will be quoted as R_1 and R_2.

Proceeding now to $r = 2$, we start with six planes in [13], and from these can be derived a double-six of [2]'s and [3]'s. Mr. Grace proves that the six [6]'s, which join the [2]'s to the [3]'s which they do not meet, have a common transversal line. Thence the points of contact of the normal rational curve which osculates the planes can be constructed as before (G, § 15, and § 1.4 of this paper).

In this paper the problem is attacked from a rather different point of view, viz. that of the properties of the rational curves which lie in the spaces of the figure. This makes the case $r = 1$ awkward to deal with, because one of the rational curves reduces to a point (the point common to the five [4]'s mentioned above), but in the case $r = 2$ we have three proper kinds of curves: the line which meets the six [6]'s, a definite conic in each of the planes, and a definite twisted cubic in each of the [3]'s of the double-six. This case is discussed at length (§ 1) in such a way that generalization of all the theorems found is at once possible (these are given in § 3.1 and § 3.2), and it is found that there are theorems for cases not included in this "main-line", of which those concerning six lines which are tangent to a curve in [8] are examples, which can be easily deduced from the main-line theorems by projection. (A particular case is discussed in § 2, and the general case is dealt with in § 3.3.)

1. *The double-six of planes and* [3]*'s in* [13].

1.1. The figure determined by six planes in [13].

Let $h_1, \ldots, h_5$ be five arbitrary planes in [13]. Then $[h_2, h_3, h_4, h_5]$ is a [11] and therefore meets h_1 in a point. We get in h_2, h_3, h_4, h_5 similar points, and all these five points lie in the five [11]'s which can be drawn through the sets of four of the planes, and they, therefore, lie in a [3]. On the other hand, any [3] which meets the five planes lies in the space which contains any four of them, so that this is the only possible such [3].

1.11. *Any five planes of* [13] *have a unique transversal* [3].

Now take six planes $h_1, \ldots, h_6$: each set of five of these has a unique transversal [3], so that we get six [3]'s, $k_1, \ldots, k_6$, such that h_ρ and k_σ have a common point, say $A_{\rho\sigma}$, if $\rho \neq \sigma$. If we express algebraically that sets of five points $A_{\rho\tau}$ which have the same ρ lie in planes, and that the sets of five with the same σ lie in [3]'s, we get eighteen syzygies, of which, since the thirty points are to lie in [13], only sixteen can be independent. From

this it is not difficult to see that the syzygies can be put in the form

$$\left.\begin{array}{ll} & \sum_{\sigma} A_{\rho\sigma} = 0, \\ & \sum_{\sigma} a_{\sigma} A_{\rho\sigma} = 0 \\ \text{(for points of the plane } h_{\rho}\text{)}, & \\ \text{and} & \sum_{\rho} A_{\rho\sigma} = 0 \\ \text{(for points of the [3] } k_{\sigma}\text{)}. & \end{array}\right\} \quad (\rho, \sigma = 1, \ldots, 6); (\rho \neq \sigma). \quad \text{(i)}$$

Take first the six planes $h_1, \ldots, h_6$ and the five solids $k_1, \ldots, k_5$, involving in all the twenty-five points $A_{\rho\sigma}$ ($\rho = 1, \ldots, 6$; $\sigma = 1, \ldots, 5$; $\rho \neq \sigma$). The five points in k_j are connected by a single syzygy, which may be supposed to be

$$\sum_{\rho} A_{\rho j} = 0.$$

There are five of these syzygies. If we work for the moment in terms of the twenty points $A_{\alpha\beta}$ ($\alpha, \beta = 1, \ldots, 5$; $\alpha \neq \beta$), the five points in h_6 are $\sum_{\alpha} A_{\alpha j}$. These are connected by two syzygies; take one of them to be

$$\sum_{\beta} q_{\beta} \sum_{\alpha} A_{\alpha\beta} = 0,$$

and then replace $q_{\beta} A_{\alpha\beta}$ by $A'_{\alpha\beta}$. In terms of these new symbols for the points (h_{α}, k_{β}), the syzygy connecting the points in k_j is

$$q_j^{-1} \sum_{\rho=1}^{6} A'_{\rho j} = 0.$$

Thus, omitting now the accents, we can take the syzygy connecting the points in each k_j to be

$$\sum_{\rho} A_{\rho j} = 0,$$

and, at the same time, one of those connecting the points in h_6 to be

$$\sum_{\beta=1}^{5} \sum_{\alpha=1}^{5} A_{\alpha\beta} = 0.$$

This last syzygy also signifies that the five points $\sum_{\beta} A_{i\beta}$ lie in a [3]. Each of the planes $h_1, \ldots, h_5$ contains one of the points, so that the [3] is the unique transversal-solid of the five planes, viz. k_6. Revert now to the full scheme of thirty points $A_{\rho\tau}$; we have found that two of the three sets of six syzygies connecting them can be taken in the form

$$\sum_{\rho} A_{\rho j} = 0,$$

$$\sum_{\sigma} A_{i\tau} = 0.$$

The third set of six comprises the second syzygies connecting the sets of five points in the planes $h_\cdot$: suppose that they are

$$\sum_\sigma c_{i\sigma} A_{i\sigma} = 0.$$

Of the eighteen syzygies, so far seventeen are independent in virtue of

$$\sum_\rho \sum_\sigma A_{\rho\sigma} \equiv \sum_\sigma \sum_\rho A_{\rho\sigma}.$$

There must be one further identity, say

$$\sum_\rho \lambda_\rho \sum_\sigma A_{\rho\sigma} + \sum_\sigma \mu_\sigma \sum_\rho A_{\rho\sigma} \equiv \sum_\rho \kappa_\rho \sum_\sigma c_{\rho\sigma} A_{\rho\sigma},$$

where none of the κ_ρ is zero. This gives the thirty equations

$$\lambda_\rho + \mu_\sigma = \kappa_\rho c_{\rho\sigma},$$

and thus, instead of the last set of syzygies, we may use

$$\sum_\sigma (\lambda_i + \mu_\sigma) A_{i\sigma} = 0.$$

Thus the two syzygies connecting the points in h_i are

$$\sum_\sigma A_{i\sigma} = 0$$

and

$$\sum_\sigma \mu_\sigma A_{i\sigma} = 0.$$

It is frequently more convenient to work in terms of the twenty points $A_{\alpha\beta}$ ($\alpha, \beta = 1, \ldots, 5$) only, which are connected by the six independent relations

$$\left.\begin{aligned} \sum_{\alpha,\beta} A_{\alpha\beta} &= 0, \\ \sum a'_\beta A_{\alpha\beta} &= 0, \end{aligned}\right\} \qquad \text{(ii)}$$

where

$$a'_\beta = a_\beta - a_6.$$

1.12. *If $h_1, \ldots, h_6$ are six planes in* [13], *and k_ρ is the* [3] *which meets the five planes other than h_ρ, then there is a unique line g which meets the six* [6]*'s* $[h_\rho, k_\rho]$*.

The existence of g can easily be proved by projecting from the line of intersection of the six [11]'s formed by joining pairs of the [6]'s $[h_1, k_1], \ldots, [h_4, k_4]$, on to some [11]†, but there is also an easy proof

* In the general theorem we shall get a space g containing a normal rational curve u, while in this case, of course, the curve and the "space" coincide. In the subsequent work, when considering ranges of points, g will frequently be referred to as u.

† *Cf.* G. 432, § 15.

using the symbols. From any point of the [6] $[h_\rho, k_\rho]$ we can draw exactly one line which meets h_ρ and k_ρ, so that if there is a line g meeting $[h_\rho, k_\rho]$ in G_ρ, it will be possible to find points B_ρ of h_ρ and C_ρ of k_ρ collinear with G_ρ*. Take

$$B_\rho = \sum_{\sigma=1}^{6} (a_\rho - a_\sigma)^{-1} A_{\rho\sigma} = a'^{-1}_\rho \sum_{\tau=1}^{5} a'_\tau (a'_\rho - a'_\tau)^{-1} A_{\rho\tau},$$

$$C_\rho = \sum_{\sigma=1}^{6} (a_\rho - a_\sigma)^{-1} A_{\sigma\rho} = a'^{-1}_\rho \sum_{\tau=1}^{5} a'_\tau (a'_\rho - a'_\tau)^{-1} A_{\tau\rho}.$$

These are collinear with

$$G_\rho = B_\rho + C_\rho \dagger,$$

whilst the six points G_ρ are collinear in virtue of relations like

$$a'_1(a'_4 - a'_1)(a'_5 - a'_1) G_1 + a'_2(a'_4 - a'_2)(a'_5 - a'_2) G_2 + a'_3(a'_4 - a'_3)(a'_5 - a'_3) G_3$$

$$\equiv -a'_4 a'_5 \sum_{\alpha,\beta=1}^{5} A_{\alpha\beta} + \sum_{\alpha} \left\{ a'_\alpha \left(\sum_{\beta} a'_\beta A_{\alpha\beta} \right) \right\},$$

which is a syzygy.

1.2. The derivation of the six-plane figure from a figure in [18]‡.

The form (ii) of the syzygies used in §1.1 suggests that the geometrical ideas will be simplified by first considering the figure in [18] which is determined by the twenty points $A_{\alpha\beta}$ connected by the syzygy

$$\sum_{\alpha=1}^{5} \sum_{\beta=1}^{5} A_{\alpha\beta} = 0 \quad (\alpha \neq \beta),$$

but not further restricted, and then projecting that figure from the five points $\sum_\beta a'_\beta A_{i\beta}$. All the work of this paragraph, 1.2, will be concerned with the figure in [18], so that no confusion will arise from using for points and spaces in the [18] the same letters as are used for their projections on to the [13].

The figure in [18] is a double-six of solids, h_ρ and k_ρ, in which h_ρ meets k_σ in a point $A_{\rho\sigma}$, but does not meet k_ρ. The five points in each [3] are connected by one of the simple syzygies of the scheme

$$\sum_{\sigma} A_{i\sigma} = 0$$

* B_ρ and C_ρ are the $A_{\rho\rho}$ and $A'_{\rho\rho}$ respectively of Mr. Grace's paper.

† *Cf.* R_1, 329. The second form given above is equivalent to the expression given there "as a point of $[h_1, k_1]$".

‡ *Cf.* R_1, 327, §9, in which five general lines in [6] are derived from five planes in [10].

(for points of h_i)

and $$\sum_{\rho} A_{\rho j} = 0$$

(for points of k_j).

To get the figure in [13] we project from a [4] which meets the six solids h_ρ. That the projection of six [3]'s in [18] from [4] which meets them all is six general planes of [13], and consequently that the forms (i) and (ii) adopted for the syzygies in § 1 . 1 are justified, can be checked from considerations of freedom. The six solids in [18] have freedom $6 \times 4 \times 15 = 360$; they are, in fact, just expressible in terms of twenty points in the [18]. The symbol for any other point in the [18] in terms of these twenty points contains twenty homogeneous parameters, so that the freedom of the projection of the six solids from an arbitrary point on to a [17] is 360—19*. Following up this argument, we find that the freedom of the projection of the six solids from five arbitrary points (*i.e.* from a general [4]) is 360—95. In the case under consideration, however, the [4] from which the six solids are projected belongs to an ∞^3 set. while general [4]'s in [18] have freedom 5×14. Thus the freedom of the projection of the six solids from a [4] which meets them all is $360 - 95 - 67 = 198$; this is exactly the freedom of six planes in [13]. In the general case that is considered later, it is assumed on algebraical grounds that $p+2$ general $[p-1]$'s in $[p^2+p-2]$ project from a $[p]$ which meets them all into $p+2$ general $[p-2]$'s in $[p^2-3]$, and here again the result can be deduced from considerations of freedom. The freedom of the $p+2$ $[p-1]$'s is $(p+2)p(p^2-1)$; the freedom of their projection, by an argument similar to the foregoing, is

$$(p+2)p(p^2-1) - (p+1)(p^2+p-1) - \{(p+1)(p^2-2) - (p-1)\},$$

i.e. $(p+2)(p-1)(p^2-p-1)$, which is exactly the freedom of the $p+2$ $[p-2]$'s.

Returning to the figure in [18], the properties and the processes by which they are obtained are precisely similar to those of an analogous

* The general theorem of this type is: the freedom of the projection from a general point of $[n]$, on to a prime, of a construct derived from $n+2$ points in the $[n]$ is $n(n+2)-(n+1)$. A well-known case is the double-four of lines in [4], which is determined by six points, and has therefore freedom 24. Its projection from an arbitrary point is a double-four belonging to a double-six, and therefore has freedom 19. A simpler case still is that of the solid Desargues figure of perspective triangles, which is definable by five points in [3], and the plane Desargues figure, which is its projection from an arbitrary point. The freedoms of these two are respectively 15 and 11. Another example will be found in R_2, foot-note to p. 462. [*Added 9 January*, 1932.] In fact $n+2$ points in $[n]$ define a figure which is normal in $[n+1]$, and the results above are at once obtained by applying the formulae proved for the freedoms of the projections of a manifold in *Proc. Camb. Phil. Soc.*, 27 (1931), 534.

figure in [10]*, and we need do no more than quote the results that we want:

1.21. *Through every point of h_i there pass ∞^1 [4]'s t, which meet in points the other five* [3]*'s h_ρ, the six points in a t being of the form $\sum_\sigma a_\sigma A_{i\sigma}$. There is a similar family of* [4]*'s s meeting the* [3]*'s k_σ. Every t has exactly one point common with any s.*

1.22. *There is a unique* [8] U *which meets all the spaces s and t in points. It meets each* [7] $[h_\rho, k_\rho]$ *in a* [3], *say g_ρ.*

The [8] is actually the space which contains the fifteen points $A_{\rho\sigma}+A_{\sigma\rho}$.

1.23. *There are ∞^3 planes g which meet the six* [3]*'s $g_\rho = ([h_\rho, k_\rho], U)$. The six points (g_ρ, g) which lie in any plane g lie on a conic u^2†. The locus of points in which the spaces t and s meet U is exactly the locus of the conics u^2. Through each point of this locus can be drawn exactly one space t and one space s, and, as the point traces a conic, the corresponding points (k_ρ, s) and (h_ρ, t) trace twisted cubics w_ρ^3 and v_ρ^3.*

Algebraically, if we take the space t containing the six points $\sum_\sigma a_\sigma A_{\rho\sigma}$, this meets U in the point $\sum_{\rho,\sigma} a_\rho a_\sigma A_{\rho\sigma}$, through which passes the [4]'s containing the six points $\sum_\rho a_\rho A_{\rho\sigma}$. Through the point $\sum a_\rho a_\sigma A_{\rho\sigma}$ we can draw one plane to meet the six [3]'s g_ρ, and it does so in the points $G_\rho = \sum_\sigma (a_\rho - a_\sigma)^{-1}(A_{\rho\sigma}+A_{\sigma\rho})$. The conic u^2 in this plane is the locus of points $\sum_\rho (\theta - a_\rho)^{-1} G_\rho$‡ as θ varies, and the corresponding twisted cubics v_σ^3 and w_σ^3 are $\sum_\rho (\theta - a_\rho)^{-1} A_{\sigma\rho}$ and $\sum_\rho (\theta - a_\rho)^{-1} A_{\rho\sigma}$. From G_ρ we can draw a line to meet h_ρ and k_ρ. Suppose that it does so in points B_ρ and C_ρ; then

1.24. *There is a conic u^2 in g and there are twisted cubics v_ρ^3 and w_ρ^3 in h_ρ and k_ρ which are related in such a way that*

$$G_1 G_2 G_3 G_4 G_5 G_6 \curlywedge B_1 A_{12} A_{13} A_{14} A_{15} A_{16} \curlywedge A_{21} B_2 A_{23} A_{24} A_{25} A_{26} \curlywedge \ldots$$

$$\curlywedge C_1 A_{21} A_{31} A_{41} A_{51} A_{61} \curlywedge \ldots.$$

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* R_1, 315–320.

† See also R_2, § 3, 473.

‡ If this is multiplied by $\Pi(\theta - a_\rho)$ the coefficients of θ^5, θ^4, and θ^3 are found to be syzygies.

1.3. Rational curves in the spaces of a double-six in [13].

To obtain the figure in [13] from that in [18], we project from one of the [4]'s t, namely that which contains the points $\sum_{\sigma=1}^{6} a_\sigma A_{\rho\sigma}$. The [3]'s h_ρ project into planes h_ρ, and the [3]'s k_ρ into the [3]'s k_ρ which meet sets of five of them. Through the point (t, U) there passes one of the conics u^2; this projects into the line g (or u) which meets the six [6]'s $[h_\rho, k_\rho]$, while the corresponding cubics v_ρ^3 and w_ρ^3 project into conics v_ρ^2 and cubics w_ρ^3 respectively. Thus

1.31. *The sets of six points such as* B_1, A_{12}, ..., A_{16} *lie on conics.*

Through any point of u we can draw a [4] t which meets h_1, ..., h_6, and a [3] s which meets k_1, ..., k_6, and as the point describes u so the points (h_ρ, t) and (k_ρ, s) describe v_ρ^2 and w_ρ^3. This gives us the fundamental relation

1.32. $$G_1 \dots G_6 \curlywedge B_1 A_{12} \dots A_{16} \curlywedge \dots$$
$$\curlywedge C_1 A_{21} \dots A_{61} \curlywedge \dots .$$

The parametric forms for the points of the curves u, v_ρ^2, and w_ρ^3 are precisely the same as for the corresponding curves in [18], with the a_ρ replaced by a_ρ.

The tangent lines b_ρ to the curves v_ρ^2 at the points B_ρ and the osculating planes c_ρ to w_ρ^3 at C_ρ have the following properties which will be required later.

1.33. *The* [11] *which contains the six tangent lines* b_ρ *cuts the* [3]*'s* k_ρ *in the tangents to the curves* w_ρ^3 *at the points* C_ρ.

This may also be expressed thus: the projection of b_6 from $[b_1, \dots, b_5]$ on to k_6 is the tangent to w_6^3 at C_6.

1.34. *The projection of* h_6 *from* $[b_1, \dots, b_5]$ *on to* k_6 *is* c_6, *the osculating plane of* w_6^3 *at* C_6.

1.35. *The projection of* h_6 *from* $B_1, \dots, B_5$ *and a* [4] t, *which meets* $h_1, \dots, h_6$ *and* u, *on to* k_6 *is the tangent line to* w_6^3 *at* C_6.

Similar algebraic methods are used to establish these three results. The proofs of 1.34 and 1.35 are given here in detail, and the reader will

have little difficulty with 1.33. We work with the twenty points $A_{\alpha\beta}$ ($\alpha, \beta = 1, \ldots, 5$) connected by the six relations

$$\left.\begin{aligned} \sum_{\alpha,\beta} A_{\alpha\beta} &= 0, \\ \sum_{\beta} a'_\beta A_{\alpha\beta} &= 0. \end{aligned}\right\}$$

In this scheme, a general point P of the conic v_1^2 is $\sum_\beta \{a'_\beta/(\phi - a'_\beta)\} A_{1\beta}$. B_1 is the point $\phi = a'_1$, and the tangent line b_1 at the point passes through $B_1^{(1)} = (\partial P/\partial\phi)_{\phi=a'_1} = \sum \{a'_\beta/(a'_1 - a'_\beta)^2\} A_{1\beta}$. In the same way we see that the osculating plane to w_6^3 at C_6 contains the three points

$$C_6 = \sum_{\alpha=1}^{5} a_\alpha'^{-1} A_{\alpha 6}, \quad C_6^{(1)} = \sum a_\alpha'^{-2} A_{\alpha 6}, \quad C_6^{(2)} = \sum a_\alpha'^{-3} A_{\alpha 6}.$$

To prove 1.34 we show that the projection of the point A_{61} from $[b_1, \ldots, b_5]$ on to k_6 is a point of c_6. Then similarly $A_{62}, \ldots, A_{65}$ project into points of c_6 and thus h_6 projects into c_6. We have to prove, that is, that $([b_1, \ldots, b_5, A_{61}], k_6)$ is a point of c_6. To find this point (as a point of k_6) we have to solve the syzygetic relation

$$\sum_{\alpha=1}^{5} \lambda_\alpha B_\alpha + \sum \mu_\alpha B_\alpha^{(1)} - \tau A_{61} - \sum \kappa_\alpha A_{\alpha 6} + \sum_\alpha \left(\nu_\alpha \sum_\beta a'_\beta A_{\alpha\beta} \right) = 0^*.$$

This gives the twenty equations

$$\lambda_\alpha a'_\beta (a'_\alpha - a'_\beta)^{-1} + \mu_\alpha a'_\beta (a'_\alpha - a'_\beta)^{-2} + \tau\epsilon_\beta + \kappa_\alpha + \nu_\alpha a'_\beta = 0,$$

where $\epsilon_\beta = 1 \quad \text{if} \quad \beta = 1$

and $\epsilon_\beta = 0 \quad \text{if} \quad \beta \neq 1.$

For each value of α there are four equations each involving the same five parameters. If $\alpha = 1$, there is no term in τ, so that $\lambda_1 = \mu_1 = \nu_1 = \kappa_1 = 0$. We easily find that the other equations are satisfied by

$$\kappa_\alpha = (1 - a'_1 a'^{-1}_\alpha)^3.$$

Thus the point of k_6 is $\sum_{\alpha=2}^{5} (1 - a'_1 a'^{-1}_\alpha)^3 A_{\alpha 6}$, which is clearly a point of

$$c_6 = \left[\sum_{\alpha=1}^{5} a_\alpha'^{-1} A_{\alpha 6}, \quad \sum a_\alpha'^{-2} A_{\alpha 6}, \quad \sum a_\alpha'^{-3} A_{\alpha 6} \right].$$

* The terms in the κ_α, whose differences only are significant, allow for the syzygy $\sum_{\alpha,\beta} A_{\alpha\beta} = 0$.

To prove 1.35 we proceed thus: a [4] which meets $h_1, \ldots, h_6$, and u, meets $h_1, \ldots, h_5$ in the points $\sum_{\beta=1}^{5} \frac{a'_\beta}{c-a'_\beta} A_{i\beta}$, so that we have to show that the point in which the [10],

$$\left[B_1, \ \ldots, \ B_5, \ \sum_{\beta=2}^{5} \frac{a'_\beta}{c-a'_\beta} A_{1\beta}, \ \ldots, \ \sum_{\beta=1}^{4} \frac{a'_\beta}{c-a'_\beta} A_{5\beta}, \ A_{61}\right],$$

meets the solid k_6 is a point of the line $C_6 C_6^{(1)}$. That is, we have to solve

$$\sum_{\alpha,\beta} \lambda_\alpha \frac{a'_\beta}{a'_\alpha - a'_\beta} A_{\alpha\beta} + \sum_{\alpha,\beta} \mu_\alpha \frac{a'_\beta}{c-a'_\beta} A_{\alpha\beta} + \sum_{\alpha,\beta} \nu_\alpha a'_\beta A_{\alpha\beta} + \tau \sum_{\alpha=2}^{5} A_{\alpha 1} + \sum_{\alpha} \kappa_\alpha \sum_{\beta} A_{\alpha\beta} = 0,$$

but are only concerned ultimately with the values of κ_α. We find, as before,

$$\kappa_1 = 0$$

and then

$$\kappa_2 = \frac{a'_3 a'_4 a'_5 (a'_2 - a'_1)(c-a'_1)}{c a'_2 (a'_1 - a'_3)(a'_1 - a'_4)(a'_1 - a'_5)} \tau,$$

so that

$$\kappa_2 : \kappa_3 : \kappa_4 : \kappa_5 = \frac{(a'_2 - a'_1)^2}{a'^2_2} : \ldots : \ldots : \frac{(a'_5 - a'_1)^2}{a'^2_5}.$$

The point common to the [10] and k_6 is

$$\sum_{\alpha=2}^{5} A_{\alpha 6} - 2a'_1 \sum_{\alpha=2}^{5} a'^{-1}_\alpha A_{\alpha 6} + a'^2_1 \sum_{\alpha=2}^{5} a'^{-2}_\alpha A_{\alpha 6},$$

which is clearly a point of $C_6 C_6^{(1)}$.

1.4. The normal rational curve which osculates six planes in [13].

1.41. *If there is a normal rational curve osculating the six planes $h_1, \ldots, h_6$, then its points of osculation are $B_1, \ldots, B_6$, and the tangent lines at those points are $b_1, \ldots, b_6$.*

Suppose that $h_\alpha = [X_\alpha, Y_\alpha, Z_\alpha]$, $\alpha = 1, \ldots, 5$, are five osculating planes to a normal rational curve in [13], X_α being the points of osculation, and $X_\alpha Y_\alpha$ the tangent lines at those points. The points Z_α, being any other points of the planes, may be supposed to be the points of intersection of the planes with the [3] which meets them all, and we can take the syzygy as

$$\sum_{\alpha} Z_\alpha = 0.$$

This is the only syzygy between the fifteen points. If r_α is the parameter of the point X_α on the curve, a general point of the curve may be taken to be

$$P = \sum_{\alpha=1}^{5} \{(\theta - r_\alpha)^{-3} X_\alpha + (\theta - r_\alpha)^{-2} Y_\alpha + (\theta - r_\alpha)^{-1} Z_\alpha\}$$

—this is a curve of order thirteen since the coefficient of θ^{14}, when the expression is multiplied by $\Pi(\theta-r_\alpha)^3$, is ΣZ_α; also clearly the point $\theta=r_\alpha$ is X_α, and $X_\alpha Y_\alpha$ is the tangent line there and $X_\alpha Y_\alpha Z_\alpha$ the osculating plane. By suitable selection of the r_α we may assign the parameter $\theta=0$ to any specified point of the curve. Then if, when P is expressed in any manner as a power series in θ, we get

$$P=X_6+\theta Y_6+\theta^2 Z_6+\ldots,$$

X_6 is the point $\theta=0$ of the curve, $X_6 Y_6$ the tangent line, and $h_6=[X_6\,Y_6\,Z_6]$ the osculating plane at that point. The simplest method of obtaining such an expression is to expand the terms $(\theta-r_\alpha)^{-t}$ as they stand. This gives

$$X_6=\sum_{\alpha=1}^{5}\{r_\alpha^{-3}X_\alpha+r_\alpha^{-2}Y_\alpha+r_\alpha^{-1}Z_\alpha\},$$

$$Y_6=\Sigma\{3r_\alpha^{-4}X_\alpha+2r_\alpha^{-3}Y_\alpha+r_\alpha^{-2}Z_\alpha\},$$

$$Z_6=\Sigma\{6r_\alpha^{-5}X_\alpha+3r_\alpha^{-4}Y_\alpha+r_\alpha^{-3}Z_\alpha\}.$$

To determine the point common to h_i and the [3] which also meets h_2, h_3, h_4, h_6, we must find a symbol of the form

$$\lambda X_6+\mu Y_6+\nu Z_6+\rho(Z_1+Z_2+Z_3+Z_4+Z_5)$$

which does not involve X_5, Y_5, or Z_5, since this gives a syzygy

$$(\text{point of } h_1)+\ldots+(\text{point of } h_4)=(\text{point of } h_6)+(\text{known syzygy}).$$

The equations obtained are

(from X_5) $$r_5^2\lambda+3r_5\mu+6\nu=0,$$

(from Y_5) $$r_5^2\lambda+2r_5\mu+3\nu=0,$$

(from Z_5) $$r_5^3\rho+r_5^2\lambda+r_5\mu+\nu=0.$$

Substituting in these values for λ, μ, ν, ρ, we find that "the point of h_1" is

$$\frac{3r_5(2r_5-r_1)}{r_1^2(r_5-r_1)^2}X_1+\frac{3r_5}{r_1(r_5-r_1)}Y_1+Z_1.$$

This is the A_{15} of the original figure, while Z_1 is A_{16}. We therefore have the scheme

$$A_{1\beta}=\frac{3r_\beta(2r_\beta-r_1)}{r_1^2(r_\beta-r_1)^2}X_1+\frac{3r_\beta}{r_1(r_\beta-r_1)}Y_1+Z_1.$$

These five points lie on the conic whose general point is

$$P_1=\frac{3\phi(2\phi-r_1)}{r_1^2(\phi-r_1)^2}X_1+\frac{3\phi}{r_1(\phi-r_1)}Y_1+Z_1;$$

this is the conic u_1^2. If we take $r_a = a'_a$, the points $A_{12}, \ldots, A_{16}$ correspond to the values $a'_2, \ldots, a'_5, 0$ of the parameter, and therefore B_1 corresponds to the value $\phi = a'_1 = r_1$. Thus $B_1 \equiv X_1$, and $b_1 = [X_1 Y_1]$.

The work above carries out in detail the course suggested by Mr. Grace for proving the uniqueness of the curve*, and, as suggested further by him, it can obviously be extended to cover the general theorem stated here explicity in § 3 . 24.

The proof of the converse theorem,

1 . 42. *It is possible to find a normal rational curve which osculates* $h_1, \ldots, h_6$ *at* $B_1, \ldots, B_6$ *and has* $b_1, \ldots, b_6$ *as its tangent lines at those points,*

presents more difficulty, unless we accept the principle enunciated by Mr. Grace in § 6 of his paper, in which case the proof of uniqueness is a proof of existence. The proof of the existence which follows here is of a more definitely geometric nature.

The conditions that the curve is to be made to satisfy, viz. of osculating planes at assigned points with assigned tangents, are all of the nature of making the curve pass through assigned points. On the other hand, there are apparently many more of them than the curve can be made to satisfy, but we show that they are all derivable from a set of n conditions which should be just sufficient to fix the curve (or at worst give a finite number of curves). This second set of conditions is not obviously linear, and the difficulty arises that they may be a set of conditions of that exceptional type which does not admit of a solution in the general case (*cf.* the problem of drawing twisted cubics to have two given lines as chords, and to pass through four given points). This possibility is, however, ruled out, because the conditions must be equivalent to exactly n of the linear conditions (the other linear conditions being deducible from these), while if they were of the no-solution type, there is no set of n linear conditions equivalent to them. Since a normal rational curve in [13] can be made to pass through sixteen points, n, the number of conditions which it can be made to satisfy, is 192.

Consider first the curves which touch the lines $b_1, \ldots, b_6$ at $B_1, \ldots, B_6$, and meet $h_1, \ldots, h_4$ each in a further point. There are ∞^8 of these. Project them from $[b_1, \ldots, b_5]$ on to k_6. They are projected into cubics which pass through $A_{16}, \ldots, A_{46}$, and have at C_6 the same tangent line as w_6^3 (*cf.* 1 . 33). Thus they all project into w_6^3. Since w_6^3 passes through A_{56} the $[10]$ $[b_1, \ldots, b_4, h_5]$ must contain eleven points of each of them, ten points altogether on $b_1, \ldots, b_5$ and one other which projects into A_{56}.

* G, § 12 (and § 9).

We have, that is, ∞^8 "extra points" in the [10], and a finite number of them must, therefore, lie in the plane h_5*. Thus a finite number of curves can be drawn to touch $b_1, \ldots, b_6$ at $B_1, \ldots, B_6$ and to have another point in each of $h_1, \ldots, h_5$. Now project these curves from $[b_1, \ldots, b_4, b_6]$ on to k_5; they all project into w_5^3, by an argument similar to the above. But h_5, which contains an "extra point" of each of the curves, projects into the osculating plane to w_5^3 at C_5 (1.34). Therefore h_5 must osculate all the curves, and similarly $h_1, \ldots, h_4$ must. Thus we can draw at least one curve which osculates $h_1, \ldots, h_5$ at $B_1, \ldots, B_5$, and has $b_1, \ldots, b_6$ as tangent lines at $B_1, \ldots, B_6$. Since all these conditions are linear (although not of course independent) there can be only one curve. Similarly we can draw a curve which osculates $h_2, \ldots, h_6$ at $B_2, \ldots, B_6$ and touches $b_1, \ldots, b_6$ at $B_1, \ldots, B_6$. These two curves have sixteen common points, and therefore coincide entirely, *i.e.* there is a curve which satisfies the conditions 1.42.

2. *Figures derived from seven planes in* [16].

2.1. The condition for the existence of certain conics in the planes.

It can be proved as in 1.1 that

2.11. *Six planes in* [16] *have a unique transversal* [4].

Thus seven planes will determine a double-seven of planes and [4]'s, say h'_ρ and k'_ρ respectively, where $(h'_\rho, k'_\sigma) = A'_{\rho\sigma}$ for $\rho \neq \sigma$†. It is not, however, true as in 1.21 that the seven [7]'s $[h'_\rho, k'_\rho]$ have a transversal line, but we have the theorem:

2.12. *A necessary and sufficient condition that the seven* [7]*'s* $[h'_\rho, k'_\rho]$ *should have a common transversal line is that the sets of six points like* $A'_{12}, \ldots, A'_{17}$ *should lie on conics.*

* This is the apparently weak point in the argument: although there are ∞^8 curves each meeting the [10] in an "extra point", it does not follow that these points form a manifold of dimension eight. (*Cf.* the cubic curve problem: there are ∞^2 twisted cubics in [3] which have two given lines as chords and pass through three given points, and the curves might therefore be expected to fill all space, whereas actually they lie on a quadric and ∞^1 pass through each of its points.) The difficulty is, however, cleared up at the next step. From consideration of the related ranges on the curves and on w_5^3, all the planes through b_5 and an "extra point" are osculating planes, because they must project from $[b_1, \ldots. b_4, b_6]$ into c_5.

† To avoid confusion we put dashes on the symbols used for this figure, keeping the undashed symbols for the main-line figures.

To prove this we consider the " main-line " figure next after that of [2]'s and [3]'s in [13], namely that of [3]'s and [4]'s in [22] (*cf.* 3.21–4). This figure consists of seven [3]'s h_ρ, and seven [4]'s k_ρ, with $(h_\rho, k_\sigma) = A_{\rho\sigma}$ for $\rho \neq \sigma$. The seven [8]'s $[h_\rho, k_\rho]$ are all met by a plane g, say in points G_ρ. Suppose that the transversal line from G_ρ to h_ρ and k_ρ meets those spaces respectively in B_ρ and C_ρ, then

$C_1, A_{21}, \ldots, A_{71}$ lie on a normal rational quartic curve w_1^4,

$B_1, A_{12}, \ldots, A_{17}$ lie on a normal rational cubic curve v_1^3,

and

$G_1, \ldots, G_7$ lie on a conic u^2,

and the ranges of seven points are related. If $h_1, \ldots, h_7$ are projected from a general [5] which meets them all, we get seven general planes in [16]: they will project into planes $h'_1, \ldots, h'_7$, which are such that the sets of points like $A'_{12}, \ldots, A'_{17}$ lie on conics if and only if the [5] meets the curves like v_1^3. The [5] has therefore to meet the seven curves v_ρ^3, and consequently it also meets the conic u^2. u^2 projects into a line which meets the seven [7]'s $[h'_\rho, k'_\rho]$. Conversely, if the line in [16] exists, the projection must have been made from a [5] which meets u^2, and which therefore also meets the curves v_ρ^3. These cubics then project into conics in the planes.

2.2. The condition that a normal rational curve should osculate the seven planes.

It is to be shown that

2.21. *A necessary and sufficient condition that there should be a normal rational curve* γ^{16} *in* [16] *which osculates the planes* $h'_1, \ldots, h'_7$ *is that there shall be a line which meets the seven* [7]*'s* $[h'_\rho, k'_\rho]$.

That the condition is necessary can be seen by projecting the figure from one of the planes, say h'_7. We then get six planes in [13] osculated by a normal rational curve γ^{13}, the projection of γ^{16}. If P'_1 is the point at which γ^{16} osculates h'_1, the points $P'_1, A'_{12}, \ldots, A'_{16}$ project into points corresponding to $B_1, A_{12}, \ldots, A_{16}$ of 1.31, and consequently they lie on a conic. Similarly $P'_1, A'_{12}, \ldots, A'_{15}, A'_{17}$ will lie on a conic, and therefore $A'_{12}, \ldots, A'_{17}$ must lie on a conic. Thus, using 2.12, the condition is necessary.

For sufficiency we have to show that, if there is a line u', in which case there will also be curves $v_\rho^{2'}$ and $w_\rho^{3'}$, then there exists a curve γ^{16}. The

extension of 1.35 to the figure in [22] gives "The projection of h_7, from $B_1, \ldots, B_6$ and a [5] which meets $v_1^3, \ldots, v_7^3$, on to k_7, is the osculating plane to w_7^4 at C_7, and further the tangent line to v_7^3 at B_7 projects into the tangent line to w_7^4 at C_7." That is, on projecting from the [5], we have in the [16] theorems analogous to 1.33 and 1.34. Now consider the family of normal rational curves in [16] which have the same tangent lines as $v_1^{2\prime}, \ldots, v_7^{2\prime}$ at $B_1', \ldots, B_7'$, and which meet $h_1', \ldots, h_5'$ each in a further point. There are ∞^{10} of these curves, so that a finite number of them can be made to meet h_6' in a further point*. From this stage the work of 1.4 can be readily adapted. Thus the condition is sufficient and, if it is satisfied, the curve is uniquely determined.

3. *The general theorems.*

The cases which have been dealt with so far are sufficiently typical to enable us to write down the general theorems at once; in fact, the algebraical expressions of the theorems can always be obtained by merely altering the ranges of the values taken by the suffixes in the summations. As far as the statement of results is concerned, it is better to start with the intermediary figure in space of dimension higher than that under consideration, analogous to that introduced in 1.2, than to discuss that figure half-way through the work, when the need for it arises. For this reason the theorems below correspond to the particular theorems in the order 1.2, 1.1, 1.3, ...

3.1. A figure in $[p^2+p-2]$†.

3.11. *$p+1$ general $[p-1]$'s in $[p^2+p-2]$ have a unique transversal $[p-1]$.*

Thus $p+2$ general $[p-1]$'s in $[p^2+p-2]$ define a double-$(p+2)$ of such spaces, in which, if the spaces be called h_ρ and k_ρ, h_ρ and k_σ have a common point $A_{\rho\sigma}$, if $\rho \neq \sigma$.

3.12. *There are ∞^p $[p]$'s t (s), which meet all the spaces h_ρ (k_ρ), ∞^1 of them passing through any point of each h_ρ (k_ρ). Every t has exactly one point common with every s.*

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* By considering the projection on to k_7' from the space that contains the tangent lines at $B_1', \ldots, B_6'$, we see that the join of that space to A_{67}', which is a space of twelve dimensions and contains h_6', contains thirteen points of each of the curves, of which twelve are fixed—on the tangents. There are thus ∞^{10} "extra points" of which a finite number will lie in h_6'.

† p is used for the $r+2$ of Mr. Grace's paper.

3.13. *There is a* $[\frac{1}{2}p(p+1)-2]$ U, *which meets every space* s *and* t. *It is met by the* $p+2$ $[2p-1]$*'s* $[h_\rho, k_\rho]$ *in* $[p-1]$*'s* g_ρ. *There are* ∞^{p-1} $[p-2]$*'s in* U *which meet all* $p+2$ *of the spaces* g_ρ, *and each does so in the points of rational* $(p-2)$*-ic curve* u^{p-2}*.

3.14. *The spaces* t *and* s *meet* U *in points of the locus of the curves* u^{p-2}. *Those which do so along a particular curve* u^{p-2} *trace in* h_ρ *and* k_ρ *respectively rational* $(p-1)$*-ic curves* v_ρ^{p-1} *and* w_ρ^{p-1}, *in such a way that to the point* $(u^{p-2}, g_\rho) = G_\rho$ *of* u^{p-2} *corresponds* $A_{\sigma\rho}$ *of* v_σ^{p-1}, *and* $A_{\rho\sigma}$ *of* w_σ^{p-1}, *and* B_ρ *of* v_ρ^{p-1}, *and* C_ρ *of* w_ρ^{p-1}, *where* B_ρ *and* C_ρ *are the points of* h_ρ *and* k_ρ *which are collinear with* G_ρ.

3.2. The main-line theorems: a figure in $[p^2-3]$.

3.21. $p+1$ *general* $[p-2]$*'s in* $[p^2-3]$ *have a unique transversal* $[p-1]$.

Thus $p+2$ general $[p-2]$'s determine a double-$(p+2)$, consisting of the $[p-2]$'s h_ρ and the $[p-1]$'s k_ρ transversal to sets of $p+1$ of them. h_ρ meets k_σ in $A_{\rho\sigma}$, but does not meet k_ρ. By considering the figure as a projection of that of 3.1 from a space t, we deduce from 3.13, 4:

3.22. *There is a single* $[p-3]$ g *which meets all the* $[2p-2]$*'s* $[h_\rho, k_\rho]$. *It does so in points* G_ρ *which lie on a rational* $(p-3)$*-ic curve* u^{p-3}.

3.23. *If* B_ρ *and* C_ρ *are the unique points of* h_ρ *and* k_ρ *respectively which are collinear with* G_ρ, *then*

$B_1, A_{12}, \ldots, A_{1,p+2}$ *lie on a rational* $(p-2)$*-ic curve* v_1^{p-2},

$C_1, A_{21}, \ldots, A_{p+2,1}$ *lie on a rational* $(p-1)$*-ic curve* w_1^{p-1},

and $G_1, \ldots, G_{p+2}$ *lie on a rational* $(p-3)$*-ic curve* u^{p-3},

and the ranges of $p+2$ *points are related.*

3.24. *There is a unique normal rational curve in* $[p^2-3]$ *which osculates the* $p+2$ *spaces* h_ρ. *It does so at the points* B_ρ, *and moreover osculates the curves* v_ρ^{p-2} *at those points*†.

3.3. Conditions that $p+2$ $[p-m-2]$'s in $[p^2-3-m(p+1)]$ should osculate a normal rational curve‡.

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* *Cf.* R_2, foot-note, p. 474.

† Theorems corresponding to 1.33, 4, 5 are true, but their main interest would be in the proofs of 3.24 and 3.33, so that there does not seem any need to state them explicitly.

‡ $m = 0, \ldots, p-3$. When $m = p-3$ the rational curve $u^{0'}$ and the space g reduce to a point. Apart from their context the results can be expressed more neatly by putting $p+2 = q$, $p-m-2 = n$: we obtain then conditions that q $[n]$'s in $[(n+1)(q-1)-2]$ should osculate a normal rational curve.

3.31. *$p+1$ general $[p-m-2]$'s in $[p^2-3-m(p+1)]$ have a unique transversal $[p-1]$.*

From this we get the usual double figure of $[p-m-2]$'s h_ρ and $[p-1]$'s k_ρ, with $(h_\rho, k_\sigma) = A_{\rho\sigma}$, $\rho \neq \sigma$.

We can derive the $[p-m-2]$'s from the $[p-1]$'s of 3.1 by projecting from $m+1$ of the $[p]$'s t. We shall get a transversal $[p-m-3]$ to the $p+2$ $[2p-m-2]$'s $[h_\rho, k_\rho]$ if and only if all these spaces t meet the same curve u^{p-2}, whence, as in 2.11,

3.32. *There is a $[p-m-3]$ g transversal to the $p+2$ $[2p-m-2]$'s $[h_\rho, k_\rho]$ if and only if the sets of $p+1$ points like $A_{12}, \ldots, A_{1,p+2}$ lie on rational $(p-m-2)$-ic curves*.*

Then we can apply the argument of 2.12 to show

3.33. *A necessary and sufficient condition that there should be a normal rational curve in $[p^2-3-m(p+1)]$ which osculates the $p+2$ $[p-m-2]$'s h_ρ is that the $p+1$ points $A_{12}, \ldots, A_{1,p+2}$ lie on a rational $(p-m-2)$-ic curve, i.e. that there exists a transversal space g.*

* It is sufficient that one of the sets should lie on a rational curve of order $p-m-2$.

THE CESÀRO SUMMATION OF TRIGONOMETRIC SERIES

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1. Let $f(t)$ be a periodic function of period 2π and integrable in $(0, 2\pi)$ in the sense of Lebesgue. Write

$$\phi(t) = f(x+t)+f(x-t)-2S,$$

$$\gamma_{1+k}(t) = \int_0^1 (1-u)^k \cos tu\, du. \tag{1}$$

It is well known that a necessary and sufficient condition that the Fourier series of $f(t)$ should be summable (C, k), $k > 0$, to S at the point $t = x$ is that

$$\lim_{\omega\to\infty} \omega \int_0^\eta \phi(t)\, \gamma_{1+k}(\omega t)\, dt = 0.$$

In this paper we are concerned first with obtaining sufficient conditions that there should be summability for all indices greater than an assigned number $p \geqslant 0$; *i.e.* we wish to obtain sufficient conditions in order that

$$\lim_{\omega\to\infty} \omega \int_0^\eta \phi(t)\, \gamma_{1+p+\delta}(\omega t)\, dt = 0 \tag{2}$$

for all $\delta > 0$. In the second half of the paper (§§ 12–23) we give the corresponding investigation for the series allied with a Fourier series.

We use the following notation:

$$\phi_0(t) = \phi(t),$$

$$\phi_1(t) = \frac{1}{t}\int_0^t \phi(u)\, du,$$

$$\dots \qquad \dots \qquad \dots$$

$$\phi_r(t) = \frac{1}{t}\int_0^t \phi_{r-1}(u)\, du,$$

where r is a positive integer. If $p = n+\alpha$, where n is zero or a positive integer and $0 < \alpha < 1$, we write

$$\phi_p(t) = \frac{1}{t^\alpha}\int_0^t \phi_n(u)(t-u)^{\alpha-1}\,du.$$

The properties of the functions $\phi_p(t)$ have been considered in a previous paper†. We shall have occasion to use some of the results there obtained. The first theorem which we prove is that equation (2) is satisfied if

$$\frac{1}{t}\int_0^t \phi_p(u)\,du = o(1), \quad \frac{1}{t}\int_0^t |\phi_p(u)|\,du = O(1),$$

as $t \to 0$. We thus have

THEOREM I. *The Fourier series of the absolutely integrable function $f(t)$ is summable (C, k) to S at $t = x$ for all $k > p$, provided that*

$$\frac{1}{t}\int_0^t \phi_p(u)\,du = o(1), \quad \frac{1}{t}\int_0^t |\phi_p(u)|\,du = O(1).$$

For $p = 0$ we obtain a special case of a theorem of Hardy and Littlewood‡. The point of the theorem is that it applies to all $p \geqslant 0$. The best previous result of this kind is due to Hardy and Littlewood§. This condition is

$$\frac{1}{t^p}\int_0^t \phi(u)(t-u)^{p-1}\,du = o(1),$$

which, by L, Theorem 7, is equivalent to

$$\phi_p(t) = o(1).$$

We need the following properties of $\gamma_{1+k}(t)$, which may be obtained from (1) by integration by parts:

$$\gamma_{1+k}(t) = O\left(\frac{1}{t^{1+k}}\right) \quad (0 \leqslant k \leqslant 1), \tag{3}$$

$$= O\left(\frac{1}{t^2}\right) \quad (k \geqslant 1), \tag{4}$$

† Verblunsky, "On the limit of a function at a point", *Proc. London Math. Soc.*, 32 (1931), 163–199. Referred to as L.

‡ Hardy and Littlewood, *Journal London Math. Soc.*, 1 (1926), 134–138 (137).

§ Hardy and Littlewood, *Proc. Camb. Phil. Soc.*, 23 (1927), 681–684.

as $t \to \infty$. Further, $\gamma_{1+k}(t)$ and its derivatives are uniformly bounded for all t.

In what follows we shall use C to denote a constant independent of the variables in the context, but not necessarily the same at each occurrence.

2. We shall first prove the theorem for $p = 0$. It is convenient to enunciate the result as a lemma.

LEMMA 1. *If*

$$\phi_1(t) = \frac{1}{t}\int_0^t \phi(u)\,du = o(1), \qquad \phi_1^*(t) = \frac{1}{t}\int_0^t |\phi(u)|\,du = O(1),$$

then

$$\lim_{\omega\to\infty} \omega \int_0^\eta \phi(t)\,\gamma_{1+\delta}(\omega t)\,dt = 0$$

for all $\delta > 0$.

We have

$$\lim_{\omega\to\infty} \omega \int_0^\eta \phi(t)\,\gamma_{1+\delta}(\omega t)\,dt = \lim_{\omega\to\infty} \omega \left(\int_0^{\lambda/\omega} + \int_{\lambda/\omega}^\eta\right) \phi(t)\,\gamma_{1+\delta}(\omega t)\,dt$$

$$= \lim_{\omega\to\infty} (I_1 + I_2).$$

We have, on integrating by parts,

$$I_1 = \lambda\phi_1\left(\frac{1}{\omega}\right)\gamma_{1+\delta}(\lambda) - \omega^2 \int_0^{\lambda/\omega} t\phi_1(t)\,\gamma'_{1+\delta}(\omega t)\,dt$$

$$= o(1) + o\left(\omega^2 \int_0^{\lambda/\omega} t\,dt\right)$$

$$= o(1),$$

and

$$|I_2| \leqslant C\omega \int_{\lambda/\omega}^\eta |\phi(t)|\,\frac{dt}{(\omega t)^{1+m}} \qquad [m = \min(1, \delta)]$$

$$\leqslant \frac{C}{\omega^m}\left[\frac{\phi_1^*(t)}{t^m}\right]_{\lambda/\omega}^\eta + \frac{C}{\omega^m}\int_{\lambda/\omega}^\eta \frac{\phi^*(t)}{t^{1+m}}\,dt$$

$$\leqslant \frac{C}{\omega^m}\,\frac{\phi_1^*(\eta)}{\eta^m} + \frac{C}{\lambda^m},$$

on neglecting the negative terms, and this tends to zero if $\lambda \to \infty$ after $\omega \to \infty$. This proves the lemma.

The proof of Theorem I for $p = n$ now follows by induction. Assume the theorem for $p = n-1$. To prove that

$$\lim_{\omega\to\infty} \omega \int_0^\eta \phi(t)\,\gamma_{1+n+\delta}(\omega t)\,dt = 0,$$

it is sufficient to prove that

$$\lim_{\omega\to\infty} \left\{\omega\eta\,\phi_1(\eta)\,\gamma_{1+n+\delta}(\omega\eta) - \omega^2 \int_0^\eta \phi_1(t)\,t\gamma'_{1+n+\delta}(\omega t)\,dt\right\} = 0,$$

i.e. by (4), that

$$\lim_{\omega\to\infty} \omega^2 \int_0^\eta t\phi_1(t)\,\gamma'_{1+n+\delta}(\omega t)\,dt = 0. \tag{5}$$

Since

$$t\gamma'_{1+k}(t) = k\gamma_k(t) - (k+1)\,\gamma_{1+k}(t),$$

(5) will be satisfied if

$$\lim_{\omega\to\infty} \omega \int_0^\eta \phi_1(t)\,\gamma_{n+\delta}(\omega t)\,dt = 0 \tag{6}$$

and

$$\lim_{\omega\to\infty} \omega \int_0^\eta \phi_1(t)\,\gamma_{1+n+\delta}(\omega t)\,dt = 0. \tag{7}$$

By the case of Theorem I for $p = n-1$, (6) is satisfied if

$$\frac{1}{t}\int_0^t \phi_n(u)\,du = o(1), \quad \frac{1}{t}\int_0^t |\phi_n(u)|\,du = O(1),$$

and (7) is satisfied *a fortiori*, since it is obtained from (6) by writing $1+\delta$ for δ.

3. We next consider Theorem I for $p = \alpha$ $(0 < \alpha < 1)$. It will be sufficient to show that, if

$$\frac{1}{t}\int_0^t \phi_\alpha(u)\,du = o(1), \quad \frac{1}{t}\int_0^t |\phi_\alpha(u)|\,du = O(1),$$

then

$$\lim_{\omega\to\infty} \omega \int_0^\eta \phi(t)\,\gamma_{1+\beta+\delta}(\omega t)\,dt = 0 \tag{8}$$

for $1 > \beta > \alpha$.

We have

$$\gamma_{1+\beta+\delta}(\omega u) = \frac{\Gamma(1+\beta+\delta)}{\Gamma(\beta)\,\Gamma(1+\delta)} \int_0^u \gamma_{1+\delta}(\omega v)\,v^{1+\delta}(u-v)^{\beta-1}\,dv,$$

and so we have to prove that

$$\lim_{\omega\to\infty} \omega \int_0^\eta \phi(t)\, \frac{dt}{t^{1+\beta+\delta}} \int_0^t \gamma_{1+\delta}(\omega v)\, v^{1+\delta}(t-v)^{\beta-1}\, dv = 0,$$

or, what is the same, that

$$\lim_{\omega\to\infty} \omega \int_0^\eta \gamma_{1+\delta}(\omega v)\, v^{1+\delta}\, dv \int_v^\eta \phi(t)\, \frac{(t-v)^{\beta-1}}{t^{1+\beta+\delta}}\, dt = 0. \tag{9}$$

Write

$$\chi(v) = v^{1+\delta} \int_v^\eta \phi(t)\, \frac{(t-v)^{\beta-1}}{t^{1+\beta+\delta}}\, dt;$$

it is sufficient, by the lemma of § 2, to show that

$$\frac{1}{y} \int_0^y \chi(v)\, dv = o(1), \tag{10}$$

and that

$$\frac{1}{y} \int_0^y |\chi(v)|\, dv = O(1). \tag{11}$$

We have

$$\int_v^\eta \phi(t)\, \frac{(t-v)^{\beta-1}}{t^{1+\beta+\delta}}\, dt = \Phi_v(\eta)\, \frac{(\eta-v)^{\beta-1}}{\eta^{1+\beta+\delta}} - \int_v^\eta \Phi_v(t)\, \frac{d}{dt}\left\{\frac{(t-v)^{\beta-1}}{t^{1+\beta+\delta}}\right\} dt,$$

where

$$\Phi_v(t) = \int_v^t \phi(u)\, du.$$

It is clear that we may replace $\chi(v)$ by

$$v^{1+\delta} \int_v^\eta \Phi_v(t)\, \frac{d}{dt}\left\{\frac{(t-v)^{\beta-1}}{t^{1+\beta+\delta}}\right\} dt = v^{1+\delta} \int_v^\eta \Phi_v(t)(\beta-1)\, \frac{(t-v)^{\beta-2}}{t^{1+\beta+\delta}}\, dt$$

$$-v^{1+\delta} \int_v^\eta \Phi_v(t)\, (1+\beta+\delta)\, \frac{(t-v)^{\beta-1}}{t^{2+\beta+\delta}}\, dt.$$

We shall confine our attention to

$$\psi(v) = v^{1+\delta} \int_v^\eta \Phi_v(t)\, \frac{(t-v)^{\beta-2}}{t^{1+\beta+\delta}}\, dt,$$

and show that (10) and (11) hold when $\chi(v)$ is replaced by $\psi(v)$. It will be seen that the same analysis may be applied to the other term, and thus (10) and (11) will be satisfied.

4. The integral equation

$$\phi_\alpha(t) = \frac{1}{t^\alpha} \int_0^t \phi(u)(t-u)^{\alpha-1}\, du$$

has the solution $$\int_0^t \phi(u)\,du = A\int_0^t \frac{\phi_\alpha(u)\,u^\alpha}{(t-u)^\alpha}\,du,$$

where A is a constant which depends only on α.

Write $$\phi_{\alpha 1}(t) = \frac{1}{t}\int_0^t \phi_\alpha(u)\,du = o(1), \tag{12}$$

$$\phi^*_{\alpha 1}(t) = \frac{1}{t}\int_0^t |\phi_\alpha(u)|\,du = O(1). \tag{13}$$

Then
$$\begin{aligned}\int_0^t \phi(u)\,du &= A\int_0^t \frac{u^\alpha\, d/du\{u\,\phi_{\alpha 1}(u)\}}{(t-u)^\alpha}\,du\\ &= A\int_0^t \frac{d/du\{u^{\alpha+1}\phi_{\alpha 1}(u)\}-\alpha u^\alpha \phi_{\alpha 1}(u)}{(t-u)^\alpha}\,du\\ &= A\,\frac{d}{dt}\int_0^t \frac{u^{\alpha+1}\phi_{\alpha 1}(u)}{(t-u)^\alpha}\,du - A\alpha\int_0^t \frac{u^\alpha\phi_{\alpha 1}(u)}{(t-u)^\alpha}\,du.\end{aligned}$$

Hence
$$\int_v^t \phi(u)\,du = A\,\frac{d}{dt}\int_0^t \frac{u^{\alpha+1}\phi_{\alpha 1}(u)}{(t-u)^\alpha}\,du - A\,\frac{d}{dv}\int_0^v \frac{u^{\alpha+1}\phi_{\alpha 1}(u)}{(v-u)^\alpha}\,du$$
$$-A\alpha\left[\int_0^t \frac{\phi_{\alpha 1}(u)\,u^\alpha}{(t-u)^\alpha}\,du - \int_0^v \frac{\phi_{\alpha 1}(u)\,u^\alpha}{(v-u)^\alpha}\,du\right].$$

Write $$\lambda(t) = \int_0^t \frac{\phi_{\alpha 1}(u)\,u^\alpha}{(t-u)^\alpha}\,du,$$

$$\lambda_1(t) = \int_0^t \frac{\phi_{\alpha 1}(u)\,u^{\alpha+1}}{(t-u)^\alpha}\,du.$$

Then, as $t\to 0$,
$$\begin{aligned}\lambda(t)-\lambda(v) &= \int_v^t \frac{\phi_{\alpha 1}(u)\,u^\alpha}{(t-u)^\alpha}\,du - \int_0^v \phi_{\alpha 1}(u)\,u^\alpha\left[\frac{1}{(v-u)^\alpha}-\frac{1}{(t-u)^\alpha}\right]du\\ &= o[t^\alpha(t-v)^{1-\alpha}] + o[t^\alpha\{(t-v)^{1-\alpha}-t^{1-\alpha}+v^{1-\alpha}\}]\\ &= o[t^\alpha(t-v)^{1-\alpha}].\end{aligned}$$

Thus, if $\mu>1$ is a positive number which remains fixed as $v\to 0$, we may write
$$\lambda(t)-\lambda(v) = o[t^\alpha(t-v)^{1-\alpha}] \quad (0<v\leqslant t\leqslant \mu v), \tag{14}$$
$$= O[t^\alpha(t-v)^{1-\alpha}] \quad (\mu v < t \leqslant \eta). \tag{15}$$

Similarly
$$\lambda_1(t)-\lambda_1(v) = o[t^{1+\alpha}(t-v)^{1-\alpha}] \quad (0<v\leqslant t\leqslant \mu v), \tag{16}$$
$$= O[t^{1+\alpha}(t-v)^{1-\alpha}] \quad (\mu v < t \leqslant \eta). \tag{17}$$

5. We have

$$\frac{1}{y}\int_0^y \psi(v)\,dv = \frac{1}{y}\int_0^y v^{1+\delta}\,dv \int_0^\eta \Phi_v(t)\,\frac{(t-v)^{\beta-2}}{t^{1+\beta+\delta}}\,dt. \tag{18}$$

Writing $$\lambda_1'(t) = \frac{d}{dt}\lambda_1(t),$$

we have

$$\Phi_v(t) = A[\lambda_1'(t)-\lambda_1'(v)] - A\alpha[\lambda(t)-\lambda(v)], \tag{19}$$

and we evaluate separately the contributions to (18) of each of the two terms in (19).

The contribution of the second term is

$$o\left[\frac{1}{y}\int_0^y v^{1+\delta}\,dv\int_v^{\mu v}\frac{(t-v)^{\beta-2}}{t^{1+\beta+\delta}}\,t^\alpha(t-v)^{1-\alpha}\,dt\right] + O\left[\frac{1}{y}\int_0^y v^{1+\delta}\,dv\int_{\mu v}^\eta \frac{(t-v)^{\beta-2}}{t^{\beta+\delta}}\,dt\right]$$

$$= o\left[\frac{1}{y}\int_0^y v^{1+\delta}\,\frac{(\mu v)^\alpha}{v^{1+\beta+\delta}}\,(\mu v-v)^{\beta-\alpha}\,dv\right]$$

$$+ O\left[\frac{1}{y}\int_0^y v^{1-\beta}u^{-\beta-\delta}\left\{\frac{1}{(\mu-1)^{1-\beta}v^{1-\beta}} - \frac{1}{(\eta-v)^{1-\beta}}\right\}dv\right]$$

$$= o(1) + O\left[\frac{1}{\mu^{1+\delta}}\right]$$

as $y\to 0$. Letting $\mu\to\infty$ after $y\to 0$, we see that this contribution is zero.

We now consider

$$\frac{1}{y}\int_0^y v^{1+\delta}\,dv\int_v^\eta \frac{(t-v)^{\beta-2}}{t^{1+\beta+\delta}}\,[\lambda_1'(t)-\lambda_1'(v)]\,dt. \tag{20}$$

We have

$$\int_v^\eta \frac{(t-v)^{\beta-2}}{t^{1+\beta+\delta}}\,\lambda_1'(t)\,dt = \int_v^\eta (t-v)^{\beta-2}\left[\frac{d}{dt}\left\{\frac{\lambda_1(t)}{t^{1+\beta+\delta}}\right\} + (1+\beta+\delta)\,\frac{\lambda_1(t)}{t^{2+\beta+\delta}}\right]dt$$

$$= \left[\frac{(t-v)^{\beta-2}\lambda_1(t)}{t^{1+\beta+\delta}}\right]_v^\eta + \int_v^\eta \frac{\lambda_1(t)}{t^{1+\beta+\delta}}\,\frac{d}{dv}\,(t-v)^{\beta-2}\,dt$$

$$+ (1+\beta+\delta)\int_v^\eta \frac{(t-v)^{\beta-2}\lambda_1(t)}{t^{2+\beta+\delta}}\,dt$$

$$= \frac{(\eta-v)^{\beta-2}\lambda_1(\eta)}{\eta^{1+\beta+\delta}} + \frac{d}{dv}\int_v^\eta \frac{\lambda_1(t)}{t^{1+\beta+\delta}}\,(t-v)^{\beta-2}\,dt$$

$$+ (1+\beta+\delta)\int_v^\eta \frac{(t-v)^{\beta-2}\lambda_1(t)}{t^{2+\beta+\delta}}\,dt.$$

The expression (20) is therefore

$$O(y^{1+\delta})+\frac{1}{y}\int_0^y v^{1+\delta}\,dv\,\frac{d}{dv}\int_v^\eta \frac{\lambda_1(t)}{t^{1+\beta+\delta}}\,(t-v)^{\beta-2}\,dt$$
$$+\frac{1}{y}\int_0^y v^{1+\delta}\,dv\,(1+\beta+\delta)\int_v^\eta (t-v)^{\beta-2}\,\frac{\lambda_1(t)}{t^{2+\beta+\delta}}\,dt$$
$$-\frac{1}{y}\int_0^y v^{1+\delta}\,dv\,\lambda_1'(v)\int_v^\eta \frac{(t-v)^{\beta-2}}{t^{1+\beta+\delta}}\,dt.$$

Neglecting the first term, the preceding expression is

$$\frac{1}{y}\left[v^{1+\delta}\int_v^\eta \frac{\lambda_1(t)(t-v)^{\beta-2}}{t^{1+\beta+\delta}}\,dt\right]_0^y-\frac{1}{y}\int_0^y \delta v^\delta\,dv\int_v^\eta \frac{\lambda_1(t)}{t^{1+\beta+\delta}}\,(t-v)^{\beta-2}\,dt$$
$$+\frac{1}{y}\int_0^y v^{1+\delta}\,dv\,(1+\beta+\delta)\int_v^\eta \frac{(t-v)^{\beta-2}}{t^{2+\beta+\delta}}\,\lambda_1(t)\,dt-\frac{1}{y}\left[v^{1+\delta}\lambda_1(v)\int_v^\eta \frac{(t-v)^{\beta-2}}{t^{1+\beta+\delta}}\,dt\right]_0^y$$
$$+\frac{1}{y}\int_0^y \lambda_1(v)\,\frac{d}{dv}\left\{v^{1+\delta}\int_v^\eta \frac{(t-v)^{\beta-2}}{t^{1+\beta+\delta}}\,dt\right\}dv. \qquad (21)$$

6. Since

$$\frac{d}{dv}\int_v^\eta \frac{(t-v)^{\beta-2}}{t^{1+\beta+\delta}}\,dt=-\frac{(\eta-v)^{\beta-2}}{\eta^{1+\beta+\delta}}-\int_v^\eta \frac{(t-v)^{\beta-2}}{t^{2+\beta+\delta}}\,(1+\beta+\delta)\,dt,$$

the expression (21) is

$$\frac{1}{y}\left[v^{1+\delta}\int_v^\eta \frac{(t-v)^{\beta-2}}{t^{1+\beta+\delta}}\{\lambda_1(t)-\lambda_1(v)\}\,dt\right]_0^y-\frac{1}{y}\int_0^y \delta v^\delta\,dv\int_v^\eta \frac{(t-v)^{\beta-2}}{t^{1+\beta+\delta}}\{\lambda_1(t)-\lambda_1(v)\}\,dt$$
$$+(1+\beta+\delta)\,\frac{1}{y}\int_0^y v^{1+\delta}\,dv\int_v^\eta \frac{(t-v)^{\beta-2}}{t^{2+\beta+\delta}}\{\lambda_1(t)-\lambda_1(v)\}\,dt$$
$$-\frac{1}{y}\int_0^y \lambda_1(v)\,v^{1+\delta}\,\frac{(\eta-v)^{\beta-2}}{\eta^{1+\beta+\delta}}\,dv$$
$$=J_1-\delta J_2+(1+\beta+\delta)\,J_3-J_4.$$

We now have

$$\lim_{y\to 0} J_1=\lim_{\mu\to\infty}\lim_{y\to 0} o\left[y^\delta\int_\mu^{\mu y}\frac{(t-y)^{\beta-2}}{t^{1+\beta+\delta}}\,t^{1+\alpha}(t-y)^{1-\alpha}\,dt\right]$$
$$+\lim_{\mu\to\infty}\lim_{y\to 0} O\left[y^\delta\int_{\mu y}^\eta \frac{(t-y)^{\beta-2}}{t^{1+\beta+\delta}}\,t^{1+\alpha}(t-y)^{1-\alpha}\,dt\right]$$
$$=\lim_{\mu\to\infty}\lim_{y\to 0} o\left[y^{\alpha-\beta}\int_y^{\mu y}(t-y)^{\beta-\alpha-1}\,dt\right]$$
$$+\lim_{\mu\to\infty}\lim_{y\to 0} O\left[y^\delta\int_{\mu y}^\eta (t-y)^{\beta-\alpha-1}\,t^{\alpha-\beta-\delta}\,dt\right]. \qquad (22)$$

The first term of (22) is zero. To evaluate the second we have

$$\begin{aligned}\int_{\mu y}^{\eta}(t-y)^{\beta-\alpha-1}t^{\alpha-\beta-\delta}\,dt &= \int_{y}^{\eta/y} y^{\beta-\alpha-1}(\tau-1)^{\beta-\alpha-1}(y\tau)^{\alpha-\beta-\delta}\,y\,d\tau\\ &= y^{-\delta}\int_{\mu}^{\eta/y}(\tau-1)^{\beta-\alpha-1}\tau^{\alpha-\beta-\delta}\,d\tau\\ &= O\left[y^{-\delta}\int_{\mu}^{\eta/y}\tau^{-1-\delta}\,d\tau\right]\\ &= O\left[\frac{1}{(\mu y)^{\delta}}\right].\end{aligned}$$

Hence the second term of (22) vanishes. Thus

$$\lim_{y\to 0} J_1 = 0.$$

Next,

$$\begin{aligned}\lim_{y\to 0} J_2 = &\lim_{\mu\to\infty}\lim_{y\to 0} o\left[\int_0^y v^{\delta}\,dv\int_v^{\mu v}\frac{(t-v)^{\beta-2}}{t^{1+\beta+\delta}}\,t^{1+\alpha}(t-v)^{1-\alpha}\,dt\right]\\ &+\lim_{\mu\to\infty}\lim_{y\to 0} O\left[\int_0^y v^{\delta}\,dv\int_{\mu v}^{\eta}\frac{(t-v)^{\beta-2}}{t^{1+\beta+\delta}}\,t^{1+\alpha}(t-v)^{1-\alpha}\,dt\right]\\ &= 0\end{aligned}$$

as before. Further,

$$\lim_{y\to 0} J_3 = 0,$$

since J_3 is obtained from J_2 by writing $1+\delta$ for δ. Finally

$$\begin{aligned}\lim_{y\to 0} J_4 &= \lim_{y\to 0} O\left[\frac{1}{y}\int_0^y v^{1+\delta}v^2\,dv\right]\\ &= 0,\end{aligned}$$

since

$$\lambda_1(v) = O(v^2).$$

This proves (10).

7. To prove (11) we must consider the expression

$$\frac{1}{y}\int_0^y v^{1+\delta}\,dv\left|\int_0^{\eta}\frac{(t-v)^{\beta-2}}{t^{1+\beta+\delta}}\,\Phi_v(t)\,dt\right|,$$

and, by (19), it is sufficient to show that

$$\frac{1}{y}\int_0^y v^{1+\delta}\,dv\int_v^{\eta}\frac{(t-v)^{\beta-2}}{t^{1+\beta+\delta}}\,|\lambda(t)-\lambda(v)|\,dt = O(1), \tag{23}$$

$$\frac{1}{y}\int_0^y v^{1+\delta}\,dv\int_v^{\eta}\frac{(t-v)^{\beta-2}}{t^{1+\beta+\delta}}\,|\lambda_1'(t)-\lambda_1'(v)|\,dt = O(1). \tag{24}$$

The equation (23) follows by the argument which was used in discussing the contribution to (18) of the second term in (19). We consider (24).

We have

$$\left| \frac{d}{du} \{u^{\alpha+1} \phi_{\alpha 1}(u)\} \right| \leqslant \frac{d}{du} \{u^{\alpha+1} \phi^*_{\alpha 1}(u)\}$$

for nearly all u. For the first member is, for nearly all u,

$$\left| (\alpha+1) u^\alpha \phi_{\alpha 1}(u) + u^{\alpha+1} \left[-\frac{1}{u^2} \int_0^u \phi_\alpha(v)\, dv + \frac{\phi_\alpha(u)}{u} \right] \right|$$

$$= |\alpha u^\alpha \phi_{\alpha 1}(u) + u^\alpha \phi_\alpha(u)|$$

$$\leqslant \alpha u^\alpha \phi^*_{\alpha 1}(u) + u^\alpha |\phi_\alpha(u)|,$$

while the second member is, for nearly all u,

$$(\alpha+1) u^\alpha \phi^*_{\alpha 1}(u) + u^{\alpha+1} \left[-\frac{1}{u^2} \int_0^u |\phi_\alpha(v)|\, dv + \frac{|\phi_\alpha(u)|}{u} \right]$$

$$= \alpha u^\alpha \phi^*_{\alpha 1}(u) + u^\alpha |\phi_\alpha(u)|,$$

which proves the assertion.

Since

$$\lambda'_1(v) - \lambda'_1(t) = \int_0^v \frac{d}{du} \{u^{\alpha+1} \phi_{\alpha 1}(u)\} \left[\frac{1}{(v-u)^\alpha} - \frac{1}{(t-u)^\alpha} \right] du - \int_v^t \frac{d\{u^{\alpha+1} \phi_{\alpha 1}(u)\}}{du} \frac{du}{(t-u)^\alpha},$$

we have

$$|\lambda'_1(t) - \lambda'_1(v)| \leqslant \int_0^v \frac{d}{du} \{u^{\alpha+1} \phi^*_{\alpha 1}(u)\} \left[\frac{1}{(v-u)^\alpha} - \frac{1}{(t-u)^\alpha} \right] du + \int_v^t \frac{d\{u^{\alpha+1} \phi^*_{\alpha 1}(u)\}}{du} \frac{du}{(t-u)^\alpha},$$

and since

$$\frac{d}{dt} \int_0^t \frac{u^{\alpha+1} \phi^*_{\alpha 1}(u)}{(t-u)^\alpha}\, du = \int_0^t \frac{d\{u^{\alpha+1} \phi^*_{\alpha 1}(u)\}}{du} \frac{du}{(t-u)^\alpha},$$

we have

$$|\lambda'_1(t) - \lambda'_1(v)| \leqslant 2 \int_v^t \frac{d\{u^{\alpha+1} \phi^*_1(u)\}}{du} \frac{du}{(t-u)^\alpha} - \left[\frac{d}{dt} \int_0^t \frac{u^{\alpha+1} \phi^*_{\alpha 1}(u)}{(t-u)^\alpha}\, du - \frac{d}{dv} \int_0^v \frac{u^{\alpha+1} \phi^*_{\alpha 1}(u)}{(v-u)^\alpha}\, du \right].$$

Further,

$$\frac{d}{dt}\int_v^t \frac{u^{\alpha+1}\phi_{\alpha 1}^*(u)}{(t-u)^\alpha}\,du$$

$$= \frac{d}{dt}\left[v^{\alpha+1}\phi_{\alpha 1}^*(v)\frac{(t-v)^{1-\alpha}}{1-\alpha}+\int_v^t \frac{d}{du}\{u^{\alpha+1}\phi_{\alpha 1}^*(u)\}\frac{(t-u)^{1-\alpha}}{1-\alpha}\,du\right]$$

$$= v^{\alpha+1}\phi_{\alpha 1}^*(v)(t-v)^{-\alpha}+\int_v^t \frac{d}{du}\{u^{\alpha+1}\phi_{\alpha 1}^*(u)\}\frac{du}{(t-u)^\alpha}$$

$$\geqslant \int_v^t \frac{d}{du}\{u^{\alpha+1}\phi_{\alpha 1}^*(u)\}\frac{du}{(t-u)^\alpha}.$$

Hence

$$|\lambda_1'(t)-\lambda_1'(v)| \leqslant 2\frac{d}{dt}\int_v^t \frac{u^{\alpha+1}\phi_{\alpha 1}^*(u)}{(t-u)^\alpha}\,du$$

$$-\left[\frac{d}{dt}\int_0^t \frac{u^{\alpha+1}\phi_{\alpha 1}^*(u)}{(t-u)^\alpha}\,du-\frac{d}{dv}\int_0^v \frac{u^{\alpha+1}\phi_{\alpha 1}^*(u)}{(v-u)^\alpha}\,du\right]. \quad (25)$$

8. Write $$\lambda_1^*(t)=\int_0^t \frac{u^{\alpha+1}\phi_{\alpha 1}^*(u)}{(t-u)^\alpha}\,du.$$

Then $$\lambda_1^*(t)-\lambda_1^*(v)=O[t^{1+\alpha}(t-v)^{1-\alpha}] \quad (0\leqslant t\leqslant\eta). \quad (26)$$

The contribution to (24) of the second term of (25), namely

$$\frac{1}{y}\int_0^y v^{1+\delta}\,dv\int_v^\eta \frac{(t-v)^{\beta-2}}{t^{1+\beta+\delta}}\,[\lambda_1'^*(t)-\lambda_1'^*(v)]\,dt,$$

may be reduced to the form

$$\frac{1}{y}\left[v^{1+\delta}\int_v^\eta \frac{(t-v)^{\beta-2}}{t^{1+\beta+\delta}}\,[\lambda_1^*(t)-\lambda_1^*(v)]\right]_v^y-\frac{1}{y}\int_0^y \delta v^\delta\,dv\int_v^\eta \frac{(t-v)^{\beta-2}}{t^{1+\beta+\delta}}\{\lambda_1^*(t)-\lambda_1^*(v)\}\,dt$$

$$+\frac{1}{y}\int_0^y v^{1+\delta}\,dv\,(1+\beta+\delta)\int_v^\eta \frac{(t-v)^{\beta-2}}{t^{2+\beta+\delta}}\{\lambda_1^*(t)-\lambda_1^*(v)\}\,dt$$

$$-\frac{1}{y}\int_0^y \lambda_1^*(v)\,v^{1+\delta}\frac{(\eta-v)^{\beta-2}}{\eta^{1+\beta+\delta}}\,dv$$

$$= J_1^*-\delta J_2^*+(1+\beta+\delta)J_3^*-J_4^* \quad (27)$$

in precisely the same way as (20) is reduced to the analogous form. And to show that (27) is $O(1)$ as $y\to 0$ in virtue of (26) and

$$\lambda_1^*(v)=O(v^2)$$

is similar to and simpler than the argument of §6, the number μ not being introduced.

To complete the proof of (11) we must consider

$$\frac{1}{y}\int_0^y v^{1+\delta}\,dv\int_v^\eta \frac{(t-v)^{\beta-2}}{t^{1+\beta+\delta}}\,dt\,\frac{d}{dt}\int_v^t \frac{u^{\alpha+1}\phi_{n1}^*(u)}{(t-u)^\alpha}\,du. \tag{28}$$

Write

$$\lambda^*(t,v)=\int_v^t \frac{u^{\alpha+1}\phi_{n1}^*(u)}{(t-u)^\alpha}\,du$$

$$=O[t^{1+\alpha}(t-v)^{1-\alpha}]. \tag{29}$$

Then

$$\int_v^\eta \frac{(t-v)^{\beta-2}}{t^{1+\beta+\delta}}\,\frac{d}{dt}\,\lambda^*(t,v)\,dt$$

$$=\int_v^\eta (t-v)^{\beta-2}\left[\frac{d}{dt}\left\{\frac{\lambda^*(t,v)}{t^{1+\beta+\delta}}\right\}+(1+\beta+\delta)\,\frac{\lambda^*(t,v)}{t^{2+\beta+\delta}}\right]dt$$

$$=\left[\frac{(t-v)^{\beta-2}\lambda^*(t,v)}{t^{1+\beta+\delta}}\right]_v^\eta+\int_v^\eta \frac{\lambda^*(t,v)}{t^{1+\beta+\delta}}\,\frac{d}{dv}\,(t-v)^{\beta-2}\,dt$$

$$+(1+\beta+\delta)\int_v^\eta (t-v)^{\beta-2}\,\frac{\lambda^*(t,v)}{t^{2+\beta+\delta}}\,dt$$

$$=\frac{(\eta-v)^{\beta-2}\lambda^*(\eta,v)}{\eta^{1+\beta+\delta}}+\frac{d}{dv}\int_v^\eta \frac{(t-v)^{\beta-2}\lambda^*(t,v)}{t^{1+\beta+\delta}}\,dt$$

$$+(1+\beta+\delta)\int_v^\eta \frac{(t-v)^{\beta-2}\lambda^*(t,v)}{t^{2+\beta+\delta}}\,dt.$$

Thus (28) is

$$\frac{1}{y}\int_0^y v^{1+\delta}\,\frac{(\eta-v)^{\beta-2}}{\eta^{1+\beta+\delta}}\,\lambda^*(\eta,v)\,dv+\frac{1}{y}\left[v^{1+\delta}\int_v^\eta \frac{(t-v)^{\beta-2}\lambda^*(t,v)}{t^{1+\beta+\delta}}\,dt\right]_0^y$$

$$-\frac{1}{y}\int_0^y \delta v^\delta\,dv\int_v^\eta \frac{(t-v)^{\beta-2}}{t^{1+\beta+\delta}}\,\lambda^*(t,v)\,dt$$

$$+(1+\beta+\delta)\,\frac{1}{y}\int_0^y v^{1+\delta}\,dv\int_v^\eta \frac{(t-v)^{\beta-2}}{t^{2+\beta+\delta}}\,\lambda^*(t,v)\,dt. \tag{30}$$

This expression is $O(1)$ in virtue of (29) in precisely the same way as is (27) in virtue of (26).

This proves Theorem I for $0<p<1$. The proof for the other non-integer values of p follows by induction by the method given in § 2.

9. Clearly, if

$$\frac{1}{t}\int_0^t \phi_{p+\delta}(u)\,du=o(1),\qquad \frac{1}{t}\int_0^t |\phi_{p+\delta}(u)|\,du=O(1)$$

for all $\delta > 0$, then

$$\lim_{\omega \to \infty} \omega \int_0^\eta \phi(t) \gamma_{1+k}(\omega t)\, dt = 0$$

for all $k > p$. We can thus enunciate

THEOREM II. *The Fourier series of the absolutely integrable function $f(t)$ is summable (C, k) to S at $t = x$ for $k > p$ provided that*

$$\frac{1}{t} \int_0^t \phi_{p+\delta}(u)\, du = o(1), \quad \frac{1}{t} \int_0^t |\phi_{p+\delta}(u)|\, du = O(1)$$

for all $\delta > 0$.

10. THEOREM III. *If*

$$\frac{1}{t} \int_0^t |\phi_p(u)|\, du = O(1), \tag{31}$$

then the Fourier series of $f(t)$ at $t = x$ is either summable (C, k) for all $k > p$, or for no $k > 0$. A necessary and sufficient condition that it should be summable to S is that

$$\frac{1}{t} \int_0^t \phi_{p+\delta}(u)\, du = o(1) \tag{32}$$

for all $\delta > 0$; or that

$$\phi_p(u) = o(1) \quad (H, 1+\delta)$$

for all $\delta > 0$.

For it is necessary, in order that there should be summability for some $k > 0$, that

$$\phi(t) = o(1) \quad (H, r)$$

for some positive integer r, in which case (32) follows by L, Theorem 11. Further, (31) implies, by L, Theorem 9b, that

$$\frac{1}{t} \int_0^t |\phi_{p+\delta}(u)|\, du = O(1)$$

for all $\delta > 0$. Hence, by Theorem II, there is summability for $k > p$.

To show that the alternative condition mentioned in the enunciation is also necessary and sufficient, we observe that, if

$$\phi(t) = o(1) \quad (H, r),$$

then, by L, Theorem 8,

$$\phi_p(t) = o(1) \quad (H, 1+\delta).$$

Hence the condition is necessary. Further, the preceding relation implies

$$\phi_p(t) = o(1) \quad (C, 1+\delta)$$

by L, Theorem 5; and this, by L, Theorem 10, implies (32). The condition is thus sufficient.

11. THEOREM IV. *If*

$$\frac{1}{t}\int_0^t |\phi_p(u)|^q du = O(1) \quad (q > 1), \tag{33}$$

then the Fourier series of $f(t)$ at $t = x$ is either summable (C, k) for all $k > p$ or for no $k > 0$. A necessary and sufficient condition that it should be summable to S is that

$$\phi_p(u) = o(1) \quad \left(H, \frac{1}{q}+\delta\right) \tag{34}$$

for all $\delta > 0$.

As for Theorem III, it is necessary, in order that there should be summability for some $k > 0$, that

$$\phi(t) = o(1) \quad (H, r).$$

This condition together with (33) implies, by L, Theorem XII, the relation (34). If (34) is satisfied, then

$$\frac{1}{t}\int_0^t \phi_p(u)\, du = o(1),$$

and then there is summability for all $k > p$ by Theorem I.

12. From this point we are concerned with the series allied with a Fourier series. Let $f(t)$ be a periodic function of period 2π and integrable in $(0, 2\pi)$ in the sense of Lebesgue. Write

$$\chi(t) = f(x+t) - f(x-t), \tag{35}$$

$$\Theta(t) = \frac{1}{\pi}\int_t^\infty \frac{\chi(u)}{u}\, du, \tag{36}$$

$$\psi(t) = \Theta(t) - S. \tag{37}$$

We use the notation

$$\psi_0(t) = \psi(t),$$

$$\psi_1(t) = \frac{1}{t}\int_0^t \psi_0(u)\, du,$$

$$\dots \quad \dots \quad \dots \quad \dots$$

$$\psi_r(t) = \frac{1}{t}\int_0^t \psi_{r-1}(u)\, du,$$

where r is a positive integer. If $p = n+a$, where n is zero or a positive integer, and $0 < a < 1$, we write

$$\psi_p(t) = \frac{1}{t^a} \int_0^t \psi_n(u)(t-u)^{a-1}\, du.$$

We further write $\psi_0(t) = \psi(t)$,

and define $\chi_r(t)$, $\chi_p(t)$ in a similar manner.

If the Fourier series of $f(t)$ is

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt), \tag{38}$$

then the series

$$\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) \tag{39}$$

is said to be the series allied with (38).

The first theorem which we prove gives a sufficient condition that the series (39) should be summable (C, k) to S at the point $t = x$ for all values of k greater than an assigned number $p+1$ where $p \geqslant 0$. We have

THEOREM V. *The series allied with the Fourier series of an absolutely integrable function $f(t)$ is summable $(C, 1+p+\delta)$ for all $\delta > 0$ to S at the point $t = x$ if*

$$\frac{1}{t} \int_0^t \psi_p(u)\, du = o(1), \quad \frac{1}{t} \int_0^t |\psi_p(u)|\, du = O(1),$$

as $t \to 0$.

13. We write

$$\gamma_{1+k}(t) = \int_0^1 (1-u)^k \cos tu\, du. \tag{40}$$

From this we easily deduce that, as $t \to \infty$,

$$\gamma_{1+k}(t) = O\left(\frac{1}{t^{1+k}}\right) \quad (0 \leqslant k \leqslant 1) \tag{41}$$

$$= O\left(\frac{1}{t^2}\right) \quad (k \geqslant 1). \tag{42}$$

Further, $\gamma_{1+k}(t)$ and its derivatives are uniformly bounded for all t, and, as $t \to \infty$,

$$\gamma'_{1+k}(t) = O\left(\frac{1}{t^{1+k}}\right) \quad (0 \leqslant k \leqslant 2) \tag{43}$$

$$= O\left(\frac{1}{t^3}\right) \quad (k \geqslant 2). \tag{44}$$

We write
$$\sigma_{1+k}(t) = \int_0^1 (1-u)^k \sin tu\, du$$
$$= \frac{t\gamma_{2+k}(t)}{1+k} \tag{45}$$
$$= \frac{1-k\gamma_k(t)}{t}. \tag{46}$$

We easily find that, as $t \to \infty$,
$$\sigma_{1+k}(t) = O\left(\frac{1}{t}\right) \quad (k \geqslant 0), \tag{47}$$
$$\sigma'_{1+k}(t) = O\left(\frac{1}{t^{1+k}}\right) \quad (0 \leqslant k \leqslant 1) \tag{48}$$
$$= O\left(\frac{1}{t^2}\right) \quad (k \geqslant 1). \tag{49}$$

Finally, $\sigma_{1+k}(t)$ and its derivatives are uniformly bounded for all t.

14. The Fourier series of $f(t)$ being given by (38), it follows that the Fourier series of
$$\tfrac{1}{2}\{f(x+t)-f(x-t)\}$$
is
$$\sum_{m=1}^{\infty} (b_m \cos mx - a_m \sin mx) \sin mt.$$

This series has no constant term. The function $\sigma_{1+k}(\omega t)$ is, for a given value of ω, of bounded variation in the interval $(0, \infty)$ of t and tends to zero as t tends to ∞. Hence, by a theorem due to Hardy†,
$$\sum_{m=1}^{\infty} (b_m \cos mx - a_m \sin mx) \int_0^{\infty} \sigma_{1+k}(\omega t) \sin mt\, dt$$
$$= \int_0^{\infty} \tfrac{1}{2}\{f(x+t)-f(x-t)\}\, \sigma_{1+k}(\omega t)\, dt.$$

Since
$$\sigma_{1+k}(\omega t) = \frac{1}{\omega} \int_0^{\omega} \left(1-\frac{u}{\omega}\right)^k \sin tu\, du,$$
we have, by Fourier's integral theorem,
$$\int_0^{\infty} \sigma_{1+k}(\omega t) \sin mt\, dt = \frac{\pi}{2\omega}\left(1-\frac{m}{\omega}\right)^k \quad (m < \omega),$$
$$= 0 \quad (m \geqslant \omega).$$

† *Cf.* Hobson, *Theory of functions of a real variable*, 2 (1926), 583.

Hence

$$\sum_{1\leqslant m<\omega}\left(1-\frac{m}{\omega}\right)^k(b_m\cos mx-a_m\sin mx)=\frac{\omega}{\pi}\int_0^\infty\{f(x+t)-f(x-t)\}\,\sigma_{1+k}(\omega t)\,dt.$$

Thus the condition that the allied series may be summable (C, k) to S at $t=x$ is that

$$\lim_{\omega\to\infty}\frac{\omega}{\pi}\int_0^\infty\chi(t)\,\sigma_{1+k}(\omega t)\,dt=S. \tag{50}$$

15. In the proof of Theorem V we are concerned only with values of $k\geqslant 1$. We have

$$\frac{\omega}{\pi}\int_0^\infty\chi(t)\,\sigma_{1+k}(\omega t)\,dt=-\omega\int_0^\infty\frac{d}{dt}\{\Theta(t)\}\,t\sigma_{1+k}(\omega t)\,dt$$

$$=-\Big[\omega t\,\Theta(t)\,\sigma_{1+k}(\omega t)\Big]_0^\infty-\omega k\int_0^\infty\Theta(t)\,\gamma_k'(\omega t)\,dt.$$

If we now observe that, for small $t>0$,

$$\Theta(t)=O\left(\frac{1}{t}\right),$$

and for $t\to\infty$

$$\Theta(t)=o\left(\frac{1}{t}\right),$$

it follows that (50) may be written

$$\lim_{\omega\to\infty}\omega k\int_0^\infty\Theta(t)\,\gamma_k'(\omega t)\,dt=-S.$$

Let $\eta>0$ be an arbitrarily assigned number. Clearly,

$$\lim_{\omega\to\infty}\omega k\int_\eta^\infty\Theta(t)\,\gamma_k'(\omega t)\,dt=0.$$

Further,

$$\lim_{\omega\to\infty}\omega k\int_0^\eta\gamma_k'(\omega t)\,dt=-1,$$

and we thus arrive at the condition

$$\lim_{\omega\to\infty}\omega\int_0^\eta\psi(t)\,\gamma_k'(\omega t)\,dt=0. \tag{51}$$

In order to prove Theorem V it is therefore sufficient to prove

THEOREM V′. *If*

$$\frac{1}{t}\int_0^t \psi_p(u)\,du = o(1), \quad \frac{1}{t}\int_0^t |\psi_p(u)|\,du = O(1),$$

then

$$\lim_{\omega\to\infty} \omega \int_0^\eta \psi(t)\,\gamma'_{1+p+\delta}(\omega t)\,dt = 0$$

for all $\delta > 0$.

The proof is strictly analogous to that of Theorem I. It will be sufficient to indicate the main steps.

16. We first need the following lemma.

LEMMA 2. *If*

$$\psi_1(t) = \frac{1}{t}\int_0^t \psi(u)\,du = o(1), \quad \psi_1^*(t) = \frac{1}{t}\int_0^t |\psi(u)|\,du = O(1),$$

then

$$\lim_{\omega\to\infty} \omega \int_0^\eta \psi(t)\,\gamma'_{1+\delta}(\omega t)\,dt = 0.$$

The proof is similar to that of Lemma 1, § 2. This proves Theorem V′ for $p = 0$. The proof for $p = n$, a positive integer, then follows by induction, the equation

$$t\gamma''_{1+k}(t) = k\gamma'_k(t) - (k+2)\,\gamma'_{1+k}(t)$$

being employed.

To prove Theorem V′ for $p = \alpha$ $(0 < \alpha < 1)$, it is sufficient to show that, if

$$\frac{1}{t}\int_0^t \psi_\alpha(u)\,du = o(1), \quad \frac{1}{t}\int_0^t |\psi_\alpha(u)|\,du = O(1),$$

then

$$\lim_{\omega\to\infty} \omega \int_0^\eta \psi(t)\,\gamma'_{1+\beta+\delta}(\omega t)\,dt = 0 \tag{52}$$

for $\beta > \alpha$. Since

$$\gamma_{1+\beta+\delta}(\omega u) = \frac{\Gamma(1+\beta+\delta)}{\Gamma(\beta)\,\Gamma(1+\delta)}\,\frac{1}{u^{1+\beta+\delta}}\int_0^u \gamma_{1+\delta}(\omega v)\,v^{1+\delta}(u-v)^{\beta-1}\,dv,$$

we have, on differentiating with respect to ω,

$$\gamma'_{1+\beta+\delta}(\omega u) = \frac{\Gamma(1+\beta+\delta)}{\Gamma(\beta)\,\Gamma(1+\delta)}\,\frac{1}{u^{2+\beta+\delta}}\int_0^u \gamma'_{1+\delta}(\omega v)\,v^{2+\delta}(u-v)^{\beta-1}\,dv.$$

The equation (52) now becomes

$$\lim_{\omega\to\infty}\omega\int_0^\eta\psi(t)\,\frac{dt}{t^{2+\beta+\delta}}\int_0^t\gamma'_{1+\delta}(\omega v)\,v^{2+\delta}(t-v)^{\beta-1}\,dv=0,$$

or

$$\lim_{\omega\to\infty}\omega\int_0^\eta\gamma'_{1+\delta}(\omega v)\,v^{2+\delta}\,dv\int_0^\eta\psi(t)\,\frac{(t-v)^{\beta-1}}{t^{2+\beta+\delta}}\,dt=0.$$

Write

$$H(v)=v^{2+\delta}\int_v^\eta\psi(t)\,\frac{(t-v)^{\beta-1}}{t^{2+\beta+\delta}}\,dt;$$

it is sufficient, by Lemma 2, to show that

$$\frac{1}{y}\int_0^y H(v)\,dv=o(1),\quad \frac{1}{y}\int_0^y|H(v)|\,dv=O(1),$$

as $y\to 0$. The proof is precisely similar to that of equations (10), (11) of § 3.

Finally, the proof of Theorem V′ for $p=n+a$, where n is a positive integer, again follows by induction.

We obtain as an immediate corollary

THEOREM VI. *The series allied with the Fourier series of an absolutely integrable function $f(t)$ is summable (C, k) to S at $t=x$ for all $k>p+1$ if*

$$\frac{1}{t}\int_0^t\psi_{p+\delta}(u)\,du=o(1),\quad \frac{1}{t}\int_0^t|\psi_{p+\delta}(u)|\,du=O(1)$$

for all $\delta>0$.

17. We further have

THEOREM VII. *At a point at which*

$$\frac{1}{t}\int_0^t|\psi_p(u)|\,du=O(1),\tag{53}$$

the series

$$\sum_{n=1}^{\infty}(b_n\cos nx-a_n\sin nx)$$

is either summable (C, k) for all $k>p+1$ or is not summable for any $k>0$. A necessary and sufficient condition for the series to be summable for $k>p+1$ is that

$$\frac{1}{t}\int_0^t\psi_{p+\delta}(u)\,du=o(1)\tag{54}$$

for all $\delta>0$.

In the first place, it is necessary, in order that there should be summability for some $k > 0$, that

$$\psi(t) = o(1) \quad (H, r) \tag{55}$$

for some positive integer r. If (55) is satisfied, then, by (53) and L, Theorem XI, (54) holds for all $\delta > 0$. Further, (53) implies, by L, Theorem IX b, that

$$\frac{1}{t}\int_0^t |\psi_{p+\delta}(u)|\,du = O(1)$$

for all $\delta > 0$. Hence, by Theorem VI, there is summability for all $k > p+1$.

18. We now obtain a result of a different character.

THEOREM VIII. *At a point at which*

$$\frac{1}{t}\int_0^t \chi_p(u)\,du = o(1), \quad \frac{1}{t}\int_0^t |\chi_p(u)|\,du = O(1),$$

the series

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx)$$

is either summable (C, k) *for all* $k > p$ *or is not summable* (C, k) *for any* $k > 0$.

In virtue of (50) what we have to show is that, under the conditions of the theorem,

$$\lim_{\omega\to\infty} \frac{\omega}{\pi}\int_0^\infty \chi(t)\{\sigma_{1+k}(\omega t) - \sigma_{1+l}(\omega t)\}\,dt = 0, \tag{56}$$

where $k > p, l > p$. This is the same as showing that

$$\lim_{\omega\to\infty} \frac{1}{\pi}\int_0^\infty \frac{\chi(t)}{t}\{k\gamma_k(\omega t) - l\gamma_k(\omega t)\}\,dt = 0. \tag{57}$$

Clearly, if $\eta > 0$ is an arbitrarily assigned number,

$$\lim_{\omega\to\infty} \int_\eta^\infty \frac{\chi(t)}{t}\gamma_k(\omega t)\,dt = 0,$$

with a similar equation when k is replaced by l. Thus (57) is equivalent to

$$\lim_{\omega\to\infty} \frac{\omega}{\pi}\int_0^\eta \chi(t)\{\sigma_{1+k}(\omega t) - \sigma_{1+l}(\omega t)\}\,dt = 0. \tag{58}$$

We therefore have to prove

THEOREM VIII′. *If*

$$\frac{1}{t}\int_0^t \chi_p(u)\,du = o(1), \quad \frac{1}{t}\int_0^t |\chi_p(u)|\,du = O(1),$$

then (58) *is satisfied for* $k > p$, $l > p$.

19. We find it convenient to give the result for $p = 0$ as a lemma.

LEMMA 3. *If*

$$\chi_1(t) = \frac{1}{t}\int_0^t \chi(u)\,du = o(1), \quad \chi_1^*(t) = \frac{1}{t}\int_0^t |\chi(u)|\,du = O(1),$$

then

$$\lim_{\omega\to\infty} \frac{\omega}{\pi}\int_0^\eta \chi(t)\{\sigma_{1+k}(\omega t) - \sigma_{1+l}(\omega t)\}\,dt = 0$$

for $k > 0$, $l > 0$.

Write

$$F(t) = \sigma_{1+k}(t) - \sigma_{1+l}(t) = \frac{k\gamma_k(t) - l\gamma_l(t)}{t}.$$

Then $F'(t)$ is uniformly bounded, and

$$F(t) = O\left(\frac{1}{t^{1+m}}\right) \quad [m = \min(k, l, 2)] \tag{59}$$

as $t\to\infty$. We have

$$\begin{aligned}\lim_{\omega\to\infty} \omega\int_0^\eta \chi(t)\,F(\omega t)\,dt &= \lim_{\omega\to\infty}\left[\omega\int_0^{\lambda/\omega} + \omega\int_{\lambda/\omega}^\eta \chi(t)\,F(\omega t)\,dt\right] \\ &= \lim_{\omega\to\infty}(J_1 + J_2).\end{aligned}$$

We have, on integrating by parts,

$$\begin{aligned}J_1 &= \chi_1\left(\frac{\lambda}{\omega}\right)F(\lambda) - \omega^2\int_0^{\lambda/\omega} t\chi_1(t)\,F'(\omega t)\,dt \\ &= o(1) + o\left[\omega^2\int_0^{\lambda/\omega} t\,dt\right] \\ &= o(1)\end{aligned}$$

and

$$\begin{aligned}|J_2| &\leqslant C\omega\int_{\lambda/\omega}^\eta |\chi(t)|\,\frac{dt}{(\omega t)^{1+m}} \\ &\leqslant \frac{C}{\omega^m}\left[\frac{\chi_1^*(t)}{t^m}\right]_{\lambda/\omega}^\eta + \frac{C}{\omega^m}\int_{\lambda/\omega}^\eta \frac{\chi_1^*(t)}{t^{1+m}}\,dt \\ &\leqslant \frac{C}{\omega^m}\frac{\chi_1^*(\eta)}{\eta^m} + \frac{C}{\lambda^m},\end{aligned}$$

neglecting negative terms, and this tends to zero if $\lambda \to \infty$ after $\omega \to \infty$. This proves the lemma.

20. The proof of Theorem VIII′ for $p = n$ is by induction. Assume it for $p = n-1$. To prove that

$$\lim_{\omega \to \infty} \omega \int_0^\eta \chi(t) \{\sigma_{1+n+\delta}(\omega t) - \sigma_{1+n+\delta'}(\omega t)\} dt = 0 \quad (\delta > 0, \ \delta' > 0)$$

it is sufficient to prove that

$$\lim_{\omega \to \infty} \omega^2 \int_0^\eta \chi_1(t)\, t \{\sigma'_{1+n+\delta}(\omega t) - \sigma'_{1+n+\delta'}(\omega t)\} dt = 0. \tag{60}$$

In virtue of the relation

$$t\sigma'_{1+k}(t) = k\sigma_k(t) - (k+1)\sigma_{1+k}(t),$$

equation (60) will be satisfied if

$$\lim_{\omega \to \infty} \omega \int_0^\eta \chi_1(t) \{\sigma_{1+n+\delta}(\omega t) - \sigma_{1+n+\delta'}(\omega t)\} dt = 0, \tag{61}$$

$$\lim_{\omega \to \infty} \omega \int_0^\eta \chi_1(t) \{\sigma_{n+\delta}(\omega t) - \sigma_{1+n+\delta}(\omega t)\} dt = 0, \tag{62}$$

$$\lim_{\omega \to \infty} \omega \int_0^\eta \chi_1(t) \{\sigma_{n+\delta'}(\omega t) - \sigma_{1+n+\delta'}(\omega t)\} dt = 0. \tag{63}$$

If now $$\frac{1}{t}\int_0^t \chi_n(u)\, du = o(1), \quad \frac{1}{t}\int_0^t |\chi_n(u)|\, du = O(1),$$

then, in virtue of Theorem VIII′ for $p = n-1$, the equations (61), (62), (63) will be satisfied. The theorem is therefore true for $p = n$.

21. To prove Theorem VIII′ for $p = \alpha$ $(0 < \alpha < 1)$, it is sufficient to show that, if

$$\frac{1}{t}\int_0^t \chi_\alpha(u)\, du = o(1), \tag{64}$$

and

$$\frac{1}{t}\int_0^t |\chi_\alpha(u)|\, du = O(1), \tag{65}$$

then

$$\lim_{\omega \to \infty} \omega \int_0^\eta \chi(t) \{\sigma_{1+\beta+\delta}(\omega t) - \sigma_{1+\beta+\delta'}(\omega t)\} dt = 0, \tag{66}$$

where $\beta > \alpha$, $\delta > 0$, $\delta' > 0$.

Writing $$B(a, b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)},$$

we have

$$\sigma_{1+\beta+\delta}(\omega u) = \frac{1}{B(\beta, 1+\delta)\, u^{1+\beta+\delta}} \int_0^u \sigma_{1+\delta}(\omega v)\, v^{1+\delta}(u-v)^{\beta-1}\, dv.$$

Hence (66) is equivalent to

$$\lim_{\omega\to\infty}\left[\omega\int_0^\eta \chi(t)\,dt\left\{\frac{1}{B(\beta, 1+\delta)\, u^{1+\beta+\delta}}\int_0^t \sigma_{1+\delta}(\omega v)\, v^{1+\delta}(t-v)^{\beta-1}\,dv\right.\right.$$

$$\left.\left. - \frac{1}{B(\beta, 1+\delta')\, u^{1+\beta+\delta'}}\int_0^t \sigma_{1+\delta'}(\omega v)\, v^{1+\delta'}(t-v)^{\beta-1}\,dv\right\}\right] = 0,$$

or to

$$\lim_{\omega\to\infty}\left[\omega\int_0^\eta v^{1+\delta}\,\sigma_{1+\delta}(\omega v)\,\frac{dv}{B(\beta, 1+\delta)}\int_0^\eta \chi(t)\,\frac{(t-v)^{\beta-1}}{t^{1+\beta+\delta}}\,dt\right.$$

$$\left. - \omega\int_0^\eta v^{1+\delta'}\,\sigma_{1+\delta'}(\omega v)\,\frac{dv}{B(\beta, 1+\delta')}\int_v^\eta \chi(t)\,\frac{(t-v)^{\beta-1}}{t^{1+\beta+\delta'}}\,dt\right] = 0. \quad (67)$$

22. We now write

$$\Delta_\delta(v) = v^{1+\delta}\int_v^\eta \chi(t)\,\frac{(t-v)^{\beta-1}}{t^{1+\beta+\delta}}\,dt.$$

Then the analysis of § 3 *et seq.* shows that the conditions (64) and (65) imply

$$\frac{1}{y}\int_0^y \Delta_\delta(v)\,dv = o(1), \quad (68)$$

$$\frac{1}{y}\int_0^y |\Delta_\delta(v)|\,dv = O(1) \quad (69)$$

as $y\to 0$. We have similar equations in δ'. By the argument used for J_1 in § 19, the condition (68) implies

$$\lim_{\omega\to\infty}\omega\int_0^{\lambda/\omega}\Delta_\delta(v)\,\sigma_{1+\delta}(\omega v)\,dv = 0.$$

It remains then to show that

$$\lim_{\omega\to\infty}\left[\omega\int_{\lambda/\omega}^\eta \Delta_\delta(v)\,\sigma_{1+\delta}(\omega v)\,\frac{dv}{B(\beta, 1+\delta)} - \omega\int_{\lambda/\omega}^\eta \Delta_{\delta'}(v)\,\sigma_{1+\delta'}(\omega v)\,\frac{dv}{B(\beta, 1+\delta')}\right] = 0, \quad (70)$$

or that

$$\lim_{\omega\to\infty}\Big[\omega\int_{\lambda/\omega}^{\eta}\chi(t)\,dt\,\Big\{\frac{1}{B(\beta,\,1+\delta)\,t^{1+\beta+\delta}}\int_{\lambda/\omega}^{t}\sigma_{1+\delta}(\omega v)(t-v)^{\beta-1}v^{1+\delta}\,dv$$

$$-\frac{1}{B(\beta,\,1+\delta')\,t^{1+\beta+\delta'}}\int_{\lambda/\omega}^{t}\sigma_{1+\delta'}(\omega v)\,(t-v)^{\beta-1}v^{1+\delta'}\,dv\Big\}\Big]=0. \quad (71)$$

Now
$$\sigma_{1+\delta}(\omega v)=\frac{1-\delta\gamma_\delta(\omega v)}{\omega v}.$$

Hence (71) will be satisfied if

$$\lim_{\omega\to\infty}\Big[\int_{\lambda/\omega}^{\eta}\chi(t)\,dt\,\Big\{\frac{1}{t^{1+\beta+\delta}\,B(\beta,\,1+\delta)}\int_{\lambda/\omega}^{t}v^{\delta}(t-v)^{\beta-1}\,dv$$

$$-\frac{1}{t^{1+\beta+\delta'}\,B(\beta,\,1+\delta')}\int_{\lambda/\omega}^{t}v^{\delta'}(t-v)^{\beta-1}\,dv\Big\}\Big]=0 \quad (72)$$

and

$$\lim_{\omega\to\infty}\Big[\int_{\lambda/\omega}^{\eta}\chi(t)\,dt\,\Big\{\frac{1}{t^{1+\beta+\delta}\,B(\beta,\,1+\delta)}\int_{\lambda/\omega}^{t}\delta\gamma_\delta(\omega v)\,v^{\delta}(t-v)^{\beta-1}\,dv$$

$$-\frac{1}{t^{1+\beta+\delta'}\,B(\beta,\,1+\delta')}\int_{\lambda/\omega}^{t}\delta'\gamma_{\delta'}(\omega v)\,v^{\delta'}(t-v)^{\beta-1}\,dv\Big\}\Big]=0. \quad (73)$$

23. Concerning (73) we observe that

$$\lim_{\omega\to\infty}\int_{\lambda/\omega}^{\eta}\chi(t)\,\frac{dt}{t^{1+\beta+\delta}}\int_{\lambda/\omega}^{t}\gamma_\delta(\omega v)\,v^{\delta}(t-v)^{\beta-1}\,dv$$

$$=\lim_{\omega\to\infty}\int_{\lambda/\omega}^{\eta}v^{\delta}\gamma_\delta(\omega v)\,dv\int_{v}^{\eta}\chi(t)\,\frac{(t-v)^{\beta-1}}{t^{1+\beta+\delta}}\,dt$$

$$=\lim_{\omega\to\infty}\omega\int_{\lambda/\omega}^{\eta}\gamma_\delta(\omega v)\,\Delta_\delta(v)\,\frac{dv}{\omega v}$$

$$=0,$$

in virtue of (69), by the same argument as for J_2 in §19. Similarly for the second term in (73).

Finally, to prove (72) we notice that

$$\frac{1}{B(\beta,\,1+\delta)\,t^{1+\beta+\delta}}\int_{0}^{t}v^{\delta}(t-v)^{\beta-1}\,dv=1,$$

and it is therefore sufficient to show that

$$\lim_{\omega\to\infty}\int_{\lambda/\omega}^{\eta}\chi(t)\,\frac{dt}{t^{1+\beta+\delta}}\int_{0}^{\lambda/\omega}v^{\delta}(t-v)^{\beta-1}\,dv=0,$$

with a similar relation in δ'. We have

$$\frac{1}{t^{1+\beta+\delta}}\int_0^{\lambda/\omega} v^{\delta}(t-v)^{\beta-1}\,dv = \int_0^{\lambda/\omega t} v^{\delta}(1-v)^{\beta-1}\,dv$$

$$= G\left(\frac{\omega t}{\lambda}\right),$$

where $G(t)$ tends to zero as $t\to\infty$.

Hence

$$\left|\lim_{\omega\to\infty}\int_{\lambda/\omega}^{\eta}\chi(t)\,G\left(\frac{\omega t}{\lambda}\right)dt\right| \leqslant \lim_{\omega\to\infty} C\int_{\lambda/\omega}^{\lambda^2/\omega}|\chi(t)|\,dt + \lim_{\omega\to\infty}\int_{\lambda^2/\omega}^{\eta}|\chi(t)|\,G\left(\frac{\omega t}{\lambda}\right)dt$$

$$\leqslant G(\lambda)\int_0^{\eta}|\chi(t)|\,dt,$$

and this is zero if $\lambda\to\infty$ after $\omega\to\infty$.

This proves Theorem VIII′ for $p=\alpha$. The proof for $p=n+\alpha$ is by induction, as in § 20.

NOTE ON THE INTEGRATION OF TRIGONOMETRIC SERIES

By S. VERBLUNSKY.

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1. The following theorem concerning the integration of trigonometric series is due to Hobson*:

If at every point x of the open interval (α, β), except those points which belong to a set E which contains no perfect component, the series

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

converges to the value $f(x)$ of a function which is integrable L in every interval interior to (α, β), then

$$\int_{x_1}^{x_2} f(x)\,dx = \left[\tfrac{1}{2}a_0 x + \sum_{n=1}^{\infty} \frac{a_n \sin nx - b_n \cos nx}{n}\right]_{x_1}^{x_2}, \tag{2}$$

where x_1, x_2 are any interior points of the interval (α, β).

We observe that, as a consequence of the convergence of (1) in a set of positive measure, the relations

$$a_n = o(1), \quad b_n = o(1) \tag{3}$$

hold. If we assume that (3) holds, we can, in the above theorem, replace convergence by summability by Poisson's method. This is a simple consequence of a theorem due to Rajchman and Zygmund (*cf.* § 2).

We may ask how far an analogous result is true when the conditions (3) are replaced by

$$a_n = o(n), \quad b_n = o(n). \tag{4}$$

This problem also admits of a general solution.

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* Hobson, *Journal London Math. Soc.*, 2 (1927), 164–166.

The enunciations of the two theorems are as follows:

THEOREM I. *Let* $\bar{P}(x)$, $\underline{P}(x)$ *denote the upper and lower Poisson sums of* (1), *the coefficients of which satisfy* (3). *If* $\bar{P}(x)$, $\underline{P}(x)$ *are integrable* L *in* (a, b) *and finite except at the points of a set* E *which contains no perfect component, then, if* $f(x)$ *is any function which satisfies*

$$\bar{P}(x) \geqslant f(x) \geqslant \underline{P}(x)$$

in (a, b), *we have*

$$\int_{x_1}^{x_2} f(x)\,dx = \left[\tfrac{1}{2}a_0 x + \sum_{n=1}^{\infty} \frac{a_n \sin nx - b_n \cos nx}{n}\right]_{x_1}^{x_2}, \tag{5}$$

where $a < x_1 < x_2 < b$.

THEOREM II. *Let* (1) *be a trigonometric series whose coefficients satisfy* (4). *If the upper and lower Poisson sums of* (1) *are finite and integrable* L *in* $a \leqslant x \leqslant b$, *and if* $f(x)$ *is any function which satisfies*

$$\bar{P}(x) \geqslant f(x) \geqslant \underline{P}(x)$$

in (a, b), *then* (5) *holds.*

If $$P(r, x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) r^n,$$

the result remains true if the Poisson sums are not both finite at the points of an enumerable set E, *provided that, at each point* $x \subset E$,

$$\lim_{r \to 1} (1-r) P(r, x) = 0. \tag{6}$$

The remainder of the note (§§ 4-6) is concerned with the discussion and generalisation of a theorem due to Paley*.

2. To prove Theorem I we may suppose without loss of generality that $a_0 = 0$. The function

$$F(x) = -\sum_{n+1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2}$$

is continuous, and by a theorem of Rajchman and Zygmund† the intervals

$$\{\bar{D}^2 F(x),\ \underline{D}^2 F(x)\},\quad \{\bar{P}(x),\ \underline{P}(x)\}$$

* Paley, *Journal London Math. Soc.*, 5 (1930), 182–189.

† Rajchman and Zygmund, *Math. Zeitschrift*, 25 (1926), 261–273 (268).

have a point in common for each x. If we write

$$\phi(x) = \text{Min}\{\bar{D}^2 F(x),\ \bar{P}(x)\},$$

we have $$\bar{D}^2 F(x) \geqslant \phi(x) \geqslant \underline{D}^2 F(x),$$

while, at the points of E, we have, by (3),

$$\lim_{h \to 0} \frac{F(x+h)+F(x-h)-2F(x)}{h} = 0.$$

It follows, by a classical theorem of de la Vallée Poussin, that

$$F(x) = \int_a^x dy \int_a^y \phi(t)\, dt + Ax + B \quad (a < x < b). \tag{7}$$

From the relation

$$-\Sigma \frac{a_n \cos nx + b_n \sin nx}{n^2} r^n = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(t) \frac{1-r^2}{1-2r\cos(t-x)+r^2}\, dt$$

and from the expression (7) it follows that the series obtained by the formal differentiation of

$$-\Sigma \frac{a_n \cos nx + b_n \sin nx}{n^2}$$

is uniformly summable by Poisson's method in $(a+\epsilon,\ b-\epsilon)$ to

$$\int_a^x \phi(t)\, dt + A,$$

and hence, by Tauber's theorem, is uniformly convergent to this function. This gives the relation (5) with $\phi(x)$ in the first member.

The relation (7) shows that $D^2 F(x)$ exists almost everywhere in (a, b). By a result of Rajchman and Zygmund*, when $D^2 F(x)$ exists, so does $P(x)$ and the two are equal. Thus $\phi(x)$ and $f(x)$ are equivalent in (a, b), and this completes the proof.

3. Theorem II will be deduced from the following result which we state as a lemma.

LEMMA 1. *Let* (1) *be a trigonometric series whose coefficients satisfy* (4). *If the upper and lower Poisson sums of* (1) *are finite and integrable* L *in* $a \leqslant x \leqslant b$, *then the difference between* (1) *and the Fourier series of any function which satisfies*

$$\bar{P}(x) \geqslant f(x) \geqslant \underline{P}(x)$$

* Rajchman and Zygmund, *loc. cit.*, 272, foot-note.

in (a, b), *and is zero elsewhere, is uniformly summable* $(C, 1)$ *to zero in* $(a+\epsilon, b-\epsilon)$, *and the series conjugate to this difference is uniformly summable* $(C, 1)$ *in the same range.*

This result was proved in a previous paper*. It follows from the arguments of that paper† that the result of the lemma holds if the Poisson sums are not both finite at the points of an enumerable set E, provided that at each point $x \subset E$ condition (6) is satisfied.

Having chosen $f(x)$, let

$$f(x) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + \beta_n \sin nx).$$

Then, by Lemma 1,

$$\Sigma\{(a_n-\alpha_n)\cos nx + (b_n-\beta_n)\sin nx\} \tag{8}$$

is uniformly summable $(C, 1)$ in $(a+\epsilon, b-\epsilon)$ to

$$-\tfrac{1}{2}(a_0-\alpha_0);$$

and

$$\Sigma\{(b_n-\beta_n)\cos nx - (a_n-\alpha_n)\sin nx\}$$

is uniformly summable $(C, 1)$ in $(a+\epsilon, b-\epsilon)$. We choose ϵ so that $(a+\epsilon, b-\epsilon)$ contains (x_1, x_2). It follows that

$$-\Sigma\left\{\frac{(b_n-\beta_n)\cos nx - (a_n-\alpha_n)\sin nx}{n}\right\} \tag{9}$$

is uniformly convergent in (x_1, x_2).

Let $S_n(x)$ denote the n-th Cesàro mean of order 1 of (8), and $\Sigma_n(x)$ the corresponding mean of (9). Then, from the uniform convergence of $S_n(x)$, we have

$$\lim_{n\to\infty} \int_{x_1}^{x_2} S_n(x)\,dx = -\tfrac{1}{2}(a_0-\alpha_0)(x_1-x_2).$$

But

$$\int_{x_1}^{x_2} S_n(x)\,dx = \Sigma_n(x_2) - \Sigma_n(x_1),$$

so that

$$\lim \int_{x_1}^{x_2} S_n(x)\,dx = \left[\Sigma\left\{\frac{(a_n-\alpha_n)\sin nx - (b_n-\beta_n)\cos nx}{n}\right\}\right]_{x_1}^{x_2}$$

* Verblunsky, *Proc. London Math. Soc.*, (2), 31 (1930), 370–386 (379).

† *Ibid.*, *cf.* the proof of Theorem V.

by the convergence of (9). Thus

$$\left[\tfrac{1}{2}(a_0-\alpha_0)x+\Sigma\left\{\frac{(a_n-\alpha_n)\sin nx-(b_n-\beta_n)\cos nx}{n}\right\}\right]_{x_1}^{x_2}=0.$$

Finally,

$$\int_{x_1}^{x_2} f(x)\,dx=\left[\tfrac{1}{2}\alpha_0 x+\Sigma\,\frac{\alpha_n\sin nx-\beta_n\cos nx}{n}\right]_{x_1}^{x_2}.$$

Hence the result.

4. For the purpose of the subsequent discussion it will be necessary to use a theorem which is analogous to Theorem II, but in which the condition (4) is replaced by the condition (F), *i.e.* the condition for

$$\Sigma\,\frac{a_n\cos nx+b_n\sin nx}{n^2}$$

to be the Fourier series of a continuous function.

We accordingly enunciate

THEOREM III. *Let* (1) *be a trigonometric series which satisfies the condition F. If the upper and lower Poisson sums of* (1) *are finite and integrable D in* $a\leqslant x\leqslant b$, *and if* $f(x)$ *is any function which satisfies*

$$\bar{P}(x)\geqslant f(x)\geqslant \underline{P}(x),$$

then

$$\int_{x_1}^{x_2} f(x)\,dx=\left[\tfrac{1}{2}a_0x+\sum_{n=1}^{\infty}\frac{a_n\sin nx-b_n\cos nx}{n}\right]_{x_1}^{x_2}\quad (C,\,1),\qquad (10)$$

where in the second member the sum $(C,\,1)$ *of the series is taken. The result remains true if the Poisson sums are not both finite at the points of an enumerable set E, provided that at each point of E the condition* (6) *is satisfied.*

Here again we require a preliminary lemma analogous to Lemma 1.

LEMMA 2. *Let* (1) *be a trigonometric series which satisfies the condition* (F). *If the upper and lower Poisson sums are finite and integrable D in* $a\leqslant x\leqslant b$, *then the difference between* (1) *and the Fourier-Denjoy series of any function which satisfies*

$$\bar{P}(x)\geqslant f(x)\geqslant \underline{P}(x)$$

in $(a,\,b)$, *and is zero elsewhere, is uniformly summable* $(C,\,2)$ *to zero in* $(a+\epsilon,\,b-\epsilon)$, *and the series conjugate to this difference is uniformly*

summable $(C, 2)$ *in the same range. The result remains true if the Poisson sums are not both finite at the points of an enumerable set* $\bar{E}$, *provided that at each point of* E *the condition* (6) *is satisfied.*

We observe that in Theorem III and Lemma 2 we require the upper and lower Poisson sums only to be integrable D, while in Theorem II and Lemma 1 we require them to be integrable L. Lemma 1 remains true when integrability L is replaced by integrability D. The reason for the restriction appears from the last equation of § 3. The series in the second member is not necessarily convergent if $f(x)$ is only integrable D. In any case it is summable $(C, 1)$, and this is sufficient for the purpose of Theorem III.

The proof of Lemma 2 is similar to that of Lemma 1, except that the fact that condition (F) is satisfied only enables us to assert that

$$a_n = o(n^2), \quad b_n = o(n^2),$$

and consequently, when applying the theorem of M. Riesz, we can infer only summability $(C, 2)$.

We now proceed to the proof of Theorem III. Having chosen $f(x)$, let

$$f(x) \sim \tfrac{1}{2}a_0 + \Sigma\,(a_n \cos nx + \beta_n \sin nx).$$

Then, by Lemma 2,

$$\Sigma\{(a_n - \alpha_n)\cos nx + (b_n - \beta_n)\sin nx\} \tag{11}$$

is uniformly summable $(C, 2)$ in $(a+\epsilon, b-\epsilon)$ to

$$-\tfrac{1}{2}(a_0 - \alpha_0),$$

and

$$\Sigma\{(b_n - \beta_n)\cos nx - (a_n - \alpha_n)\sin nx\}$$

is uniformly summable $(C, 2)$ in $(a+\epsilon, b-\epsilon)$. We choose ϵ so that $(a+\epsilon, b-\epsilon)$ contains (x_1, x_2). Then

$$-\Sigma\left\{\frac{(b_n - \beta_n)\cos nx - (a_n - \alpha_n)\sin nx}{n}\right\} \tag{12}$$

is uniformly summable $(C, 1)$ in (x_1, x_2).

Let $S_n(x)$ denote the n-th Cesàro mean of order 2 of (11), and $\Sigma_n(x)$ the corresponding mean of (12). Then, from the uniform convergence of $S_n(x)$, we have

$$\lim_{n\to\infty} \int_{x_1}^{x_2} S_n(x)\,dx = -\tfrac{1}{2}(a_0 - \alpha_0)(x_1 - x_2).$$

But
$$\int_{x_1}^{x_2} S_n(x)\,dx = \Sigma_n(x_2) - \Sigma_n(x_1).$$

Hence
$$\lim \int_{x_1}^{x_2} S_n(x)\,dx = \left[\Sigma \left\{\frac{(a_n-\alpha_n)\sin nx - (b_n-\beta_n)\cos nx}{n}\right\}\right]_{x_1}^{x_2} \quad (C, 1)$$

by the summability $(C, 1)$ of (12). Thus

$$\left[\tfrac{1}{2}(a_0-\alpha_0)x + \Sigma \left\{\frac{(a_n-\alpha_n)\sin nx - (b_n-\beta_n)\cos nx}{n}\right\}\right]_{x_1}^{x_2} = 0 \quad (C, 1).$$

Finally,
$$\int_{x_1}^{x_2} f(x)\,dx = \left[\tfrac{1}{2}a_0 x + \Sigma \frac{a_n \sin nx - \beta_n \cos nx}{n}\right]_{x_1}^{x_2} \quad (C, 1).$$

Hence the result.

5. We can now discuss the theorem of Paley, which is enunciated as follows.

THEOREM A. *Suppose that* $0 < \gamma < 1$, *and that*

$$\Sigma\,(a_n \cos nx + b_n \sin nx) \tag{13}$$

is summable (C, γ) *to zero except in a closed enumerable set* E, *then*

$$\Sigma\,(a_n \cos nx + b_n \sin nx)\, r^n = -\frac{1}{2\pi}\int_0^{2\pi} \Phi(t)\,\frac{\partial}{\partial t}\left\{\frac{1-r^2}{1-2r\cos(t-x)+r^2}\right\} dt, \tag{14}$$

where $\Phi(t)$ *is a function which is integrable* D *in* $(0, 2\pi)$ *and which, moreover, is constant in each of the intervals complementary to* E.

The significance of the enunciation becomes more apparent when we consider the following lemma.

LEMMA 3. *Let* $f(x)$ *be a periodic function of period* 2π *which is integrable* D *in* $(0, 2\pi)$. *Let*

$$\Sigma\,(a_n \cos nx + b_n \sin nx)$$

be its Fourier-Denjoy series. Then

$$\Sigma\, n(b_n \cos nx - a_n \sin nx)\, r^n = -\frac{1}{2\pi}\int_0^{2\pi} f(t)\,\frac{\partial}{\partial t}\left\{\frac{1-r^2}{1-2r\cos(t-x)+r^2}\right\} dt.$$

Thus we can say that, under the conditions of Theorem A, the series (13) is obtained by term-by-term differentiation of the Fourier-Denjoy series of $\Phi(t)$, and in our subsequent generalisations of Theorem A, this statement will replace equation (14).

To prove Lemma 3, we observe that

$$\sum_{n=1}^{m} (a_n \cos nx + b_n \sin nx) r^n = \frac{1}{2\pi} \int_0^{2\pi} f(t) \left[1 + 2 \sum_{n=1}^{m} r^n \cos n(t-x) \right] dt.$$

Now $$\left| \frac{\partial}{\partial t} \left\{ 2 \sum_{n=m+1}^{\infty} r^n \cos n(t-x) \right\} \right| < 2 \sum_{n=1}^{\infty} n r^n.$$

Hence, by a known theorem*,

$$\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) r^n = \frac{1}{2\pi} \int_0^{2\pi} f(t) \left\{ \frac{1-r^2}{1-2r\cos(t-x)+r^2} \right\} dt.$$

It follows that

$$\sum_{n=1}^{\infty} n(b_n \cos nx - a_n \sin nx) r^n$$

$$= \frac{1}{2\pi} \frac{d}{dx} \int_0^{2\pi} f(t) \left\{ \frac{1-r^2}{1-2r\cos(t-x)+r^2} \right\} dt$$

$$= \frac{1}{2\pi} \lim_{h \to 0} \int_0^{2\pi} f(t) \left[\frac{2 \sum_{n=1}^{\infty} r^n \cos n(t-x-h) - 2 \sum_{n=1}^{\infty} r^n \cos n(t-x)}{h} \right] dt. \quad (15)$$

Since

$$\left| \frac{\partial}{\partial t} \left\{ \frac{\sum_{n=1}^{\infty} r^n \cos n(t-x-h) - \sum_{n=1}^{\infty} r^n \cos n(t-x)}{h} \right\} \right| < \sum_{n=1}^{\infty} n^2 r^n,$$

it follows, by the same theorem as before, that we can take the limit under the integral sign in (15) and this gives

$$\sum_{n=1}^{\infty} n(b_n \cos nx - a_n \sin nx) r^n = \frac{1}{2\pi} \int_0^{2\pi} f(t) \frac{\partial}{\partial x} \left\{ \frac{1-r^2}{1-2r\cos(t-x)+r^2} \right\} dt,$$

from which the required result immediately follows.

A generalised form of Theorem A can now be enunciated as follows.

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* *Cf.* Hobson, *Theory of functions of a real variable*, 2 (1926), 437.

THEOREM IV. *Let the trigonometric series*

$$\Sigma (a_n \cos nx + b_n \sin nx) \tag{16}$$

satisfy the condition (F), *and let its upper and lower Poisson sums be finite except at an enumerable set* E, *and integrable* D *in* $(0, 2\pi)$. *Suppose further that almost everywhere*

$$\bar{P}(x) \geqslant 0 \geqslant \underline{P}(x).$$

Then the series (16) *is obtained by term-by-term differentiation of the Fourier-Denjoy series of a function* $\Phi(x)$, *which is constant in each of the intervals contiguous to* E.

It should be remembered that the set of points at which not both Poisson sums are finite is an ordinary inner limiting set. Being enumerable, it contains no component dense in itself. Thus the derivative E' of E is enumerable, and the contiguous intervals of E together make up, except for the points of $E - EE'$, the contiguous intervals of E'.

Let (α_n, β_n) be a contiguous interval of E. Let

$$\alpha_n < x_1 < x_2 < \beta_n.$$

In Theorem III we can take $f(x) = 0$. Then

$$\Sigma \frac{a_n \sin nx - b_n \cos nx}{n} \tag{17}$$

is uniformly summable $(C, 1)$ in (x_1, x_2) to a constant C_n. We denote by $\Phi(t)$ the function which is equal to C_n in (α_n, β_n). Since (16) satisfies the condition (F),

$$a_n = o(n^2), \quad b_n = o(n^2).$$

Hence (17) satisfies the conditions of Theorem II in (α_n, β_n) and therefore

$$\int_{x_1}^{x_2} \Phi(t)\, dt = -\left[\Sigma \frac{a_n \cos nx + b_n \sin nx}{n^2}\right]_{x_1}^{x_2}.$$

But there exists a continuous function $F(x)$ such that

$$F(x) \sim -\Sigma \frac{a_n \cos nx + b_n \sin nx}{n^2}.$$

Hence

$$\int_{x_1}^{x_2} \Phi(t)\, dt = F(x_2) - F(x_1).$$

This relation is sufficient to enable us to assert* that $\Phi(t)$ is integrable D.

* *Cf.* Lebesgue, *Leçons sur l'integration* (1928), 14.

It remains then to show that (17) is the Fourier-Denjoy series of $\Phi(t)$. We have

$$-\frac{a_n}{n^2} = \frac{1}{\pi}\int_0^{2\pi} F(t)\cos nt\,dt$$

$$= \frac{1}{\pi}\int_0^{2\pi} F(t)\frac{d}{dt}\left(\frac{\sin nt}{n}\right)dt$$

$$= -\frac{1}{n\pi}\int_0^{2\pi}\Phi(t)\sin nt\,dt,$$

and

$$-\frac{b_n}{n^2} = \frac{1}{\pi}\int_0^{2\pi} F(t)\sin nt\,dt$$

$$= -\frac{1}{\pi}\int_0^{2\pi} F(t)\frac{d}{dt}\left(\frac{\cos nt}{n}\right)dt$$

$$= \frac{1}{n\pi}\int_0^{2\pi}\Phi(t)\cos nt\,dt,$$

since $F(0) = F(2\pi)$. This completes the proof.

Theorem IV may be further extended by omitting the condition

$$\bar{P}(x) \geqslant 0 \geqslant \underline{P}(x).$$

In this case, if $f(x)$ is any integrable D function which satisfies

$$\bar{P}(x) \geqslant f(x) \geqslant \underline{P}(x)$$

almost everywhere, then $\Phi(t)$ is of the form

$$C_n + \int_{x_n}^{x} f(t)\,dt$$

in (a_n, β_n). The proof is the same as before.

6. It would be interesting to determine whether Theorem IV remains true when the condition (F) is replaced by

$$a_n = o(n), \quad b_n = o(n).$$

As before, let (a_n, β_n) denote an interval contiguous to E, and let

$$a_n < x_1 < x_2 < \beta_n.$$

In Theorem II take $f(x) = 0$. Then

$$\sum \frac{a_n \sin nx - b_n \cos nx}{n}$$

is uniformly convergent in (x_1, x_2) to a constant C_n. Denote by $\Phi(t)$ the function which in (α_n, β_n) is equal to C_n. Then

$$\int_{x_1}^{x_2} \Phi(t)\,dt = -\left[\Sigma \frac{a_n \cos nx - b_n \sin nx}{n^2}\right]_{x_1}^{x_2}$$

$$= F(x_2) - F(x_1),$$

where

$$F(x) \sim -\Sigma \frac{a_n \cos nx + b_n \sin nx}{n^2}.$$

In order that $\Phi(t)$ should be integrable D, it is necessary and sufficient for $F(x)$ to be continuous. To prove that the theorem is false, it would be sufficient to construct a discontinuous function $F(x)$, whose Fourier constants are $o(1/n)$, and which is linear in the intervals contiguous to E.

The theorem is certainly false when condition (F) is replaced by

$$a_n = O(n), \quad b_n = O(n),$$

as is shown by the series

$$\Sigma n \sin nx$$

which is everywhere summable by Poisson's method to zero. The series

$$\Sigma \cos nx$$

is not a Denjoy-Fourier series, since

$$\Sigma \frac{\sin nx}{n}$$

is discontinuous at the origin.

ON DOUBLE ORTHOGONAL SERIES

By RALPH PALMER AGNEW.

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1. *Introduction.* A vast amount of research has been done in the theory of simple trigonometrical series†. Numerous writers have been interested in developing the theory of double trigonometric series‡. In recent years, much has been written on the theory of simple orthogonal series§, but apparently no one has considered general double orthogonal series.

It is the object of this paper to begin an investigation in the theory of double series of general orthogonal functions.

Sections 4–7 of this paper are devoted largely to a consideration of orthogonal series of which the coefficients satisfy the condition

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}^2 < \infty.$$

We show that this condition is sufficient to ensure essentially uniform convergence of certain double subsequences of the double sequence of partial sums of a series, and give some of the consequences of this result. In §§ 8–15 we show that the condition

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} [\log(i+1)]^2 [\log(j+1)]^2 a_{ij}^2 < \infty$$

is sufficient to ensure essentially uniform convergence of the complete double sequence of partial sums of a series. In § 16 we give a theorem

* This paper embodies two papers which were presented to the American Mathematical Society, 30 December, 1930, and 28 February, 1931.

† Many results and references are to be found in Hobson's *Theory of functions of a real variable*, 2 (1926), 476–698.

‡ Hobson, *loc. cit.*, 698–719.

§ See Hobson, *loc. cit.*, 753–770, and, for example, several papers in *Studia Mathematica*, 1 (1929) and 2 (1930), and the references there given.

pertaining to summability of orthogonal series. In § 17 we give two important consequences of results of the paper.

2. *Definitions and notation.* Let A be a set, in space of d dimensions*, having a d-dimensional measure $m(A) > 0$. We take, for convenience, a single variable x to represent a point of A. Let a double sequence $\{\phi_{mn}(x)\}$ of real measurable functions be defined, normal and orthogonal over A. Thus

$$\int_A \phi_{mn}^2(x)\,dx = 1 \quad (m, n = 1, 2, 3, \ldots),$$

$$\int_A \phi_{mn}(x)\,\phi_{m'n'}(x)\,dx = 0 \text{ unless } m' = m \text{ and } n' = n,$$

the integrals being Lebesgue integrals. Let $\{a_{mn}\}$ be a double sequence of real constants. Then the double series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}\,\phi_{ij}(x) \tag{2.1}$$

(convergent or divergent) will be called a double orthogonal series.

For each point x of A, convergence (as defined by Pringsheim) of (2.1) is equivalent to existence of the double limit $\lim_{m,\,n\to\infty} s_{mn}(x)$, where

$$s_{mn}(x) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\,\phi_{ij}(x). \tag{2.2}$$

We shall, when convenient, write s_{mn} or $s(m, n)$ for $s_{mn}(x)$ and write ϕ_{ij} for $\phi_{ij}(x)$.

3. *A note on double series of non-negative terms.* The following lemma will be used in § 4.

LEMMA 3.1. *If* $b_{ij} \geqslant 0$ *and*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} b_{ij} < \infty, \tag{3.11}$$

then there exist real sequences $1 \leqslant p_1 \leqslant p_2 \leqslant \ldots$ *and* $1 \leqslant q_1 \leqslant q_2 \leqslant \ldots$, *each tending to infinity, such that*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} p_i q_j b_{ij} < \infty.$$

* Doubtless the case $d = 2$ is of most interest; however, all results of this paper are independent of d, and none of the demonstrations is complicated by supposing d to be an arbitrary integer.

It is known* that if Σu_α is a convergent simple series of non-negative constant terms, then there exists a monotone increasing sequence $0 \leqslant r_1 \leqslant r_2 \leqslant \ldots$ with $\lim r_\alpha = \infty$ such that $\Sigma r_\alpha u_\alpha$ converges. Obviously the convergence of $\Sigma r_\alpha u_\alpha$ will not be affected by replacing by 1 each r_α which is less than 1; hence we may assume that $1 \leqslant r_1 \leqslant r_2 \leqslant \ldots$. It involves only a change of notation to say that, if Σu_α is a convergent simple series of non-negative constant terms, then there exists a sequence $1 \leqslant r_1 \leqslant r_2 \leqslant \ldots$ with $\lim r_\alpha = \infty$ such that $\Sigma r_\alpha^2 u_\alpha$ converges.

With the aid of the preceding remarks, we can easily prove Lemma 3.1. It follows from (3.11) that the diagonal series†

$$b_{11}+b_{21}+b_{12}+b_{31}+b_{22}+b_{13}+b_{41}+b_{32}+b_{23}+b_{14}+\ldots$$

is convergent; hence there is a sequence $1 \leqslant r_1 \leqslant r_2 \leqslant \ldots$ such that the series

$$r_1^2 b_{11}+r_2^2 b_{21}+r_3^2 b_{12}+r_4^2 b_{31}+r_5^2 b_{22}+r_6^2 b_{13}+r_7^2 b_{41}+\ldots$$

is convergent. Each term of the last series is of the form $r_\alpha^2 b_{ij}$, where $\alpha \geqslant i$ and $\alpha \geqslant j$. Therefore the series

$$r_1 r_1 b_{11}+r_2 r_1 b_{21}+r_1 r_2 b_{12}+r_3 r_1 b_{31}+r_2 r_2 b_{22}+\ldots$$

and the corresponding double series

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} r_i r_j b_{ij}$$

converge. On setting $p_i = r_i$, $q_j = r_j$, we complete the proof of Lemma 3.1.

4. *Convergence of subsequences of s_{mn}.* Suppose that the real sequence $\{a_{mn}\}$ is such that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}^2 < \infty; \tag{4.01}$$

then, by Lemma 3.1, we can choose real sequences

$$1 \leqslant p_1 \leqslant p_2 \leqslant p_3 \leqslant \ldots \quad \text{and} \quad 1 \leqslant q_1 \leqslant q_2 \leqslant q_3 \leqslant \ldots,$$

each of which tends to infinity, such that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} p_i q_j a_{ij}^2 < \infty. \tag{4.02}$$

Let $P_1 < P_2 < P_3 < \ldots$ and $Q_1 < Q_2 < Q_3 < \ldots$ be increasing sequences of positive integers so chosen that, for each index α,

$$p_{P_\alpha} \geqslant \alpha, \quad q_{Q_\alpha} \geqslant \alpha. \tag{4.03}$$

* Bromwich, *An introduction to the theory of infinite series* (1908), 41.

† Bromwich, *loc. cit.*, 77.

We show that the subsequence $s(P_m, Q_n)$ of s_{mn} converges almost everywhere over A as m and n tend to infinity by proving the following theorem*.

THEOREM 4.1. *If the coefficients in a double orthogonal series satisfy* (4.01), *then, corresponding to each pair of sequences* $\{P_\alpha\}$ *and* $\{Q_\alpha\}$ *defined as above, there is a function* $f(x)$ *such that*

$$\lim_{m,\, n\to\infty} s(P_m, Q_n) = f(x) \tag{4.11}$$

essentially uniformly† *over* A‡.

Using (2.2), we find, for $m < r$, $n < r$,

$$s_{P_r Q_r} - s_{P_m Q_n} = \sum_{i=1}^{P_r} \sum_{j=Q_n+1}^{Q_r} a_{ij}\phi_{ij} + \sum_{i=P_m+1}^{P_r} \sum_{j=1}^{Q_r} a_{ij}\phi_{ij} - \sum_{i=P_m+1}^{P_r} \sum_{j=Q_n+1}^{Q_r} a_{ij}\phi_{ij},$$

and, on applying the inequality $(\alpha_1+\alpha_2-\alpha_3)^2 \leqslant 3(\alpha_1^2+\alpha_2^2+\alpha_3^2)$, we obtain

$$(s_{P_r Q_r} - s_{P_m Q_n})^2 \leqslant 3\Big(\sum_{i=1}^{P_r} \sum_{j=Q_n+1}^{Q_r} a_{ij}\phi_{ij}\Big)^2 + 3\Big(\sum_{i=P_m+1}^{P_r} \sum_{j=1}^{Q_r} a_{ij}\phi_{ij}\Big)^2 + 3\Big(\sum_{i=P_m+1}^{P_r} \sum_{j=Q_n+1}^{Q_r} a_{ij}\phi_{ij}\Big)^2, \tag{4.12}$$

the right member being a function of r, m, n, and x. Given integers R and r satisfying the relation $1 \leqslant R < r$, consider all pairs of values of m and n which satisfy the relations $R \leqslant m < r$, $R \leqslant n < r$. For each such pair of values of m and n, the three non-negative terms of the right member of (4.12) are included among those of $W = W(R, r, x)$, where

$$W = 3\sum_{n=R}^{r-1}\Big(\sum_{i=1}^{P_r} \sum_{j=Q_n+1}^{Q_r} a_{ij}\phi_{ij}\Big)^2 + 3\sum_{m=R}^{r-1}\Big(\sum_{i=P_m+1}^{P_r} \sum_{j=1}^{Q_r} a_{ij}\phi_{ij}\Big)^2 + 3\sum_{m=R}^{r-1}\sum_{n=R}^{r-1}\Big(\sum_{i=P_m+1}^{P_r} \sum_{j=Q_n+1}^{Q_r} a_{ij}\phi_{ij}\Big)^2 \tag{4.13}$$

is independent of m and n. Thus we have, for each point x of A,

$$(s_{P_r Q_r} - s_{P_m Q_n})^2 \leqslant W(R, r, x) \quad (R \leqslant m < r, \quad R \leqslant n < r). \tag{4.14}$$

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* This includes an extension to double series of a theorem on simple series given by Rademacher. See *Math. Annalen*, 87 (1922), 114.

† A sequence is said to converge essentially uniformly over A if it converges uniformly over a subset of A of measure as close as we please to $m(A)$. Hence essentially uniform convergence over a set implies convergence almost everywhere over the set. See Weyl, *Math. Annalen*, 67 (1909), 225.

‡ It should be noted that $\{P_\alpha\}$ and $\{Q_\alpha\}$ are independent of the functions $\{\phi_{mn}\}$, being dependent only upon $\{a_{mn}\}$. The sequences $\{p_\alpha\}$, $\{q_\alpha\}$, $\{P_\alpha\}$, and $\{Q_\alpha\}$ are of course not uniquely determined in terms of a given sequence $\{a_{mn}\}$; questions relating to uniqueness of $f(x)$ are relegated to § 6.

Employing the normal orthogonal properties of $\{\phi_{mn}\}$, we obtain from (4.13)

$$(4.15)\qquad \int_A W\,dx = 3\sum_{n=R}^{r-1}\sum_{i=1}^{P_r}\sum_{j=Q_n+1}^{Q_r} a_{ij}^2 + 3\sum_{m=R}^{r-1}\sum_{i=P_m+1}^{P_r}\sum_{j=1}^{Q_r} a_{ij}^2 + 3\sum_{m=R}^{r-1}\sum_{n=R}^{r-1}\sum_{i=P_m+1}^{P_r}\sum_{j=Q_n+1}^{Q_r} a_{ij}^2.$$

Let the three terms of the right member of (4.15) be denoted by $3W_1$, $3W_2$, and $3W_3$ respectively. We find that

$$W_1 = \sum_{i=1}^{P_r}\sum_{n=R}^{r-1}\sum_{u=n}^{r-1}\sum_{j=Q_u+1}^{Q_{u+1}} a_{ij}^2,$$

and, after changing the order of the summations with respect to n and u, we have

$$W_1 = \sum_{i=1}^{P_r}\sum_{u=R}^{r-1}\sum_{n=R}^{u}\sum_{j=Q_u+1}^{Q_{u+1}} a_{ij}^2 \leqslant \sum_{i=1}^{P_r}\sum_{u=R}^{r-1} u \sum_{j=Q_u+1}^{Q_{u+1}} a_{ij}^2.$$

When $j > Q_u$, $q_j \geqslant q_{Q_u} \geqslant u$. Hence

$$W_1 \leqslant \sum_{i=1}^{P_r}\sum_{u=R}^{r-1}\sum_{j=Q_u+1}^{Q_{u+1}} q_j a_{ij}^2 = \sum_{i=1}^{P_r}\sum_{j=Q_R+1}^{Q_r} q_j a_{ij}^2,$$

so that

$$(4.16)\qquad W_1 \leqslant \sum_{i=1}^{P_r}\sum_{j=Q_R+1}^{Q_r} p_i q_j a_{ij}^2.$$

The method by which (4.16) was obtained suffices to show that

$$(4.161)\qquad W_2 \leqslant \sum_{i=P_R+1}^{P_r}\sum_{j=1}^{Q_r} p_i q_j a_{ij}^2$$

and

$$(4.162)\qquad W_3 \leqslant \sum_{i=P_R+1}^{P_r}\sum_{j=Q_R+1}^{Q_r} p_i q_j a_{ij}^2.$$

It follows from (4.16), (4.161), and (4.162) that

$$W_1 + W_2 + W_3 \leqslant 3\left(\sum_{i=1}^{P_r}\sum_{j=1}^{Q_r} p_i q_j a_{ij}^2 - \sum_{i=1}^{P_R}\sum_{j=1}^{Q_R} p_i q_j a_{ij}^2\right),$$

and hence from (4.15) that

$$\int_A W(R, r, x)\,dx \leqslant 9\left(\sum_{i=1}^{P_r}\sum_{j=1}^{Q_r} p_i q_j a_{ij}^2 - \sum_{i=1}^{P_R}\sum_{j=1}^{Q_R} p_i q_j a_{ij}^2\right)$$

$$\leqslant 9\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} p_i q_j a_{ij}^2 - \sum_{i=1}^{P_R}\sum_{j=1}^{Q_R} p_i q_j a_{ij}^2\right).$$

Given $\delta > 0$ we can, by (4.02), choose $R = R(\delta)$ so great that the last member of the preceding inequality is less than δ^3. Then, for each $r > R$, $\int_A W(R, r, x)\,dx < \delta^3$. Since $W \geqslant 0$, the set of points for which $W \geqslant \delta^2$ has measure less than δ. Hence there is a set B_r, of measure greater than or equal to $m(A)-\delta$, such that

$$W(R, r, x) < \delta^2, \quad x \text{ in } B_r.$$

Using this relation and (4.14) we see that, when x is in B_r,

$$|s_{P_r Q_r} - s_{P_m Q_n}| < \delta \quad (R \leqslant m < r, \ \ R \leqslant n < r).$$

Hence, when x is in B_r,

$$|s_{P_{m'} Q_{n'}} - s_{P_m Q_n}| < 2\delta \quad (R \leqslant m', \ n', \ m, \ n < r). \tag{4.17}$$

Let C_r be the set of *all* points x for which (4.17) holds. Then $C_r \supset B_r$ so that $m(C_r) \geqslant m(A)-\delta$. Consider the sets C_{R+1}, C_{R+2}, ... Evidently $C_{R+1} \supset C_{R+2} \supset C_{R+3}$, ...; hence the set D of points common to C_{R+1}, C_{R+2}, ... has measure greater than or equal to $m(A)-\delta$. For x in D we have

$$|s_{P_{m'} Q_{n'}} - s_{P_m Q_n}| < 2\delta \quad (R \leqslant m', \ n', \ m, \ n).$$

Given an arbitrarily small positive number ϵ, let $\delta_1+\delta_2+\delta_3+\ldots$ be a series of positive numbers which converges to ϵ. For each $\alpha > 0$ there is an index R_α and a set D_α of measure greater than or equal to $m(A)-\delta_\alpha$ such that, for x in D_α,

$$|s_{P_{m'} Q_{n'}} - s_{P_m Q_n}| < 2\delta_\alpha \quad (R_\alpha \leqslant m', \ n', \ m, \ n). \tag{4.18}$$

The sets D_α have in common a set E of measure greater than or equal to $m(A)-\epsilon$. When x is in E, (4.18) holds for each index α. Therefore the sequence $\{s(P_m, Q_n)\}$ converges uniformly over the set E whose measure is as close to $m(A)$ as we please. On defining $f(x)$ to be equal to $\lim_{m, n \to \infty} s(P_m, Q_n)$ for each point x for which this limit exists, and $f(x) = 0$ for all other points x, we complete the proof of Theorem 4.1.

A modification of the proof of the preceding theorem suffices to prove

THEOREM 4.2. *If the coefficients in a double orthogonal series satisfy* (4.01), *and* $\{P_\alpha\}$ *and* $\{Q_\alpha\}$ *are defined as in* §4, *then* $\lim_{n \to \infty} s(m, Q_n)$ *exists essentially uniformly over* A *for each* m, *and* $\lim_{m \to \infty} s(P_m, n)$ *exists essentially uniformly over* A *for each* n.

It follows from Theorems 4.1 and 4.2 that there is a set of measure $m(A)$ over which the following limits

$$\lim_{m\to\infty}\lim_{n\to\infty} s(P_m, Q_n), \quad \lim_{n\to\infty}\lim_{m\to\infty} s(P_m, Q_n), \quad \lim_{m,n\to\infty} s(P_m, Q_n)$$

exist and are equal*.

Two corollaries of Theorem 4.1 are of especial interest. If the real sequence $\{a_{mn}\}$ is such that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \log(i+1)\log(j+1)\,a_{ij}^2 < \infty, \tag{4.21}$$

then† we may take $p_i = \log(i+1)$, $q_j = \log(j+1)$, $P_i = 2^i$, and $Q_j = 2^j$, and Theorem 4.1 gives

THEOREM 4.3. *If the coefficients in a double orthogonal series satisfy* (4.21), *then there exists a function* $f(x)$ *such that*

$$\lim_{m,n\to\infty} s(2^m, 2^n) = f(x)$$

essentially uniformly over A‡.

If the coefficients $\{a_{mn}\}$ are such that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \log\log(i+3)\log\log(j+3)\,a_{ij}^2 < \infty, \tag{4.31}$$

then we may take $P_i = 2^{2^i}$ and $Q_j = 2^{2^j}$ and thus obtain

THEOREM 4.4. *If the coefficients in a double orthogonal series satisfy* (4.31), *then there exists a function* $f(x)$ *such that*

$$\lim_{m,n\to\infty} s(2^{2^m}, 2^{2^n}) = f(x)$$

essentially uniformly over A§.

5. *Relations between* $f(x)$ *and the coefficients* a_{mn}.

Theorem 4.1 and an application to double series of classical methods enable us to establish the following theorem:

* Bromwich, *loc. cit.*, 75.

† The logarithms are assumed to have the base 2.

‡ This is a generalization to double series of a result given for simple series by Plancherel, *Comptes rendus*, 157 (1913), 539, the result being of fundamental importance in the theory of convergence of orthogonal series.

§ This is a generalization of a result used by Kaczmarz, *Math. Zeitschrift*, 26 (1927), 99, in a paper on Cesàro summability of simple orthogonal series.

‖ See Weyl, *Math. Annalen*, 67 (1909), 227.

THEOREM 5.1. *If the coefficients in a double orthogonal series satisfy* (4.01), *and* $f(x)$ *is defined as in Theorem* 4.1, *then* $f(x)$ *is measurable and of summable square, and is such that*

$$\int_A \phi_{mn}(x) f(x)\, dx = a_{mn} \tag{5.11}$$

and

$$\int_A f^2(x)\, dx = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}^2 \tag{5.12}$$

and

$$\lim_{m,n\to\infty} \int_A |f(x) - s_{mn}(x)|^2 dx = 0. \tag{5.13}$$

Thus each double orthogonal series of which the coefficients satisfy (4.01) is the orthogonal development of a measurable function of summable square to which the series converges in the mean. It is easy to show that (4.01) is also necessary in order that a series may be the orthogonal development of such a function, and necessary in order that a series may converge in the mean.

6. *Uniqueness of* $f(x)$. As was pointed out in §4, a sequence $\{a_{mn}\}$ satisfying (4.01) does not uniquely determine a set of sequences $\{p_\alpha\}$, $\{q_\alpha\}$, $\{P_\alpha\}$, and $\{Q_\alpha\}$. If $f(x)$ and $g(x)$ are two functions determined as in Theorem 4.1 by two such sets of sequences, then $f(x)$ and $g(x)$ are almost everywhere equal, since it follows from Theorem 5.1 that $\Sigma\Sigma a_{ij}\phi_{ij}$ converges in the mean to both $f(x)$ and $g(x)$.

7. *Further properties of series for which* (4.01) *holds.* As a corollary of Theorem 4.1 we obtain

THEOREM 7.1. *If the coefficients in a double orthogonal series satisfy* (4.01), *and* $f(x)$ *is a function to which the series converges in the mean, then*

$$\lim_{m,n\to\infty} s(t_m, u_n) = f(x) \tag{7.11}$$

essentially uniformly, provided only that the increasing sequences $\{t_\alpha\}$ *and* $\{u_\alpha\}$ *of integers tend to infinity sufficiently rapidly.*

In fact (7.11) holds if, for all sufficiently great α, $t_\alpha \geqq P_\alpha$ and $u_\alpha \geqq Q_\alpha$ where $\{P_\alpha\}$ and $\{Q_\alpha\}$ are a specified pair of sequences defined as in §4.

Again, it follows from Theorem 4.1 that the series must converge to $f(x)$ almost everywhere in the set of values of x for which it converges.

We have seen that (4.01) is sufficient to ensure essentially uniform convergence of certain double subsequences of the double sequence of

partial sums of a series. However, the complete double sequence of partial sums of a series for which (4.01) holds may be everywhere non-convergent. We proceed to show that a condition slightly stronger than (4.01) is sufficient to ensure convergence.

8. *Convergence of s_{mn}.*

THEOREM 8.1. *If the coefficients in a double orthogonal series satisfy the condition*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}[\log(i+1)]^2[\log(j+1)]^2 a_{ij}^2 < \infty, \tag{8.11}$$

the series converges essentially uniformly over A.*

After having proved Theorem 8.1, we shall find it easy to establish the following useful theorem:

THEOREM 8.2. *If the coefficients in a double orthogonal series satisfy* (8.11), *there is a subset of A of measure $m(A)$ over which each row (column) is convergent and the series of values of the rows (columns) converges to the value of the double series.*

We proceed with the proof of Theorem 8.1, which occupies §§ 9–14.

9. Since (8.11) implies (4.21), it follows from Theorem 4.3 there must be a function $f(x)$ and, corresponding to a given $\delta > 0$, a subset B of A of measure $m(B) \geqq m(A) - \delta$ over which $s(2^p, 2^q)$ converges uniformly to $f(x)$ as $p \to \infty$, $q \to \infty$. Let $R = R(\delta)$ be an integer so great that, for x in B,

$$|s(2^p, 2^q) - f| < \delta \quad (p \geqq R,\ q \geqq R). \tag{9.13}$$

We establish Theorem 8.1 by showing that, under the hypothesis (8.11), the double series (2.1) converges essentially uniformly to f.

10. With R defined as in § 9, choose an integer $r > R$, and consider a pair of integers m and n satisfying the relations $2^R \leqq m < 2^r$, $2^R \leqq n < 2^r$. Let p and q be functions of m and n defined by the relations $2^p \leqq m < 2^{p+1}$, $2^q \leqq n < 2^{q+1}$; then $R \leqq p < r$ and $R \leqq q < r$. To simplify typography, we define

$$t(\alpha, \beta;\ \gamma, \delta) = \sum_{i=\alpha+1}^{\beta}\sum_{j=\gamma+1}^{\delta} a_{ij}\phi_{ij}.$$

* This theorem is a generalization to double orthogonal series of a theorem on simple orthogonal series given by Rademacher, *Math. Annalen*, 87 (1922), 112–138, and by Menchoff, *Fundamenta Math.*, 4 (1923), 82–105.

Then we may write the identity*

$$(10.1)\quad s_{mn}-s(2^p, 2^q) = t(0, 2^r;\ 2^q, n)-t(2^p, 2^r;\ 2^q, n) + t(2^p, m;\ 0, 2^r)-t(2^p, m;\ 2^q, 2^r) + t(2^p, m;\ 2^q, n),$$

and, on applying the inequality

$$(10.11)\qquad (\alpha_1+\alpha_2+\ldots+\alpha_h)^2 \leqslant h(\alpha_1^2+\alpha_2^2+\ldots+\alpha_h^2),$$

we obtain the relation

$$(10.2)\quad [s_{mn}-s(2^p, 2^q)]^2 \leqslant 5t^2(0, 2^r;\ 2^q, n)+5t^2(2^p, 2^r;\ 2^q, n) + 5t^2(2^p, m;\ 0, 2^r)+5t^2(2^p, m;\ 2^q, 2^r) + 5t^2(2^p, m;\ 2^q, n).$$

Let the terms of the right member be denoted by $5\psi_1$, $5\psi_2$, $5\psi_3$, $5\psi_4$, and $5\psi_5$ respectively.

11. The non-negative functions ψ_λ are functions of some or all of the indices m, n, p, and q as well as of r and x. We proceed to find functions Ω_1, Ω_2, ..., Ω_5 which are, for $2^R \leqslant m < 2^r$, $2^R \leqslant n < 2^r$, independent of m, n, p, and q, being functions of R, r, and x only, such that

$$(11.1)\qquad \psi_\lambda \leqslant \Omega_\lambda(R, r, x) \quad (2^R \leqslant m < 2^r,\ \ 2^R \leqslant n < 2^r)$$

for $\lambda = 1, 2, 3, 4, 5$.

Given integers m and n, restricted as above, let n be written in the dyadic scale; thus

$$(11.2)\qquad n = 2^q+\nu_1 2^{q-1}+\nu_2 2^{q-2}+\ldots+\nu_{q-1}2+\nu_q,$$

where ν_i is 0 or 1 for each i and $R \leqslant q < r$. From the definition of ψ_1 we find, on defining $\nu_0 = 0$,

$$\psi_1 = \Big[\sum_{k=1}^{q} t(0, 2^r;\ 2^q+\nu_1 2^{q-1}+\ldots+\nu_{k-1}2^{q-k+1},\ 2^q+\nu_1 2^{q-1}+\ldots+\nu_k 2^{q-k})\Big]^2,$$

and on using the inequality (10.11) we obtain

$$(11.3)\quad \psi_1 \leqslant q\sum_{k=1}^{q} t^2(0, 2^r;\ 2^q+\nu_1 2^{q-1} + \ldots+\nu_{k-1}2^{q-k+1},\ 2^q+\nu_1 2^{q-1}+\ldots+\nu_k 2^{q-k}).$$

Each non-vanishing term of the right member of (11.3) is of the form

$$qt^2(0, 2^r;\ 2^q+\delta 2^\gamma,\ 2^q+\delta 2^\gamma+2^{\gamma-1}),$$

* In case $m = 2^p$ or $n = 2^q$, certain sums of the right member vanish.

where γ has one of the values 1, 2, ..., $q-1$, and δ has one of the values 0, 1, 2, ..., $2^{q-\gamma}-1$; hence, since all of the terms considered are non-negative, it follows from (11.3) that

$$(11.31) \qquad \psi_1 \leqslant q \sum_{\gamma=1}^{q} \sum_{\delta=0}^{2^{q-\gamma}-1} t^2(0,\ 2^r;\ 2^q+\delta 2^\gamma,\ 2^q+\delta 2^\gamma+2^{\gamma-1}).$$

Finally, since q must have one of the values R, $R+1$, ..., $r-1$, we obtain

$$(11.32) \qquad \psi_1 \leqslant \sum_{q=R}^{r-1} q \sum_{\gamma=1}^{q} \sum_{\delta=0}^{2^{q-\gamma}-1} t^2(0,\ 2^r;\ 2^q+\delta 2^\gamma,\ 2^q+\delta 2^\gamma+2^{\gamma-1}).$$

The right member is a function of R, r, and x alone; let it be denoted by $\Omega_1(R, r, x)$. With this definition, (11.1) holds for $\lambda = 1$.

A slight modification of the derivation of (11.32) suffices to show that

$$\psi_2 \leqslant \sum_{q=R}^{r-1} q \sum_{\gamma=1}^{q} \sum_{\delta=0}^{2^{q-\gamma}-1} t^2(2^p,\ 2^r;\ 2^q+\delta 2^\gamma,\ 2^q+\delta 2^\gamma+2^{\gamma-1}).$$

Since p must have one of the values R, $R+1$, ..., $r-1$, we find that

$$(11.4) \qquad \psi_2 \leqslant \sum_{q=R}^{r-1} q \sum_{\gamma=1}^{q} \sum_{\delta=0}^{2^{q-\gamma}-1} \sum_{p=R}^{r-1} t^2(2^p,\ 2^r;\ 2^q+\delta 2_\gamma,\ 2^q+\delta 2^\gamma+2^{\gamma-1}).$$

On defining $\Omega_2(R, r, x)$ to be the right member, we have (11.1) for $\lambda = 2$.

On writing m in the dyadic scale,

$$m = 2^p+\mu_1 2^{p-1}+\mu_2 2^{p-2}+\ldots+\mu_{p-1} 2+\mu_p,$$

and, employing the method by which (11.32) was obtained, we find that

$$(11.5) \qquad \psi_3 \leqslant \sum_{p=R}^{r-1} p \sum_{\alpha=1}^{p} \sum_{\beta=0}^{2^{p-\alpha}-1} t^2(2^p+\beta 2^\alpha,\ 2^p+\beta 2^\alpha+2^{\alpha-1};\ 0,\ 2^r).$$

Letting $\Omega_3(R, r, x)$ represent the right member, we have (11.1) for $\lambda = 3$.

Again, we find that

$$(11.6) \qquad \psi_4 \leqslant \sum_{p=R}^{r-1} p \sum_{\alpha=1}^{p} \sum_{\beta=0}^{2^{p-\alpha}-1} \sum_{q=R}^{r-1} t^2(2^p+\beta 2^\alpha,\ 2^p+\beta 2^\alpha+2^{\alpha-1};\ 2^q,\ 2^r),$$

and, on letting $\Omega_4(R, r, x)$ represent the right member, we have (11.1) for $\lambda = 4$.

A combination of the methods by which (11.32) and (11.5) were obtained enables us to show that

$$(11.7) \qquad \psi_5 \leqslant \sum_{p=R}^{r-1} \sum_{q=R}^{r-1} pq \sum_{\alpha=1}^{p} \sum_{\beta=0}^{2^{p-\alpha}-1} \sum_{\gamma=1}^{q} \sum_{\delta=0}^{2^{q-\gamma}-1} [t^2(2^p+\beta 2^\alpha,\ 2^p+\beta 2^\alpha+2^{\alpha-1};\ 2^q+\delta 2^\gamma,\ 2^q+\delta 2^\gamma+2^{\gamma-1})].$$

Letting $\Omega_5(R, r, x)$ be the right member, we have (11.1) for $\lambda = 5$.

12. Let $\Omega(R, r, x)$ be defined by

$$\Omega(R, r, x) = 5(\Omega_1+\Omega_2+\Omega_3+\Omega_4+\Omega_5). \tag{12.1}$$

It follows from (10.2), the definitions of $\psi_1, \psi_2, \ldots, \psi_5$, and the relations (11.1) and (12.1), that

$$[s_{mn}-s(2^p, 2^q)]^2 \leqslant \Omega(R, r, x) \tag{12.2}$$

for every pair of integers m and n satisfying the relations $2^R \leqslant m < 2^r$, $2^R \leqslant n < 2^r$, p and q being functions of m and n as defined in § 10.

13. In this section, we shall find an upper bound for the integral over A of the non-negative function $\Omega(R, r, x)$. Using the normal orthogonal properties of $\{\phi_{mn}\}$, we find that

$$\int_A \Omega_1 dx = \sum_{q=R}^{r-1} q \sum_{\gamma=1}^{q} \sum_{\delta=0}^{2^{q-\gamma}-1} \sum_{i=1}^{2^r} \sum_{j=2^q+\delta.2^\gamma+1}^{2^q+\delta.2^\gamma+2^{\gamma-1}} a_{ij}^2,$$

and, after effecting the summation with respect to δ, we have

$$\int_A \Omega_1 dx \leqslant \sum_{q=R}^{r-1} q \sum_{\gamma=1}^{q} \sum_{i=1}^{2^r} \sum_{j=2^{q-1}+1}^{2^q+1} a_{ij}^2 \leqslant \sum_{q=R}^{r-1} \sum_{i=1}^{2^r} \sum_{j=2^{q-1}+1}^{2^q+1} q^2 a_{ij}^2. \tag{13.01}$$

When $j > 2^q$, we have $q^2 < (\log j)^2$; hence we may replace q^2 by $(\log j)^2$ and effect the summation with respect to q to obtain

$$\int_A \Omega_1 dx \leqslant \sum_{i=1}^{2^r} \sum_{j=2^R+1}^{2^r} (\log j)^2 a_{ij}^2. \tag{13.1}$$

After integrating Ω_2 over A, and effecting summations with respect to δ, γ, and q as in the derivations of (13.1), we find that

$$\int_A \Omega_2 dx \leqslant \sum_{p=R}^{r-1} \sum_{i=2^p+1}^{2^r} \sum_{j=2^R+1}^{2^r} (\log j)^2 a_{ij}^2.$$

Using $$\sum_{p=R}^{r-1} \sum_{i=2^p+1}^{2^r} = \sum_{p=R}^{r-1} \sum_{h=p}^{r-1} \sum_{i=2^h+1}^{2^{h+1}} = \sum_{h=R}^{r-1} \sum_{p=R}^{h} \sum_{i=2^h+1}^{2^{h+1}}$$

and effecting the summation with respect to p, we obtain

$$\int_A \Omega_2 dx \leqslant \sum_{h=R}^{r-1} \sum_{i=2^h+1}^{2^{h+1}} \sum_{j=2^R+1} h(\log j)^2 a_{ij}^2.$$

Replacing h by $\log i$ and summing with respect to h, we have

$$\int_A \Omega_2 dx \leqslant \sum_{i=2^R+1}^{2^r} \sum_{j=2^R+1}^{2^r} (\log i)(\log j)^2 a_{ij}^2. \tag{13.2}$$

The method by which (13.1) was obtained enables us to show that

$$\int_A \Omega_3 dx \leqslant \sum_{i=2^R+1}^{2^r} \sum_{j=1}^{2^r} (\log i)^2 a_{ij}^2, \tag{13.3}$$

and, using the method by which (13.2) was obtained, we find that

$$\int_A \Omega_4\,dx \leqslant \sum_{i=2^R+1}^{2^r} \sum_{j=2^R+1}^{2^r} (\log i)^2 (\log j)\, a_{ij}^2. \tag{13.4}$$

Finally, a combination of the derivations of (13.1) and (13.3) will produce

$$\int_A \Omega_5\,dx \leqslant \sum_{i=2^R+1}^{2^r} \sum_{j=2^R+1}^{2^r} (\log i)^2 (\log j)^2\, a_{ij}^2. \tag{13.5}$$

It follows from (13.1), (13.2), ..., (13.5) respectively that, for $\lambda = 1, 2, \ldots, 5$,

$$\begin{aligned}\int_A \Omega_\lambda\,dx &\leqslant \left(\sum_{i=1}^{2^r}\sum_{j=1}^{2^r} - \sum_{i=1}^{2^R}\sum_{j=1}^{2^R}\right)[\log(i+1)]^2\,[\log(j+1)]^2\,a_{ij}^2 \\ &\leqslant \left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} - \sum_{i=1}^{2^R}\sum_{j=1}^{2^R}\right)[\log(i+1)]^2\,[\log(j+1)]^2\,a_{ij}^2.\end{aligned}$$

Using this inequality and (12.2), we find that

$$\int_A \Omega\,dx \leqslant 25\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} - \sum_{i=1}^{2^R}\sum_{j=1}^{2^R}\right)[\log(i+1)]^2\,[\log(j+1)]^2\,a_{ij}^2. \tag{13.6}$$

14. With the $\delta > 0$ chosen in § 9, we may, if necessary, by (8.11), increase the integer R to a value sufficiently great to make the right member of (13.6) less than δ^3; such an increase in R will not disturb (10.13). For this integer R we have

$$\int_A \Omega(R, r, x)\,dx < \delta^3 \quad (r > R).$$

Since $\Omega(R, r, x) \geqslant 0$, the set of all points x for which $\Omega(R, r, x) \geqslant \delta^2$ must therefore have measure less than δ; hence, corresponding to each integer $r > R$, there is a set C_r of measure greater than or equal to $m(A)-\delta$ such that

$$\Omega(R, r, x) < \delta^2, \quad x \text{ in } C_r.$$

Using this relation and (12.2), we see that, when x is in C_r,

$$|s_{mn} - s(2^p, 2^q)| < \delta \tag{14.1}$$

for every pair of integers m and n satisfying the relations $2^R \leqslant m < 2^r$, $2^R \leqslant n < 2^r$, p and q being functions of m and n as defined in § 10.

Let D_r be the set of points common to B (defined in § 9) and C_r. The set D_r has measure greater than or equal to $m(A)-2\delta$. When x is in D_r, both (9.13) and (14.1) hold; hence, for x in D_r,

$$|s_{mn} - f(x)| < 2\delta \quad (2^R \leqslant m < 2^r,\ 2^R \leqslant n < 2^r). \tag{14.2}$$

Theorem 8.1 follows from the relation (14.2) by an argument very similar to that by which Theorem 4.1 follows from the relation (4.17).

15. We turn now to the proof of Theorem 8.2. Let (2.1) be a series for which (8.11) holds. For a fixed α, we see that the elements of the α-th row of (2.1) comprise the simple series

$$\sum_{j=1}^{\infty} a_{\alpha j}\phi_{\alpha j}. \tag{15.01}$$

This is a simple orthogonal series, and, since (8.11) implies the condition

$$\sum_{j=1}^{\infty} (\log j)^2 a_{\alpha j}^2 < \infty,$$

it follows from the theorem of Rademacher and Menchoff* that (15.01) converges almost everywhere in A. An analogous argument shows that each column of (2.1) converges almost everywhere in A, and it follows from Theorem 8.1 that (2.1) itself converges almost everywhere in A.

The set H of all points x common to a countable set of subsets of A, the measure of each of which is $m(A)$, has measure $m(H)$ equal to $m(A)$. Hence there is a subset H of A of measure $m(H) = m(A)$ over which each row of (2.1) converges, each column of (2.1) converges, and (2.1) itself converges. Over the set H, the series of the values of the rows (columns) of (2.1) converges to the value of (2.1)† and Theorem 8.2 is proved.

An important theorem, which is equivalent to Theorem 8.2, is the following:

THEOREM 15.1. *If the coefficients in a double orthogonal series satisfy* (8.11), *then there is a subset of A of measure $m(A)$ over which the limits*

$$\lim_{m\to\infty}\left(\lim_{n\to\infty} s_{mn}\right), \quad \lim_{n\to\infty}\left(\lim_{m\to\infty} s_{mn}\right), \quad \lim_{m,n\to\infty} s_{mn}$$

exist and are equal.

16. We wish to point out that the method of proof of Theorem 8.1 will furnish a proof of the following theorem, which is a generalization to double series of a theorem of fundamental importance in the theory of summability of simple orthogonal series‡:

* See the papers cited in a foot-note to § 8.

† Hobson, *The theory of functions of a real variable*, 2 (1926), 51.

‡ See Kaczmarz, *Math. Annalen*, 96 (1926), 151, and *Math. Zeitschrift*, 26 (1927), 101; Borgen, *Math. Annalen*, 98 (1927), 125; Zygmund, *Bulletin de l'Academie Polonaise* (1927), 295; and Chen, *Tôhoku Math. J.*, 29 (1928), 407, and 30 (1928), 472.

THEOREM 16.1. *If the coefficients in a double orthogonal series satisfy the condition*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}[\log\log(i+3)]^2[\log\log(j+3)]^2 a_{ij}^2 < \infty, \tag{16.11}$$

then there is a function $f(x)$ such that

$$\lim_{m,n\to\infty} s(2^m, 2^n) = f(x)$$

essentially uniformly over A.

The proof of Theorem 16.1 depends upon Theorem 4.4. Theorem 4.4 plays in the proof of Theorem 16.1 the same rôle that Theorem 4.3 plays in the proof of Theorem 8.1.

In the proof of Theorem 16.1, the analogue of (9.13) is

$$|s(2^{2^{p'}}, 2^{2^q}) - f| < \delta \quad (p \geqq R,\ q \geqq R);$$

the analogue of (10.1) is

$$\begin{aligned} s(2^m, 2^n) - s(2^{2^{p'}}, 2^{2^q}) = {} & t(0, 2^{2^r};\ 2^{2^s}, 2^n) - t(2^{2^{p'}}, 2^{2^r};\ 2^{2^t}, 2^n) \\ & + t(2^{2^{p'}}, 2^m;\ 0, 2^{2^q}) - t(2^{2^{p'}}, 2^m;\ 2^{2^t}, 2^{2^q}) \\ & + t(2^{2^{p'}}, 2^m;\ 2^{2^s}, 2^n), \end{aligned}$$

where p and q are functions of m and n defined as in § 10; and the analogue of (11.2) is (11.2) itself. When the analogue of (13.01) is reached, the lower limit of the summation with respect to j will be $2^{2^r}+1$; hence $(\log\log j)^2$ appears, instead of $(\log j)^2$, in the analogue of (13.1). To save space, we omit the details of the proof of Theorem 16.1.

17. *Conclusion.* Results of this paper may be combined to give numerous theorems involving double orthogonal series. For example, we may obtain from Theorem 8.1, § 5 and § 7, the following important theorems:

THEOREM 17.1. *If the coefficients in a double orthogonal series satisfy* (8.11), *and $f(x)$ is a function to which the double series converges in the mean, then the double series converges essentially uniformly over A to $f(x)$.*

THEOREM 17.2. *If the coefficients in the orthogonal development of a function $f(x)$ in terms of a complete sequence of measurable normal orthogonal functions over A satisfy* (8.11), *then the orthogonal development of $f(x)$ converges essentially uniformly over A to $f(x)$.*

SPHERICAL HARMONICS HAVING POLYHEDRAL SYMMETRY

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The purpose of this paper is to enumerate those classes of harmonic functions of integral order which have the same planes of symmetry as one of the five regular solids. The problem is readily solved by combining two celebrated methods: the first of these is the specification of a spherical harmonic by differentiation of the reciprocal radius vector $1/r$, invented by Clerk Maxwell and completed by Sylvester; the other is the elementary theory of the groups of rotations and reflections of the regular solids, so beautifully set forth in Klein's famous lectures on the icosahedron. Since the solution of the problem required no deep knowledge of algebra or modular theory, I failed to realize how familiar some of the expressions I was obtaining must seem to the expert in those subjects. I became aware of this on reading the lucid account of the quintic equation in Perron's admirable *Algebra*. But my especial thanks are due to the Society's Referee and to the Senior Secretary for their friendly and illuminating suggestions. In revising the paper, a good deal of superfluous detail has been omitted, while a new § 12 has been inserted at the suggestion of the Referee. A number of minor corrections have been made, and I have attempted to indicate more clearly the parallelism with the classical theory of the polyhedral functions. A list of references will be found at the end of the paper.

1. The difficult electrostatic problems connected with polyhedral conductors have not yet been solved: but they suggest quite an interesting elementary problem, which seems to have been overlooked. Consider any distribution of charges lying within a sphere of finite radius, and symmetrically distributed with respect to each plane of symmetry of a regular polyhedron. It is required to specify those spherical harmonics

which may appear in the expansion of the potential at exterior points. Let this expansion be

$$V \equiv \sum_{n=0}^{\infty} V_n \equiv \sum_{n=0}^{\infty} S_n(\theta, \phi)/r^{n+1} \equiv \sum_{n=0}^{\infty} H_n(x, y, z)/r^{2n+1}, \tag{1}$$

where V_n is a solid harmonic, $S_n(\theta, \phi)$ is a spherical surface harmonic, and $H_n(x, y, z)$ is a homogeneous harmonic polynomial of degree n. We begin by proving that the terms of degree n must satisfy the conditions of symmetry. For, if possible, let V_m be the first group of terms which is not a symmetrical solid harmonic. From the identity

$$r^{m+1}\left(V - \sum_{n=0}^{m-1} V_n\right) \equiv S_m(\theta, \phi) + O\left(\frac{1}{r}\right), \tag{2}$$

whose left-hand member is symmetrical, we see by making r large that $S_m(\theta, \phi)$, and so V_m, must also be symmetrical. Thus each term of the expansion (1) must satisfy the conditions of symmetry, and the problem is reduced to specifying homogeneous solid harmonics of integral order which have the property of symmetry.

Now it is well known that *any* homogeneous polynomial of order n in the Cartesian coordinates (x, y, z) can be expanded in terms of *harmonic* homogeneous polynomials, together with the square of the radius vector. According to a theorem of Gauss, supplemented by a formula due to Dougall, the form of the expansion is

$$F_n(x, y, z) \equiv H_n(x, y, z) + r^2 H_{n-2}(x, y, z) + r^4 H_{n-4}(x, y, z) + \ldots, \tag{3}$$

and the functions H_κ can be determined by operating repeatedly with Laplace's operator $\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$.

We shall find that the number of available coefficients which remain indeterminate in our expansion, among the terms whose order does not exceed n, is only a small fraction of the full number $(n+1)^2$ of coefficients of the corresponding terms of the general Laplace series. For large values of n, the fraction tends to a fixed finite limit. As an illustration, if we take a configuration having the symmetry of the cube and consider terms of order not exceeding 12, we find that there are only seven constants at our disposal, instead of 169.

It is doubtful whether our method admits of any practical applications, but we might suggest an extension to the more complicated groups of reflections and rotations considered in crystallography; and the method might also be useful in discussing the deviation from spherical symmetry of the field of an atomic model, whose electrons and protons are supposed to be symmetrically arranged.

2. If (ξ, η, ζ) denote respectively the differential operators $(\partial/\partial x, \partial/\partial y, \partial/\partial z)$. Maxwell showed that any homogeneous harmonic polynomial of order n can be generated by a homogeneous operator with constant coefficients

$$F_n(\xi, \eta, \zeta)(1/r) \equiv H_n(x, y, z)/r^{2n+1}. \tag{4}$$

Sylvester then proved that, if all the coefficients are real, the operator on the left may be replaced by a product of n real linear operators; this resolution is unique, except for the order of the factors and a trivial constant multiplier in each factor. Consequently we may write

$$F_n(\xi, \eta, \zeta)(1/r) = \left\{ \prod_{s=1}^{n} (a_s\xi + b_s\eta + c_s\zeta) \right\} (1/r). \tag{5}$$

The two operators in (5) need not be formally identical, but their difference must be divisible by Laplace's operator $\nabla^2 = \xi^2+\eta^2+\zeta^2$, which annihilates $1/r$ and all other spherical harmonics.

Sylvester's theorem enables us to write any real harmonic of integral order in Maxwell's form.

$$H_n(x, y, z)/r^{2n+1} = C(\partial^n/\partial h_1 \partial h_2 \ldots \partial h_n)(1/r), \tag{6}$$

where $h_1, h_2, \ldots, h_n$ are Maxwell's "axes" for that harmonic, and, of course, $\partial/\partial h_s \equiv l_s\xi + m_s\eta + n_s\zeta$, where (l_s, m_s, n_s) are the real direction cosines of the axis h_s.

We next observe that, in any linear orthogonal transformation of coordinates, which does not alter the position of the origin or the scale of the figure (and which corresponds to any combination of reflections or rotations about the origin), the operators (ξ, η, ζ) are cogredient with the variables (x, y, z). Thus a harmonic polynomial, which is invariant for such an orthogonal transformation, corresponds to an invariant operator. But the great advantage of using the operator, instead of the polynomial, is that we may simplify it by using Laplace's relation, $\nabla^2 = \xi^2+\eta^2+\zeta^2 = 0$ so long as we operate only on $1/r$ or on a spherical harmonic, as will be the case throughout this paper.

3. The notion of Maxwell's "axes" will be used in a modified form, as we shall not attempt to identify the "axes" of each harmonic and to exhibit them in the form (6). We shall instead introduce a family of "axes" $(h_1, h_2, \ldots, h_n)$, which are in the aggregate invariant for any reflection or rotation which leaves the fundamental polyhedron invariant, and which are merely interchanged among themselves. We then

introduce operators which have an intrinsic geometric relation with the polyhedron, independent of the choice of our Cartesian frame of reference; an operator which is a symmetrical polynomial in $(\partial/\partial h_s)$ is clearly of this type, and generates a harmonic function having polyhedral symmetry.

Now it can be proved that, for each of the classical regular solids, a system of "axes" can be chosen, such that every invariant operator is an algebraic symmetric function of the linear operators belonging to those "axes". The following illustration shows that this fact is not at all self-evident. Consider the six diameters containing the twelve corners of a regular icosahedron; these may be sorted into twenty triads, half of which correspond to a pair of opposite faces of the icosahedron, and half do not. Any reflection or rotation which leaves the icosahedron invariant cannot possibly interchange two triads of opposite types. But if six letters correspond to the six diameters, there are many permutations of the letters which do interchange triads of opposite types and which do not correspond to any operation of the icosahedral group.

Let Φ denote either a number or an operator which must be commutative with (ξ, η, ζ); and let us form the expression

$$F(\Phi) \equiv \prod_{s=1}^{N} (\Phi - l_s\xi - m_s\eta - n_s\zeta). \tag{7}$$

The coefficients of the expanded form of $F(\Phi)$ are symmetric functions of the operators $\partial/\partial h_s$, while $F(\Phi)$ will vanish identically on substituting for Φ any of the operators $\partial/\partial h_s$. The analogy with an algebraic equation must not, however, be pressed too far, since some operators can be factorized, *e.g.* $\Phi^2 - \xi^2 \equiv (\Phi - \xi)(\Phi + \xi)$, while others cannot, *e.g.* $\Phi^2 - \eta\zeta$. Expressions such as (7) can be multiplied and added like algebraic polynomials, and it will be interesting and convenient to construct the expressions corresponding to the principal systems of "axes" of each of the regular solids. It will appear that the coefficients can be expressed in terms of two fundamental invariants in each case. A further proof is necessary that every geometrically invariant operator must be an algebraic symmetric function in the selected system of operators $\partial/\partial h_s$.

4. The tetrahedron, octahedron, and cube may be treated together as a single configuration. The origin of coordinates is placed at the centre of the cube, with the planes of reference parallel to the faces. The eight corners of the cube fall into two sets of four, forming a pair of "desmic" tetrahedra, whose edges are the diagonals of the faces of the cube, each tetrahedron being regular. The direction ratios of the vertices of the two

tetrahedra are respectively

$$\left.\begin{array}{ll} (\ 1:\ \ 1:\ \ 1) & (-1:-1:-1) \\ (\ 1:-1:-1) & (-1:\ \ 1:\ \ 1) \\ (-1:\ \ 1:-1) & (\ 1:-1:\ \ 1) \\ (-1:-1:\ \ 1) & (\ 1:\ \ 1:-1) \end{array}\right\}, \tag{8}$$

the actual direction cosines being given on dividing by $\sqrt{3}$.

The regular octahedron has its corners on the coordinate axes, in the same directions as the centres of the faces of the cube and as the middle points of the edges of the tetrahedra, while the principal diagonals of the cube are the normals at the centres of the faces of the octahedron.

We shall now show that the fundamental invariants for the tetrahedron are

$$L \equiv \xi\eta\zeta, \quad M \equiv \eta^2\zeta^2+\zeta^2\xi^2+\xi^2\eta^2, \tag{9}$$

where we have also, since $\xi^2+\eta^2+\zeta^2 = 0$, the relation

$$(\xi+\eta+\zeta)(\xi-\eta-\zeta)(-\xi+\eta-\zeta)(-\xi-\eta+\zeta) = \Sigma\,\xi^4-2\,\Sigma\eta^2\zeta^2 = -4M. \tag{10}$$

The fundamental invariants for the cube and octahedron are L^2 and M.

The proof is obtained by calculating the expressions $F(\Phi)$ defined in (7) for the various systems of lines passing through the principal points of our configuration. The calculation is very easy and details are left to the reader. The results are as follows.

Corresponding to the first and second tetrahedra in (8), the products $F(\Phi)$ are proportional to

$$9\Phi^4-8\sqrt{3}\,L\Phi-4M, \quad 9\Phi^4+8\sqrt{3}\,L\Phi-4M, \tag{11}$$

respectively. The product of these is the form corresponding to the vertices of the cube and of both tetrahedra, and to the centres of the faces of the octahedron and of both tetrahedra, namely

$$81\Phi^8-72M\Phi^4-192L^2\Phi^2+16M^2. \tag{12}$$

Corresponding to the vertices of the octahedron, the centres of the faces of the cube, and the middle points of the edges of the tetrahedra, the form is

$$\Pi(\Phi^2-\xi^2) = \Phi^6+M\Phi^2-L^2. \tag{13}$$

The symmetric form corresponding to the twelve operators of the type $(\pm\eta\pm\zeta)/\sqrt{2}$, whose "axes" pass through the middle points of the edges of the cube and of the octahedron, is found to be

$$64\Phi^{12}-32M\Phi^8+208L^2\Phi^6-28M^2\Phi^4-108L^2M\Phi^2-27L^4-4M^3. \tag{14}$$

We see, from the expressions (11) to (14), that L and M can be expressed as symmetric functions of a system of Maxwell's "axes", which is intrinsically invariant for the tetrahedron, and that L^2 and M have an invariant relation to the octahedron and cube. The group of reflections and rotations of the tetrahedron is a sub-group of that of the octahedron and cube; the latter configurations are symmetrical about the origin, while the tetrahedron is not. Symmetry about the origin, corresponding to three reflections in the coordinate planes, which changes L into $-L$, explains why L is an invariant for the tetrahedron but not for the cube and octahedron.

5. We must now prove that all geometrically invariant operators of this configuration can be expressed in terms of L and M. For the tetrahedron, we consider a reflection in a plane passing through the centre, containing one of the edges, and bisecting the opposite edge at right angles. This leaves two vertices undisturbed and interchanges the other two; hence an invariant operator, expressed in terms of the system of Maxwell's "axes" passing through the four vertices, must remain unchanged when any pair of vertices are interchanged. It must therefore be a symmetric function, expressible in terms of the coefficients of (11).

For the octahedron and cube, whose planes of symmetry are identical, we consider first a reflection in one of the coordinate planes which passes through four edges of the octahedron. We find that the operator must be an even function of (ξ, η, ζ) separately. Next we consider a reflection in a plane containing the centre and two parallel edges of the cube; this interchanges any desired pair of (ξ^2, η^2, ζ^2), so that the operator must be a symmetric function of (ξ^2, η^2, ζ^2), and so is expressible in terms of the coefficients of (13).

6. Typical harmonic polynomials of this family, generated by L, M, L^2 respectively, are

$$\left.\begin{aligned} &xyz, \\ &\Sigma x^4 - 3\Sigma y^2 z^2, \\ &2\Sigma x^6 - 15\Sigma x^4(y^2+z^2) + 180 x^2 y^2 z^2. \end{aligned}\right\} \tag{15}$$

The general Laplace series, having respectively tetrahedral and octahedral symmetry, may be exhibited in the forms

$$\left.\begin{aligned} V_1 &= [\Sigma a_{pq} L^p M^q](1/r), \\ V_2 &= [\Sigma b_{pq} L^{2p} M^q](1/r). \end{aligned}\right\} \tag{16}$$

The number of independent coefficients, among the terms of order not greater than n, is equal to the number of pairs of positive or zero integers satisfying the respective inequalities

$$\left.\begin{aligned} 3p+4q &\leqslant n, \\ 6p+4q &\leqslant n. \end{aligned}\right\} \tag{17}$$

The number in question is equal to the number of lattice-points in one or other of two triangles, and its approximate value, when n is large, is either $n^2/24$ or $n^2/48$. This must be compared with $(n+1)^2$, the number of independent coefficients in the corresponding terms of (1).

7. Our results are strictly parallel to those of the classical theory of the polyhedral functions, set out (for example) in Klein's *Vorlesungen über das Ikosaeder*. In the classical treatment, a polyhedral figure is projected from its centre on to a unit sphere. If a vertex or other interesting point has co-latitude θ and longitude ϕ. its coordinates on the unit sphere are

$$x = \sin\theta\cos\phi, \quad y = \sin\theta\sin\phi, \quad z = \cos\theta. \tag{18}$$

Let a stereographic projection now be made from the "North Pole" on to the plane through the equator; this projection is also a geometrical inversion, and so is a conformal transformation. Introducing a new complex variable, we may write

$$w = u+iv = \frac{x+iy}{1-z} = \cot\tfrac{1}{2}\theta \,.\, e^{i\phi}. \tag{19}$$

In the classical theory, a polyhedral figure on the sphere is cut up into elementary triangles, half of which are "black" and half "white". Any two "white" triangles are congruent, and can be rotated into one another by one of the rotations of the appropriate group; but to transform a "white" triangle into a "black" one a reflection in a plane of symmetry is necessary, and in general a rotation also.

If we consider the problem of conformally representing on a half-plane one of the fundamental triangles into which our polyhedron has been dissected, we find that certain polynomials occur which are called "polyhedral functions".

Broadly speaking, these are of the type

$$\prod_{s=1}^{N} (w-w_s), \tag{20}$$

where $(w_1, w_2, \ldots, w_N)$ are a set of corners, or other special points, of our

polyhedral configuration, which are interchanged by the rotations, or "covering operations", of our polyhedral group.

These correspond precisely to our operators (7), whose "axes" are directed from the centre towards that set of points. Thus the operators (11), (12), (13), and (14), corresponding to the tetrahedral and octahedral configurations, correspond to the following forms given by Klein, in the *Ikosaeder*, viz.

$$\left.\begin{aligned} \Phi,\ \Psi \text{ of (45), page 51,} \\ w \quad \text{of (51a), page 54,} \\ t \quad \text{of (51a), page 54,} \\ \chi \quad \text{of (52), page 54.} \end{aligned}\right\} \tag{21}$$

It would be tedious to discuss the configuration of the icosahedron and dodecahedron in the same detail; we shall confine ourselves to the salient features, and the reader will readily perceive how the complete analogy could be worked out.

8. The icosahedron and dodecahedron have the same planes of symmetry and form one configuration, the radii vectores from the common centre to the vertices of one figure passing through the centres of the faces of the other, and the edges of the two figures cutting at right angles at their middle points. Projecting the configuration from the centre on the unit sphere, we shall require the lengths of the edges l and l' of the icosahedron and dodecahedron respectively, measured in radians at the centre; the radius R of the small circle circumscribing a face of either figure (which is also the distance between two neighbouring vertices of the two figures); and the altitude p of the icosahedral triangles. By elementary spherical trigonometry, we find the results

$$\left.\begin{aligned} \tan l = 2, \quad \tan R = 3-\sqrt{5}, \quad \tan p = \tfrac{1}{2}(\sqrt{5}+1), \\ \tan(\pi-R-l) = 3+\sqrt{5}, \quad \tan \tfrac{1}{2}l = \tfrac{1}{2}(\sqrt{5}-1), \\ l+l'+2R = \pi, \quad l+2p = \pi, \quad \sin l' = \tfrac{2}{3}. \end{aligned}\right\} \tag{22}$$

Following Klein, we refer our figures to the following system of spherical polar coordinates: the pole is taken at a vertex of the icosahedron and the principal meridian coincides with one of the edges of the icosahedron passing through the pole; the polar distance or co-latitude is θ, and the azimuth or longitude ϕ [not to be confused with the operator Φ of (7)].

The positions of the principal points of our configuration will now be written down in terms of θ and ϕ, and from these we could easily obtain the complex numbers, which we have called w, some of which are given by Klein (54), page 55.

(i) Two vertices of the icosahedron lie at the poles, where ϕ is indeterminate and $\theta = 0$ or $\theta = \pi$ respectively. The remaining vertices are given by

$$(\theta = l, \ \phi = 2s\pi/5), \quad \{\theta = \pi - l, \ \phi = (2s+1)\pi/5\}, \tag{23}$$

where s is any integer.

(ii) The vertices of the dodecahedron are given by

$$\left.\begin{aligned} &\{\theta = R, \ \phi = (2s+1)\pi/5\}, \quad (\theta = \pi - R, \ \phi = 2s\pi/5), \\ &(\theta = l + R, \ \phi = 2s\pi/5), \quad \{\theta = \pi - l - R, \ \phi = (2s+1)\pi/5\}. \end{aligned}\right\} \tag{24}$$

(iii) The middle points of the edges, ten of which lie on the equator, are given by

$$\left.\begin{aligned} &(\theta = \tfrac{1}{2}l, \ \phi = 2s\pi/5), \quad \{\theta = \pi - \tfrac{1}{2}l, \ \phi = (2s+1)\pi/5\}, \\ &\{\theta = p, \ \phi = (2s+1)\pi/5\}, \quad (\theta = \pi - p, \ \phi = 2s\pi/5), \\ &\{\theta = \tfrac{1}{2}\pi, \ \phi = (2s+1)\pi/10\}. \end{aligned}\right\} \tag{25}$$

We shall construct the products of the operators corresponding to these sets of " axes ", and we shall then obtain by elimination the complete forms corresponding to (7). The latter can be constructed directly, but the calculation is of inordinate length, and I do not think it profitable to reproduce it. The weapon which will be employed is the following identity, where ω is a primitive fifth root of unity:

$$\prod_{s=1}^{5} (a + b\omega^s + c\omega^{-s}) \equiv a^5 + b^5 + c^5 - 5a^3bc + 5ab^2c^2. \tag{26}$$

To obtain further insight into this identity, the reader may refer to the theory of circulating determinants, such as

$$\begin{vmatrix} a & b & c & d & e \\ e & a & b & c & d \\ d & e & a & b & c \\ c & d & e & a & b \\ b & c & d & e & a \end{vmatrix}. \tag{27}$$

Further light will be thrown on the subject by any account of the classical work of Jacobi, Brioschi, and others. on the equation of the fifth degree, of which an admirable version is given in Perron's *Algebra*: but a knowledge of this theory is not necessary for the understanding of this paper.

Taking as "axes" the six diameters through the vertices of the icosahedron, whose direction cosines may be read off from (23), the invariant differential operator formed by their product has an ambiguous sign, and is of the form

$$\pm 5^{-\frac{1}{2}}\zeta \prod_{s=1}^{5} \{2\xi \cos(2s\pi/5)+2\eta \sin(2s\pi/5)+\zeta\}. \tag{28}$$

To evaluate this, we put

$$\lambda = \xi - i\eta, \quad \mu = \xi + i\eta, \quad \omega = \exp(2i\pi/5), \tag{29}$$

which gives

$$\pm 5^{-\frac{1}{2}}\zeta \prod_{s=1}^{5} \{\lambda\omega^s+\mu\omega^{-s}+\zeta\}; \tag{30}$$

the identity (26) and the relation corresponding to Laplace's equation

$$\lambda\mu+\zeta^2 \equiv \xi^2+\eta^2+\zeta^2 \equiv 0 \tag{31}$$

(which holds when operating on a spherical harmonic) now yield the form

$$\pm 5^{-\frac{1}{2}}\{\zeta(\lambda^5+\mu^5)+11\zeta^6\}. \tag{32}$$

We introduce the notation

$$P \equiv \zeta(\lambda^5+\mu^5)+11\zeta^6 \tag{33}$$

as our first invariant operator, corresponding to Klein's form f, of (55), page 56.

Our second fundamental invariant operator is derived in the same manner as the product of the ten operators, whose "axes" are the diameters passing through the vertices of the dodecahedron. We read off the direction cosines from (24) and allow for an ambiguity of sign, since each diameter corresponds to two vertices in opposite directions. The result is

$$\begin{aligned}\pm \prod_{s=1}^{5} \{\tfrac{1}{2}(\lambda\omega^s+\mu\omega^{-s}) \sin R-\zeta \cos R\} \prod_{s=1}^{5} \{\tfrac{1}{2}(\lambda\omega^s+\mu\omega^{-s}) \sin (l+R) \\ +\zeta \cos (l+R)\}\end{aligned}$$

$$= \pm(\tfrac{1}{2})^{10} \sin^5 R \sin^5(l+R) \Pi \{\lambda^5+\mu^5-(114\pm 50\sqrt{5})\zeta^5\}$$

$$= \pm 3^{-5}\, 5^{-\frac{3}{2}}\{\lambda^{10}+\mu^{10}-228\zeta^5(\lambda^5+\mu^5)+494\zeta^{10}\}, \tag{34}$$

where we have made use of the relations (22), (24), (26), and (31) to effect the reduction. We shall introduce the notation

$$Q \equiv \lambda^{10}+\mu^{10}-228\zeta^5(\lambda^5+\mu^5)+494\zeta^{10} \tag{35}$$

for our second operator. This corresponds to Klein's H, of (56), page 56.

Our third fundamental invariant operator is obtained in just the same manner from the fifteen diameters passing through the thirty points (25). The form is

$$\pm(\tfrac{1}{2})^{15}(\lambda^5-\mu^5)\prod_{r=1,\,2}\{\lambda^5+\mu^5+\zeta^5 f(\theta_r)\}\sin^5\theta_r, \tag{36}$$

where

$$\theta_1 = \tfrac{1}{2}l, \quad \theta_2 = \pi - p, \tag{37}$$

$$f(\theta) = 10\cot\theta+40\cot^3\theta+32\cot^5\theta. \tag{38}$$

A straightforward reduction yields

$$\begin{aligned}&\pm 2^{-15}\,5^{-\frac{1}{2}}(\lambda^5-\mu^5)\prod\{\lambda^5+\mu^5+\zeta^5(261\pm 125\sqrt{5})\}\\ &\quad = \pm 2^{-15}\,5^{-\frac{5}{2}}(\lambda^5-\mu^5)\{\lambda^{10}+\mu^{10}+522\zeta^5(\lambda^5+\mu^5)-10006\zeta^{10}\},\end{aligned} \tag{39}$$

and we may write the third invariant operator in the form

$$S \equiv (\lambda^5-\mu^5)\{\lambda^{10}+\mu^{10}+522\zeta^5(\lambda^5+\mu^5)-10006\zeta^{10}\}. \tag{40}$$

This corresponds to Klein's T. of (57), page 56. If we form the square of S, and make use of the expressions for P and Q given in (33) and (35) respectively, we can verify the identity

$$S^2 \equiv Q^3+1728P^5, \tag{41}$$

corresponding to Klein's relation (58), page 57.

It is important to observe that η is a factor of S, as may be seen by inspection of the first factor in (40). This corresponds to an important geometrical property of the configuration, namely that the normal at the centre to any plane containing two opposite edges of the icosahedron, or of the dodecahedron, passes through the middle points of another pair of opposite edges of each figure. The meridian plane Oxz contains two edges of the icosahedron, abutting at the poles; it also contains two edges of the dodecahedron, namely

$$\left.\begin{aligned}&\phi = 0, \quad l+R<\theta<\pi-R,\\ &\phi = \pi, \quad R<\theta<\pi-l-R,\end{aligned}\right\} \tag{42}$$

as may be verified from (24).

The distance on this meridian between the middle point of an edge of the icosahedron and that of an edge of the dodecahedron is clearly a right

angle, since we have eliminated l' by means of the relation $l+l'+2R=\pi$ of (22).

The diameters passing through the middle points of these pairs of edges are orthogonal to one another, and are the internal and external bisectors of the angles subtended at the centre by any of the four edges. They are also orthogonal to the axis Oy, which also passes through the middle points of pairs of opposite edges of the icosahedron and of the dodecahedron. Thus the fifteen diameters, passing through the thirty middle points of edges given in (25), form five orthogonal triads, and this fact will play an important part in the later stages of our investigation.

9. The complete symmetric operators of the type (7) corresponding to the icosahedral-dodecahedral configuration can be calculated directly, with very great labour; but they can also be obtained as eliminants. We observe that P, Q, and S are certainly invariant operators of the group of orthogonal transformations of our configuration, except for an ambiguity of sign, corresponding to the choice of a direction along each diameter. Let us now eliminate $\lambda^5+\mu^5$ between the equations (31), (33), and (35); the result obtained is

$$3125\zeta^{12}-250\zeta^6 P-\zeta^2 Q+P^2=0. \tag{43}$$

This identity involves only even powers of ζ, so that it is independent of the choice of direction of the operator ζ. Moreover, P and Q are invariants of the icosahedral group. For they are both of even order in the operators, and so are unchanged by a reflection in the origin, which transforms (ξ, η, ζ) into $(-\xi, -\eta, -\zeta)$. The sign of P can be fixed by taking as the positive ends of the diameters any arbitrary vertex and the five nearest vertices lying on a small circle about it; a similar convention would fix the sign of Q. The identity (43) is equally satisfied by $+\zeta$ or $-\zeta$; and while P, Q are invariants for the group of orthogonal transformations, ζ is merely one of the operators directed towards a vertex of the icosahedron. If we substitute for ζ an operator $\Phi=\partial/\partial h_s$, corresponding to any of the twelve vertices of the icosahedron, we must obtain an operator of the type (7), which admits twelve linear factors. This operator is accordingly

$$3125\Phi^{12}-250\Phi^6 P-\Phi^2 Q+P^2. \tag{44}$$

This form exhibits P and Q as symmetrical functions of the squared operators $\partial^2/\partial h_s^2$, and all ambiguity of sign is now completely removed. We may observe that, although (44) was calculated by means of an artifice,

it can also be found directly, like the forms (11) to (14), and its coefficients correspond to symmetric functions of the twelve operators, corresponding to the vertices of the icosahedron.

10. We shall now prove that any polynomial operator, having the symmetry of the icosahedral configuration, is a polynomial in P and Q.

(i) Since there is symmetry about the centre, the operator must be invariant, when (ξ, η, ζ) are replaced by $(-\xi, -\eta, -\zeta)$, so that the terms of the operator must be of even order. This condition is satisfied by P, Q, and S^2, but not by S.

(ii) Since Oxz is a plane of symmetry, the operator must be unaltered when η is replaced by $-\eta$, or when (λ, μ) are interchanged. If the operator is expressed in terms of (λ, μ, ζ), and if the product $\lambda\mu$ is everywhere replaced by $-\zeta^2$ [in accordance with (31)], the operator must be symmetrical in (λ, μ), and so is of the form $\Sigma A(\lambda^m+\mu^m)\zeta^n$.

(iii) Since the configuration is unaltered by a rotation about Oz through an angle $2s\pi/5$, where s is an integer, the operator must be unaltered when (λ, μ) are replaced by $(\lambda\omega^{-s}, \mu\omega^s)$, where $\omega = \exp(2i\pi/5)$. This requires that the exponents of (λ, μ) should be multiples of 5, and gives the form $\Sigma A(\lambda^{5k}+\mu^{5k})\zeta^n$. Since $5k+n$ is even, this form is expressible as a polynomial in $\lambda^{10}+\mu^{10}$, $\zeta(\lambda^5+\mu^5)$, and ζ^2; and this in turn may be expressed in terms of P, Q, and ζ^2. Thus our invariant operator may be written

$$\Omega \equiv f(P, Q, \zeta^2) \equiv f(P, Q, \partial^2/\partial h_1^2), \tag{45}$$

where h_1 is one of the set of twelve axes directed towards the vertices of the icosahedron.

(iv) We have now considered all the operations which do not shift the axis Oz or h_1. The expressions P and Q are known to be invariant for every operation of the group, by which h_1 may be interchanged with any other "axis" of its set. But, by hypothesis, Ω is an invariant of the group, so that the presence of h_1 in (45) cannot correspond to a real lack of symmetry. We can remove h_1 by means of a classical artifice of the theory of equations; for on interchanging h_1 with the other "axes" of the set, we leave P and Q invariant and obtain twelve apparently different forms of the same operator Ω. If we form the sum of these, and evaluate the symmetrical functions of the operators $(\partial/\partial h_1, \partial/\partial h_2, \ldots, \partial/\partial h_{12})$ by means of the coefficients of (44), we shall obtain an expression for Ω in

terms of P and Q only, as follows:

$$\Omega = \tfrac{1}{12} \sum_{s=1}^{12} f(P, Q, \partial^2/\partial h_s^2) = F(P, Q). \tag{46}$$

Before pursuing our algebraic investigations, we note the bearing of our results on the theory of spherical harmonics. The two simplest harmonics having icosahedral symmetry (but not spherical symmetry) correspond to the operators P and Q; these are

$$\left.\begin{aligned} &8P_6(\cos\theta) - 21 \sin^5\theta \cos\theta \cos 5\phi, \\ 256P_{10}(\cos\theta) - 132 \sin^5\theta &(15 \cos\theta - 170 \cos^3\theta + 323 \cos^5\theta) \cos 5\phi \\ &+ 187 \sin^{10}\theta \cos 10\phi, \end{aligned}\right\} \tag{47}$$

where P_6 and P_{10} are the usual Legendre polynomials.

The most general harmonic of the family corresponds to the most general series in P and Q, and has the form

$$V_3 = \{\Sigma a_{mn} P^m Q^n\}(1/r). \tag{48}$$

Since (P, Q) are of order (6, 10) respectively, the number of coefficients a_{mn} in (48), corresponding to terms of order not greater than a large number N, so that $6m+10n \leqslant N$, is approximately $N^2/120$, compared with $(N+1)^2$ in the full series (1).

11. An interesting proof of the identity (41) may be based on the fundamental theorem. The expression S^2, corresponding to the thirty "axes" through the middle points of edges, is evidently an invariant; being of order 30 and being expressible in terms of P and Q, it can only be of the form $S^2 = AP^5 + BQ^3$. To determine A/B, we make use of the geometrical property noted at the end of § 8, that the operator η is a factor of S, and so η^2 is a factor of S^2; moreover, by the invariant property of S^2 only even powers of η can occur in S^2, and these may be eliminated by means of (31). If we now replace the *operators* (ξ, η, ζ) by *numbers*, satisfying the relations $\xi^2 + \zeta^2 = 0$, $\eta = 0$, the expression for S^2 must vanish identically, and this will give the ratio A/B. Suitable numbers are $(1, 0, i)$, and if we denote by P^* and Q^* the result of substituting these in P and Q, we find that

$$P^* = -11 + 2i, \quad Q^* = -492 - 456i.$$

Now the required operator is real, and the values of (A, B) are real, so

that we are led to look for some peculiarity in the structure of P^* and Q^*, and we find

$$P^* = (1-2i)^3, \quad Q^* = -12(1-2i)^5,$$

$$A - 12^3 B = 0.$$

By inspection of the terms in $\lambda^{30}+\mu^{30}$, we find that

$$B = 1, \quad A = 1728,$$

and the result (41) is thus confirmed.

The peculiarity of the numbers P^* and Q^* corresponds to a peculiarity in the operators P and Q, which can be expressed in terms of three quadratic operators having a simple geometrical interpretation.

Let us write

$$\left.\begin{aligned} &\alpha \equiv 4\eta^2, \quad \beta \equiv (2\xi+\zeta)\zeta, \quad \gamma \equiv (4\xi^2-6\xi\zeta+\zeta^2), \\ &\text{so that} \qquad 4\nabla^2 \equiv \alpha+3\beta+\gamma = 0. \end{aligned}\right\} \tag{49}$$

Then it may be verified that

$$\left.\begin{aligned} P &\equiv \alpha^2\beta+\alpha\beta^2-\beta^3 \equiv 5\beta^3+5\beta^2\gamma+\beta\gamma^2, \\ Q &\equiv -\alpha^5+10\alpha^3\beta^2+20\alpha^2\beta^3+55\alpha\beta^4+12\beta^5, \\ Q &\equiv 200\beta^4\gamma+200\beta^3\gamma^2+80\beta^2\gamma^3+15\beta\gamma^4+\gamma^5, \\ Q &\equiv (20\beta^2+10\beta\gamma+\gamma^2)(10\beta^2+5\beta\gamma+\gamma^2)\gamma. \end{aligned}\right\} \tag{50}$$

The geometrical interpretation of the operators α, β, γ is as follows: The operator α is a squared Maxwellian operator, whose "axis" Oy bisects an edge of the icosahedron and of the dodecahedron. The operator β has two distinct factors, and its "axes" are drawn towards the extremities of one of the two opposite edges of the icosahedron lying in the meridian plane Oxz; while the operator γ is similarly determined by the extremities of one of the two opposite edges of the dodecahedron in the meridian plane. The operators are determined in magnitude and sign by the relations

$$\alpha = 4\frac{\partial^2}{\partial h^2}, \quad \beta = \sqrt{5}\frac{\partial^2}{\partial h_1 \partial h_2}, \quad \gamma = 3\sqrt{5}\frac{\partial^2}{\partial h_1' \partial h_2'}. \tag{51}$$

12. Following a suggestion of the Referee, we may add the expressions of α, β, γ, P, and Q in terms of the system of coordinates, two of whose principal axes bisect the edges of the icosahedron and of the dodecahedron

lying in the plane Oxz; this system is accordingly one of the five triads mentioned at the end of § 8. The formulae of transformation are

$$\xi_1+i\zeta_1=(\xi+i\zeta)\exp(\tfrac{1}{2}il), \quad \eta_1=\eta; \tag{52}$$

and, after some arithmetical labour, the following results are obtained:

$$\left.\begin{aligned} \alpha &= 4\eta_1^2, \\ \beta &= \sqrt{5}(-\xi_1^2\sin^2\tfrac{1}{2}l+\zeta_1^2\cos^2\tfrac{1}{2}l)=\tfrac{1}{2}(-\sqrt{5}\,\xi_1^2-\eta_1^2+\sqrt{5}\,\zeta_1^2), \\ \gamma &= 3\sqrt{5}(\xi_1^2\cos^2\tfrac{1}{2}l'-\zeta_1^2\sin^2\tfrac{1}{2}l')=\tfrac{1}{2}(3\sqrt{5}\,\xi_1^2-5\eta_1^2-3\sqrt{5}\,\zeta_1^2). \end{aligned}\right\} \tag{49a}$$

The values of P, Q, and S can be calculated by substituting these expressions; but it is more interesting to recalculate the direction cosines of the principal points referred to the frame of reference $Ox_1y_1z_1$. If we find the points lying within and on the boundary of the principal octant, the remaining points are evaluated (i) by symmetrical reflection in the principal planes, (ii) by cyclic interchange of the axes.

(i) The corners of the icosahedron in the principal octant lie on the boundary, and are obtained by cyclic interchange from the system of direction cosines

$$(\cos\tfrac{1}{2}l, \quad \sin\tfrac{1}{2}l, \quad 0); \tag{23a}$$

the cyclic interchange and the complete reflection yield the twelve corners.

(ii) The corners of the dodecahedron lie on the boundary and at the centre of the octant; the full set of twenty are obtained by cyclic symmetry and by complete reflection from the types

$$\left.\begin{aligned} &(\sin\tfrac{1}{2}l', \quad \cos\tfrac{1}{2}l', \quad 0), \\ &(1/\sqrt{3}, \quad 1/\sqrt{3}, \quad 1/\sqrt{3}). \end{aligned}\right\} \tag{24a}$$

(iii) The middle points of the sides lie either on the principal axes of coordinates or in cyclically symmetrical triads within each octant; the typical sets of direction cosines, from which the complete system of thirty points can be found, are

$$\left.\begin{aligned} &(1, \quad 0, \quad 0), \\ &(\cos 36^0, \quad \cos 72^0, \quad \cos 60^0). \end{aligned}\right\} \tag{25a}$$

The numerical values of the direction cosines are found from (22); the numerical coefficients of P, Q, and S are given by (32), (34), and (39).

Following (9), it is convenient to write

$$L_1 \equiv \xi_1\eta_1\zeta_1, \quad M_1 \equiv \Sigma\eta_1^2\zeta_1^2, \quad \Delta_1 \equiv \Pi(\eta_1^2-\zeta_1^2) \equiv \sqrt{(-27L_1^4-4M_1^3)}. \quad (53)$$

We can then evaluate P and Q as follows:

$$P = 5^{\frac{5}{2}}\Pi(-\xi_1^2\cos^2\tfrac{1}{2}l+\eta_1^2\sin^2\tfrac{1}{2}l),$$

$$-8P = 5\Pi\{(\sqrt{5}+1)\xi_1^2-(\sqrt{5}-1)\eta_1^2\},$$

$$2P = -11L_1^2-\sqrt{5}\,\Delta_1; \quad (33a)$$

$$Q = 3^3 5^{\frac{5}{2}}(-\Sigma\xi_1^4+2\Sigma\eta_1^2\zeta_1^2)\,\Pi(-\xi_1^2\sin^2\tfrac{1}{2}l'+\eta_1^2\cos^2\tfrac{1}{2}l'),$$

$$Q = -2^{-1}3^3 5^{\frac{5}{2}} M_1\,\Pi\{3(\xi_1^2-\eta_1^2)+\sqrt{5}\,\zeta_1^2\},$$

$$Q = M_1(5^3 38L_1^2-5^{\frac{5}{2}}6\Delta_1). \quad (35a)$$

By drawing an octant of the sphere, on which the icosahedral figure is described, forming the product of fifteen operators obtained by reflection in the three sides of the octant, and comparing the result with (36), we find the formula

$$S = -i\,2^{15}5^{\frac{5}{2}}\xi_1\eta_1\zeta_1\,\Pi(l^4\xi_1^4+m^4\eta_1^4+n^4\zeta_1^4 \\ -2m^2n^2\eta_1^2\zeta_1^2-2n^2l^2\zeta_1^2\xi_1^2-2l^2m^2\xi_1^2\eta_1^2).$$

Eliminating ζ_1 from the expression in brackets, the latter yields

$$(l^2+n^2)^2\xi_1^4+(m^2+n^2)^2\eta_1^4+\{2n^2(l^2+m^2+n^2)-2l^2m^2\}\xi_1^2\eta_1^2,$$

and, if we substitute for (l, m, n) the values from (25a), we have

$$\tfrac{5}{4}(l^2\xi_1^4+m^2\eta_1^4)+\tfrac{3}{8}\xi_1^2\eta_1^2 = \tfrac{5}{3\,2}(3-\sqrt{5})\xi_1^4+\tfrac{5}{3\,2}(3+\sqrt{5})\eta_1^4+\tfrac{3}{8}\xi_1^2\eta_1^2.$$

Hence

$$S = -i\,5^{\frac{5}{2}}\xi_1\eta_1\zeta_1\,\Pi\{(15-5\sqrt{5})\xi_1^4+12\xi_1^2\eta_1^2+(15+5\sqrt{5})\eta_1^4\}.$$

The cyclical product can be calculated by means of the identity

$$\Pi(A\xi_1^4+B\xi_1^2\eta_1^2+C\eta_1^4) \\ \equiv\{(A+B+C)^3-\tfrac{9}{2}(A+B+C)(BC+CA+AB)+\tfrac{27}{2}ABC\}L_1^4 \\ -AC(A-B+C)M_1^3-\tfrac{1}{2}(B-C)(C-A)(A-B)L_1^2\Delta_1.$$

Substituting the numerical values of A, B, C, we find that

$$S = i\,2^2 5^{\frac{5}{2}} L_1(-837L_1^4+450M_1^3+145\sqrt{5}\,L_1^2\Delta_1), \quad (40a)$$

where the second factor corresponds to the product of the four operators

of the type L_1, corresponding to the other four triads of diameters bisecting the edges.

13. By elimination between the expressions (50), we obtain three identities between (α, P, Q), (β, P, Q), and (γ, P, Q) respectively. Since P and Q are invariants, the same identities are verified when we substitute for α, β, or γ the most general operator of the same type, having the same geometrical relation to the icosahedral figure. We write the α-identity in such a form that many of the terms can be quickly verified on putting either $(\beta = 0,\ P = 0)$ or $(Q+\alpha^5 = 0)$, namely

$$(Q+\alpha^5)^3-627(Q+\alpha^5)^2\alpha^5+3751(Q+\alpha^5)\alpha^{10}+400P(Q+\alpha^5)^2\alpha^2$$
$$+12100P(Q+\alpha^5)\alpha^7-2740P^2(Q+\alpha^5)\alpha^4-37510P^2\alpha^9$$
$$+2700P^3(Q+\alpha^5)\alpha-121000P^3\alpha^6-153715P^4\alpha^3+1728P^5\equiv 0. \quad (54)$$

The symmetric operator of the type (44), but corresponding to the thirty linear operators $\Pi(\Phi-\partial/\partial h_s)$, where the axis h_s passes through the middle point of an edge, is obtained by putting $4\Phi^2$ for α in (54). On rearranging the terms, we find the result

$$2^{30}5^5\Phi^{30}+2^{26}5^5P\Phi^{24}+2^{22}5^4Q\Phi^{20}-161\,.\,2^{19}5^3P^2\Phi^{18}+129\,.\,2^{16}5^2PQ\Phi^{14}$$
$$-1183\,.\,2^{14}5^2P^3\Phi^{12}-39\,.\,2^{14}Q^2\Phi^{10}-137\,.\,2^{10}P^2Q\Phi^8-30743\,.\,2^6\,5P^4\Phi^6$$
$$+2^8\,5^2PQ^2\Phi^4+27\,.\,2^4\,5^2P^3Q\Phi^2+1728P^5+Q^3. \quad (55)$$

14. The β-identity is most readily found from the factorized form of Q in (50). We obtain first a formula which is linear in γ, and we then eliminate γ from the formula which defines P. The results are

$$\beta^2Q+(5\beta^3+P)\{25\beta^4+10\beta^3\gamma-(5\beta+\gamma)P\}=0,$$
$$\beta^4(5P^2-\beta Q-125\beta^6)(Q-125\beta^5-25\beta^2P)$$
$$+(P-5\beta^3)(P^2-5\beta^3P-50\beta^6)^2\equiv 0. \quad (56)$$

It is useless to replace β by the square of a linear operator, since the resulting expression would not factorize like (44) or (55). But if we write for β an indeterminate quadratic operator Ψ, corresponding to a pair of adjacent vertices of the icosahedron, with a numerical factor given by (51), we obtain the symmetric operator

$$3125\Psi^{15}+3125P\Psi^{12}+250P^2\Psi^9+25PQ\Psi^7-150P^3\Psi^6-Q^2\Psi^5$$
$$+5P^2Q\Psi^4-15P^4\Psi^3+P^5. \quad (57)$$

15. The γ-identity is readily obtained by putting $\kappa = 2\beta^2+\beta\gamma$, and observing that (50) then gives $Q = \gamma(10\kappa+\gamma^2)(5\kappa+\gamma^2)$. We begin by eliminating β between the expressions for κ and P, and then we eliminate κ between the resultant and Q; we thus find that

$$8P^2-(\gamma^3+10\gamma\kappa)P+\kappa\gamma^4+5\kappa^2\gamma^2-25\kappa^3 = 0,$$

$$\gamma^3(640P^2+40P\gamma^3+26Q\gamma+7\gamma^6)^2 = (8Q+\gamma^5)(2Q+40P\gamma^2+9\gamma^5)^2. \quad (58)$$

Replacing γ by an indeterminate quadratic operator, corresponding to two adjacent vertices of the dodecahedron, with a numerical multiplier given by (51), we have the symmetric operator

$$\Psi^{15}+5P\Psi^{12}+10Q\Psi^{10}-280P^2\Psi^9+120PQ\Psi^7-1600P^3\Psi^6-12Q^2\Psi^5$$
$$-640P^2Q\Psi^4-1280P^4\Psi^3+40PQ^2\Psi^2+Q^3. \quad (59)$$

16. In conclusion, we shall form the symmetrical operator having twenty linear factors, corresponding to the vertices of the dodecahedron. We begin by multiplying the four factors corresponding to the vertices in the plane Oxz, whose direction cosines may be read off from (24) and evaluated by (22); this gives

$$45\Pi\{\Phi\pm[\xi\sin(l+R)+\zeta\cos(l+R)]\}\,\Pi\{\Phi\pm[\xi\sin R-\zeta\cos R]\}$$
$$= 45\Phi^4-15(\beta+\gamma)\Phi^2+\gamma^2. \quad (60)$$

We now put
$$3\Phi^2 \equiv \delta, \quad (61)$$

and proceed to eliminate β and γ from the system of identities

$$\left.\begin{aligned} P &= 5\beta^3+5\beta^2\gamma+\beta\gamma^2, \\ Q &= (20\beta^2+10\beta\gamma+\gamma^2)(10\beta^2+5\beta\gamma+\gamma^2)\gamma, \\ 5\delta^2-5(\beta+\gamma)\delta+\gamma^2 &= 0. \end{aligned}\right\} \quad (62)$$

We know that the correct result is of order 10 in δ; but the result of elimination according to the rules of algebra is of order 30, and is at best the cube of the expression required. The direct calculation of the true eliminant is apparently at least as tedious as its construction by the trigonometrical methods of § 8. We are therefore led to employ the following artifice.

If in (62) we make $\beta = 0$, the system reduces to

$$P = 0, \quad Q = \gamma^5, \quad 5\delta^2-5\gamma\delta+\gamma^2 = 0, \quad (62a)$$

and the eliminant Δ_0 is obtained by generalizing the last equation in

accordance with the formula (26). This yields

$$\Delta_0 \equiv 3125\delta^{10} - 625Q\delta^5 + Q^2, \tag{63}$$

and a comparison with (34) and (61) indicates that these are the terms of the full eliminant Δ which are left if everywhere we put $P=0$, so that

$$\Delta \equiv \Delta_0 + P\Delta_1. \tag{64}$$

If we apply the same process to the last equation of (62), without erasing β, we obtain the identity

$$3125\delta^{10} - 625\{5(\beta+\gamma)^5 - 5(\beta+\gamma)^3\gamma^2 + (\beta+\gamma)\gamma^4\}\delta^5 + \gamma^{10} = 0, \tag{65}$$

which will presently be reduced to the required form. But, in order not to obscure the proof with complicated arithmetic, we shall at once tabulate a number of polynomials which appear in the course of the work. These are

$$\left.\begin{aligned} A_2 &= 5\beta^2 - 180\beta\gamma + 24\gamma^2, \quad B_2 = 55\beta^2 - 80\beta\gamma + 64\gamma^2, \\ A_3 &= 320\beta^3 + 240\beta^2\gamma + 27\beta\gamma^2 + 54\gamma^3, \\ A_4 &= 1600\beta^4 + 2800\beta^3\gamma + 1280\beta^2\gamma^2 + 485\beta\gamma^3 + 206\gamma^4, \\ A_5 &= 1600\beta^5 + 1600\beta^4\gamma + 960\beta^3\gamma^2 + 405\beta^2\gamma^3 + 454\beta\gamma^4 + 138\gamma^5, \\ A_6 &= (75\beta^3 + 245\beta^2\gamma + 225\beta\gamma^2 + 44\gamma^3)\gamma^3, \\ A_7 &= (105\beta^5 + 100\beta^4\gamma + 60\beta^3\gamma^2 + 15\beta^2\gamma^3 + 5\beta\gamma^4 + \gamma^5)\gamma^2. \end{aligned}\right\} \tag{66}$$

The middle coefficient of (65) is

$$\begin{aligned} -625\{5\beta^5 + 25\beta^4\gamma + 45\beta^3\gamma^2 + 35\beta^2\gamma^3 + 11\beta\gamma^4 + \gamma^5\} \\ &= -625Q - 625\{5\beta^5 - 175\beta^4\gamma - 155\beta^3\gamma^2 - 45\beta^2\gamma^3 - 4\beta\gamma^4\} \\ &= -625Q - 125PA_2 + 1375(15\beta^2\gamma^3 + 4\beta\gamma^4). \end{aligned} \tag{67}$$

We now obtain from (65) the form

$$3125\delta^{10} - 625Q\delta^5 - 125PA_2\delta^5 + 1375(15\beta + 4\gamma)\beta\gamma^3\delta^5 + \gamma^{10} = 0. \tag{68}$$

From the last equation of (62) we can calculate the formulae

$$\left.\begin{aligned} 5\delta^2 &= 5(\beta+\gamma)\delta - \gamma^2, \\ 5\delta^3 &= (5\beta^2 + 10\beta\gamma + 4\gamma^2)\delta - (\beta+\gamma)\gamma^2, \\ 25\delta^4 &= (25\beta^3 + 75\beta^2\gamma + 65\beta\gamma^2 + 15\gamma^3)\delta - (5\beta^2 + 10\beta\gamma + 4\gamma^2)\gamma^2, \\ 25\delta^5 &= (25\beta^4 + 100\beta^3\gamma + 135\beta^2\gamma^2 + 70\beta\gamma^3 + 11\gamma^4)\delta \\ &\qquad - (5\beta^3 + 15\beta^2\gamma + 13\beta\gamma^2 + 3\gamma^3)\gamma^2. \end{aligned}\right\} \tag{69}$$

From the last of these we obtain

$$25\delta^5 = (5\beta^2+5\beta\gamma+\gamma^2)(5\beta^2+15\beta\gamma+11\gamma^2)\,\delta-(5\beta^3+15\beta^2\gamma+13\beta\gamma^2+3\gamma^3)\,\gamma^2,$$

$$25\beta\delta^5 = P\{(5\beta^2+15\beta\gamma+11\gamma^2)\,\delta-(\beta+2\gamma)\,\gamma^2\}-(2\beta+\gamma)\,\beta\gamma^3. \qquad (70)$$

Substituting this expression in the last term but one of (68), we find that

$$3125\delta^{10}-625Q\delta^5-125PA_2\delta^5+55P(15\beta+4\gamma)\,\gamma^3$$
$$\times\{(5\beta^2+15\beta\gamma+11\gamma^2)\,\delta-(\beta+2\gamma)\,\gamma^2\}$$
$$+\{-55\beta\gamma^7(15\beta+4\gamma)(2\beta+\gamma)+\gamma^{10}\}=0. \qquad (71)$$

The terms in the last line reduce to

$$-55\gamma^7(30\beta^3+23\beta^2\gamma+4\beta\gamma^2)+\gamma^{10}$$
$$=-330P\gamma^7+385\beta^2\gamma^8+110\beta\gamma^9+\gamma^{10}$$
$$=-330P\gamma^7+Q^2-80PA_7. \qquad (72)$$

The first and most troublesome stage of the reduction of the discriminant, namely that to the form (64), is now completed, and we have $\Delta\equiv\Delta_0+P\Delta_1$, where we may verify that

$$\Delta_1\equiv-125A_2\delta^5+55A_6\delta-5A_5\gamma^2. \qquad (73)$$

We now make use of the last equation of (62) to eliminate γ^2 from the last term in (73), and repeat the process systematically. The successive stages of the reduction are

$$\Delta_1=-125A_2\delta^5+25A_5\delta^2+\{55A_6-25(\beta+\gamma)\,A_5\}\,\delta,$$
$$=-125A_2\delta^5+25A_5\delta^2-1600P^2\delta-5A_4\gamma^2\delta;$$

$$\Delta_1+1600P^2\delta=-125A_2\delta^5+25A_4\delta^3+25\{A_5-(\beta+\gamma)\,A_4\}\delta^2,$$
$$=-125A_2\delta^5+25A_4\delta^3-350Q\delta^2-25A_3\gamma^2\delta^2;$$

$$\Delta_1+1600P^2\delta+350Q\delta^2=-125A_2\delta^5+125A_3\delta^4+25\{A_4-5(\beta+\gamma)\,A_3\}\delta^3,$$
$$=-125A_2\delta^5+125A_3\delta^4-25B_2\gamma^2\delta^3,$$
$$=125(B_2-A_2)\,\delta^5+125\{A_3-(\beta+\gamma)\,B_2\}\delta^4,$$
$$=125(B_2-A_2)\,\delta^5+6625P\delta^4-1250(\beta+\gamma)\,\gamma^2\delta^4;$$

$$\Delta_1+1600P^2\delta+350Q\delta^2-6625P\delta^4$$
$$=1250(5\beta^2+10\beta\gamma+4\gamma^2)\,\delta^5-1250\,(\beta+\gamma)\,\gamma^2\delta^4$$
$$=6250(\beta+\gamma)\,\delta^6-1250\gamma^2\delta^5,$$
$$=6250\delta^7;$$

thus we have finally

$$\Delta_1 \equiv 6250\delta^7 + 6625P\delta^4 - 350Q\delta^2 - 1600P^2\delta. \quad (74)$$

Substituting this expression in (64) and then writing $3\Phi^2$ instead of δ, we obtain the symmetrical operator corresponding to the vertices of the dodecahedron, which completes the cycle of those which we set out to construct. The final result is

$$3^{10}5^5\Phi^{20} + 2 \cdot 3^7 5^5 P\Phi^{14} - 3^5 5^4 Q\Phi^{10} + 53 \cdot 3^4 5^3 P^2\Phi^8 - 14 \cdot 3^2 5^2 PQ\Phi^4$$
$$- 64 \cdot 3 \cdot 5^2 P^3\Phi^2 + Q^2. \quad (75)$$

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ON THE CONVERGENCE ALMOST EVERYWHERE OF RADEMACHER'S SERIES AND OF THE BOCHNER-FEJÉR SUMS OF A FUNCTION ALMOST PERIODIC IN THE SENSE OF STEPANOFF

By H. D. URSELL.

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We shall use the abbreviation p.p. (= *presque partout*) for the phrase "almost everywhere".

§ 1 contains a new proof that Rademacher's series converge p.p. In §§ 2–4 examples are constructed of S-a.p. functions whose Bochner-Fejér sums do not converge p.p. The example of § 3 is S_2-*grenz-periodisch*; that of § 4 has an integral basis for its Fourier exponents consisting of two members only.

1. Rademacher* considered series of the form

$$\Sigma a_n f_n(x), \tag{1}$$

where Σa_n^2 converges and $f_n(x) = f_0(2^n x)$ is equal to $+1$ or to -1 according as the integral part of $2^n x$ is even or odd. He showed that every such series converges p.p., and we shall first show how this result may be deduced from the well-known result that the Fejér sums of a purely periodic function converge p.p.

Given two sequences $\{g_n(x)\}$, $\{h_n(x)\}$ of step-functions defined in $0 \leqslant x \leqslant 1$, I say that they are *equi-measurable* if, for each n and every set of n real numbers $(a_1, a_2, \ldots, a_n)$, the set $G(a)$ of points x for which

$$g_i(x) = a_i \quad (i \leqslant n)$$

* Rademacher, *Math. Annalen*, 87 (1922), 112–138 (130).

has the same measure as the set $H(a)$ derived in the same way from the second sequence. In such a case *the series* $\Sigma g_n(x)$, $\Sigma h_n(x)$ (which I shall also describe as equi-measurable) *converge in sets of the same measure**.

Thus Rademacher's theorem will be proved if we can find a single sequence of integers $\nu(n)$ all different and such that the series

$$\sum_{n=1} a_n f_{\nu(n)}(x) \tag{2}$$

converges p.p. Moreover, if Rademacher's theorem is true at all, the series $\Sigma a_{\psi(n)} f_{\psi(n)}(x)$ also converges p.p. with the equi-measurable series $\Sigma a_{\psi(n)} f_n(x)$, and so Rademacher's series converges p.p. for every possible order of summation: here $\psi(n)$ denotes any permuting function of the positive integers.

We shall now show that (2) certainly converges p.p. if $\nu(n)$ increases sufficiently rapidly. Let $\Sigma\epsilon_n$ be a convergent series of positive terms and η_n the remainder after n terms, and suppose that $\nu(n)$ has already been chosen for $n \leqslant m$. The Fejér sums of the function

$$\sum_{n \leqslant m} a_n f_{\nu(n)}(x) \tag{3}$$

converge p.p. to the function; accordingly we can find an integer $N(m)$ so great that in any interval of length 1 the Fejér sum of index $N(m)$ of the function (3) differs from this function by less than ϵ_n except in a set of measure less than ϵ_n. If integers $N(n)$ have already been chosen in this way for $n \leqslant m$ we choose $\nu(m+1)$ different from all previous $\nu(n)$ and greater than any number $N(n)$ for $n \leqslant m$. We put $\nu(1) = 1$ and then determine by induction as described the whole sequence $\{\nu(n)\}$ and at the same time a corresponding sequence $\{N(n)\}$.

The series (2) so determined converges in mean to a function $f(x)$ of summable square, and the Fourier series of $f(x)$ is the formal sum of the Fourier series of the individual terms of (2). Now, owing to the high periodicity of the later terms, only the first m terms of (2) give any

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* For the set $G(n, p, \epsilon)$ of points x for which

$$|g_{n+1}(x) + \ldots + g_{n+q}(x)| \leqslant \epsilon \quad (q \leqslant p)$$

has the same measure as the set $H(n, p, \epsilon)$: since these sets decrease as p increases, the set $G(n, \epsilon) = \lim_{p \to \infty} G(n, p, \epsilon)$ has the same measure as the set $H(n, \epsilon)$: since these sets increase as n increases, the set $G(\epsilon) = \lim_{n \to \infty} G(n, \epsilon)$ has the same measure as the set $H(\epsilon)$: and since these sets decrease as ϵ decreases, the set $G = \lim_{\epsilon \to 0} G(\epsilon)$ has the same measure as the set H.

contribution to the Fejér sum of index $N(m)$ of the function $f(x)$, and hence the latter differs from (3), in any interval of length 1, by less than ϵ_n except in a set of measure less than ϵ_n. It follows that the sequence of partial sums (3) converges p.p. with the sequence of Fejér sums which approximate them so closely.

I may observe in conclusion that the restriction to step-functions is not essential to the discussion of equi-measurable series—it serves merely to fix the ideas—and that the argument which follows will apply with minor alterations to the generalisation of Rademacher's series considered by Khintchine and Kolmogoroff*.

2. We shall now construct an S_2-a.p. function whose Bochner-Fejér sums diverge p.p. Our example will be of the form

$$\sum_{n=1} a_n f_0(\lambda_n x), \tag{4}$$

where Σa_n^2 converges but $\Sigma|a_n|$ diverges. Observe first that by making λ_n increase sufficiently rapidly we can make (4) asymptotically equi-measurable with the series (1); in brief, by imposing a condition of the form

$$\lambda_{n+1} > \Lambda(\lambda_1, \lambda_2, \ldots, \lambda_n), \tag{5}$$

we can make the series (4) converge p.p. and (S_2) to a limit-function $f(x)$.

The S_2-a.p. function $f(x)$ has a Fourier series obtained by formal summation of the Fourier series of the individual terms of (4). We choose the numbers λ_n linearly independent so that the numbers $\pi\lambda_n$ form an integral basis for the Fourier exponents of $f(x)$. The Bochner-Fejér sums of (x) formed from this basis are of the form

$$\sigma(x) = \sigma_1(x) + \sigma_2(x) + \ldots, \tag{6}$$

where $\sigma_n(x)$ is a Fejér sum of the purely periodic function $a_n f_0(\lambda_n x)$ (of period $2/\lambda_n$) and the indices of these Fejér sums are arbitrary and independent subject to the restriction that only a finite number of them can differ from zero [so that (6) is effectively a finite sum]. Since (for a given value of n) $\sigma_n(x) \to a_n f_0(\lambda_n x)$ p.p. as the index of this Fejér sum tends to infinity, we can choose a particular determination of $\sigma_n(x)$, say $\sigma_n'(x)$, which differs from $a_n f_0(\lambda_n x)$ by less than ϵ_n except, in any interval of length 1, in a set of measure less than ϵ_n.

* Khintchine and Kolmogoroff, *Recueil Math.* (Moscow), 32 (1924), 668.

Now let $g_n(x)$ denote the positive part of $a_n f_0(\lambda_n x)$, so that

$$M\{g_n(x)\} = \tfrac{1}{2}|a_n| \quad \text{and} \quad g_n(x) - \tfrac{1}{2}|a_n| = \tfrac{1}{2}a_n f_0(\lambda_n x).$$

Thus, subject to (5), the series $\Sigma(g_n - \frac{1}{2}|a_n|)$ converges p.p. and in particular the series $\Sigma g_n(x)$ diverges p.p. to $+\infty$.

Consider any portion

$$\sigma_{r,\rho}(x) = \sigma_{r+1}(x) + \sigma_{r+2}(x) + \ldots + \sigma_{r+\rho}(x) \tag{7}$$

of the series (6): by giving to each term $\sigma_n(x)$ independently the two determinations $\sigma_n'(x)$ and 0 we obtain 2^ρ determinations of $\sigma_{r,\rho}(x)$, and if for each point x we take the maximum of these we obtain a function which in any interval of length 1 differs from

$$g_{r+1}(x) + g_{r+2}(x) + \ldots + g_{r+\rho}(x)$$

by less than η_r except in a set of measure less than η_r.

Divide the range of n into mutually exclusive intervals

$$(r+1,\ r+\rho) = (n_\nu + 1,\ n_{\nu+1})$$

such that

$$\tfrac{1}{2}(|a_{r+1}| + |a_{r+2}| + \ldots + |a_{r+\rho}|) > \nu + |a_1| + |a_2| + \ldots + |a_r|.$$

For each ν we take a "good approximation"

$$\Sigma_\nu(x) = \sigma_1(x) + \sigma_2(x) + \ldots + \sigma_r(x)$$

to the function

$$a_1 f_0(\lambda_1 x) + a_2 f_0(\lambda_2 x) + \ldots + a_r f_0(\lambda_r x);$$

it is sufficient for our purpose that the sums $\Sigma_\nu(x)$ should (have Fourier series which) converge formally to $f(x)$. We have

$$|\sigma_n(x)| \leqslant |a_n|, \quad |\Sigma_\nu(x)| \leqslant |a_1| + |a_2| + \ldots + |a_r|.$$

We form 2^ρ sums $\sigma(x)$ by adding $\Sigma_\nu(x)$ to the 2^ρ determinations of $\sigma_{r,\rho}(x)$ chosen above; then if for each value of x we take the maximum of these we obtain a function which in any interval of length 1 exceeds

$$\nu - \eta_r + (g_{r+1} - \tfrac{1}{2}|a_{r+1}|) + (g_{r+2} - \tfrac{1}{2}|a_{r+2}|) + \ldots + (g_{r+\rho} - \tfrac{1}{2}|a_{r+\rho}|),$$

except in a set of measure less than η_r. Combining all these batches of Bochner-Fejér sums into a single infinite sequence, and remembering that

$\Sigma(g_n - \frac{1}{2}|a_n|)$ converges p.p., we see that the sequence diverges p.p. although converging formally to $f(x)$.

3. The example of § 2 was of a special type in that it possessed an integral (but infinite) basis. In this section we shall construct another example whose Fourier exponents are all commensurable and which is therefore *S_2-grenz-periodisch**. Our example will again be of the form (4), but this time we write

$$\lambda_n = q(n)/p(n),$$

where $q(n)$ is prime to $p(n)$; the numbers $p(n)$ will be all primes and all different.

As before we divide the range of n into mutually exclusive intervals

$$(r+1,\ r+\rho) = (n_\nu+1,\ n_{\nu+1})$$

such that

$$\tfrac{1}{2}(|a_{r+1}|+|a_{r+2}|+\ldots+|a_{r+\rho}|) > \nu+|a_1|+|a_2|+\ldots+|a_r|.$$

Given any integer p we can form a function $\phi_p(x)$ by writing

$$f_0(x) \sim \sum_{-\infty}^{\infty} a_n e^{in\pi x}, \quad \phi_p(x) \sim \sum_{-\infty}^{\infty} a_{np} e^{inp\pi x}.$$

Since $a_0 = 0$, $M\{\phi_p^2\} \to 0$ as $p \to \infty$, and we can, therefore, choose the primes $p(n)$ all different and so that

$$a_n^2 M\{\phi_{p(n)}^2\} < \epsilon_n^3 2^{-\rho-1}$$

for $r < n \leqslant r+\rho$. The integers $q(n)$ will be chosen later increasing so rapidly that (5) is satisfied; then (4) will converge in mean to an *S_2-grenz-periodisch* function $f(x)$. If we form the Bochner-Fejér sum

$$\sigma_{P,Q}(x) = M\left\{f(x+t)\, K_Q\left(\frac{t}{P}\right)\right\},$$

where

$$K_m(t) = \sum_{|\mu|<m} \left(1 - \frac{|\mu|}{m}\right) e^{i\mu t\pi} = \frac{1}{m}\left(\frac{\sin \frac{1}{2}mt}{\sin \frac{1}{2}t}\right)^2,$$

then only those terms of (4) will contribute for which $q(n) < Q$, and unless

* H. Bohr, *Acta Math.*, 46 (1925), 101–214 (141).

$p(n)$ divides P the term $a_n f_0(\lambda_n x)$ will only contribute the same as would $a_n \phi_{p(n)}(\lambda_n x)$. In this case the contribution of the term to $\sigma_{P,Q}(x)$ does not exceed ϵ_n in absolute value except in a set of mean measure less than $\epsilon_n 2^{-\rho-1}$, and therefore in any interval of length 1 does not exceed ϵ_n except in a set of measure less than $\epsilon_n 2^{-\rho}$, provided that $\lambda_n > 1$ as we shall suppose. On the other hand, if $p(n)$ does divide P, and if $Q/Pq(n)$ is sufficiently great, then in any interval of length 1 the contribution of $a_n f_0(\lambda_n x)$ differs from the function by less than ϵ_n except in a set of measure less than $\epsilon_n 2^{-\rho}$.

Suppose that the numbers $q(n)$ have been chosen for $n \leqslant r+\rho$, then we form a batch of 2^ρ sums $\sigma_{P,Q}(x)$ combining these events for $r < n \leqslant r+\rho$ in every possible way. The later numbers $q(n)$ will all be chosen so as to exceed all these numbers Q just chosen; accordingly this batch of sums $\sigma_{P,Q}(x)$ will contain no contributions from terms of (4) for which $n > r+\rho$, *i.e.* for which $q(n)$ has not yet been chosen. If for each value of x we take the maximum of these sums $\sigma_{P,Q}(x)$ we see that in any interval of length 1 it exceeds

$$\nu - \eta_r + (g_{r+1} - \tfrac{1}{2}|a_{r+1}|) + (g_{r+2} + \tfrac{1}{2}|a_{r+2}|) + \ldots + (g_{r+\rho} - \tfrac{1}{2}|a_{r+\rho}|),$$

save in a set of measure less than η_r.

We choose the numbers $q(n)$ and the sums $\sigma_{P,Q}(x)$ alternately in batches. When choosing the numbers $q(r+1)$, $q(r+2)$, ..., $q(r+\rho)$ we require that $q(n) > p(n)$, that $q(n)$ is prime to $p(n)$, that (5) is satisfied, and that $q(n)$ exceeds all numbers Q so far chosen. When choosing the corresponding batch of 2^ρ sums $\sigma_{P,Q}(x)$ we require that the various values of P shall possess the arithmetical property described with respect to $p(r+1)$, $p(r+2)$, ..., $p(r+\rho)$, and that Q/P shall be sufficiently great; in order that the single infinite sequence formed by combining all these batches of sums $\sigma_{P,Q}(x)$ shall converge formally to $f(x)$ we require also that throughout the ν-th batch $Q > \nu P$ and P is a multiple of $p(1)\,p(2)\ldots p(r)$.

4. We shall now prove that even when the basis is both finite and integral the Bochner-Fejér sums do not necessarily converge p.p. The result of this section is therefore stronger than that of § 2, but the latter has, perhaps, some interest of its own as well as forming a natural introduction to § 3.

Given a continuous a.p. function possessing an integral basis of two members only,

$$f(x) \sim \Sigma a_{m,n} e^{i(m\alpha+n\beta)x}, \tag{8}$$

we know that the sequence $\{f(x+\tau_n)\}$ is uniformly convergent if (and only if) it is formally convergent; thus, by Dirichlet's theorem on Diophantine approximations, we can obtain as the limit of such a sequence an a.p. function

$$f_c(x) \sim \Sigma a_{m,n} e^{i(m\alpha+n\beta)x+in\beta c}, \tag{9}$$

and that for any assigned value of c. If we define $f(x, y)$ by the equation

$$f(x, x+c) = f_c(x), \tag{10}$$

we obtain a continuous function periodic in each variable; and

$$f(x, y) \sim \Sigma a_{m,n} e^{i(m\alpha x+n\beta y)}, \tag{11}$$

since the Bochner-Fejér sums of $f_c(x)$ converge to the function uniformly in c and x. Conversely, if $f(x, y)$ is a continuous function periodic in both x and y with incommensurable periods $2\pi/\alpha$ and $2\pi/\beta$, then τ is a translation number of $f(x, x)$ provided that $\tau\alpha$ and $\tau\beta$ are nearly multiples of 2π; hence $f(x, x)$ is continuous a.p. and by Dirichlet's theorem possesses a Fourier series of type (8). A further application of Dirichlet's theorem shows that when we construct as above a continuous function periodic in two variables from the a.p. function $f(x, x)$ we get back to $f(x, y)$.

In just the same way when an S-a.p. function (8) is given, we know that the sequence $\{f(x+\tau_n)\}$ is S-convergent if (and only if) it is formally convergent, and we can obtain as the S-limit of such a sequence an S-a.p. function (9) whatever be the value of c. $f_c(x)$ can thereby be altered at will in a set of measure zero, but we can so choose the determination of these functions that (10) gives a function $f(x, y)$ which is periodic in each variable. The latter has a Fourier series given by (11) since the Bochner-Fejér sums of $f_c(x)$ are S-convergent to the function uniformly in the parameter c. It also possesses a property which we shall refer to as C and which may be expressed as follows: each function $f(x, x+c)$ is of class L and every sequence $\{f(x, x+c_n)\}$ of such functions is S-convergent for which $\{c_n\}$ is convergent. Conversely, if $f(x, y)$ is periodic in each variable with incommensurable periods $2\pi/\alpha$ and $2\pi/\beta$ and possesses the property C, then τ is an S-translation number of $f(x, x)$ provided that $\tau\alpha$ and $\tau\beta$ are nearly multiples of 2π; hence $f(x, x)$ is S-a.p. and by Dirichlet's theorem possesses a Fourier series of the type (8). A further application of Dirichlet's theorem shows that when we construct as above a function periodic in two variables and possessing C from the S-a.p. function $f(x, x)$ we get back to $f(x, y)$.

Throughout the above L, S, and C may be replaced by L^r, S_r, and C_r with any value of $r>1$. In the case $r<1$ we observe that, if a given S_r-a.p. function $f(x)$ is the S_r-limit of a sequence of trigonometric polynomials $s(x)$ whose exponents are all of the form $m\alpha+n\beta$, α and β being incommensurable, then* the bounded function $f_N(x)$† is the S_r-limit and hence the S-limit of the sequence of continuous a.p. functions $s_N(x)$ whose exponents are all of the form $m\alpha+n\beta$. From $f_N(x)$ we determine as above a function $f_N(x, y)$ of two variables possessing the property C and bounded by N. When $N\to\infty$ the functions $f_N(x, x+c)$ are S_r-convergent uniformly in c: this follows from the general principle of S_r-convergence combined with the facts (i) that $f_N(x, x+c)$ is the S-limit and therefore also the S_r-limit of a sequence $\{f_N(x+\tau_n)\}$ and (ii) that $f_N(x)$ is S_r-convergent to $f(x)$ as $N\to\infty$. The limit functions give us a function $f(x, y)$ periodic both in x and in y and possessing the property C_r: it is now easily seen that if $\tau_n\alpha\to 0$ and $\tau_n\beta\to c\beta$ (mod 2π) as $n\to\infty$ then $\{f(x+\tau_n)\}$ is S_r-convergent and to the limit $f(x, x+c)$. Conversely, if $f(x, y)$ is periodic both in x and in y with incommensurable periods $2\pi/\alpha$ and $2\pi/\beta$, and if it possesses the property C_r, then $f_N(x, y)$ possesses C and $f(x, x)$ is the S_r-limit of a sequence of trigonometric polynomials whose exponents are all of the form $m\alpha+n\beta$: and from the S_r-a.p. function $f(x, x)$ we may reconstruct $f(x, y)$ as above.

We shall now construct a function $f(x, y)$, periodic in both x and y with incommensurable periods $2\pi/\alpha$ and $2\pi/\beta$ and possessing the property C, whose Fejér sums diverge at all points of the segment Δ of the line $y=x$ inside the fundamental rectangle

$$0\leqslant x\leqslant\frac{2\pi}{\alpha},\quad 0\leqslant y\leqslant\frac{2\pi}{\beta},$$

but converge elsewhere in the rectangle. Suppose for definiteness that $\alpha>\beta$ and put $f(x, y)=0$ at all points of the rectangle for which $x\geqslant y$. Elsewhere we shall make $f(x, y)$ non-negative, and it will be continuous at every point of the rectangle except those on Δ; to ensure its continuity on the boundary we shall make it vanish there, but we shall be concerned in the sequel only with the values of $f(x, y)$ in a narrow band bounded below by Δ. Subject to these conditions $f(x, y)$ will possess the property C provided that

$$\int_\Delta f(x, c+x)\,dx\to 0\quad\text{as}\quad c\to+0.$$

For each integer n we divide Δ into a set Δ_n of equal intervals $\Delta_{n,k}(k\leqslant n)$ and we denote by D_n and $D_{n.k}$ the projections of Δ_n and of $\Delta_{n,k}$ on the axis of x. Corresponding to Δ_n we shall choose an interval $\delta_n=(c_{n+1}, c_n)$ divided into subintervals $\delta_{n,k}=(c_{n,k+1}, c_{n,k})$. When c belongs to $\delta_{n,k}$

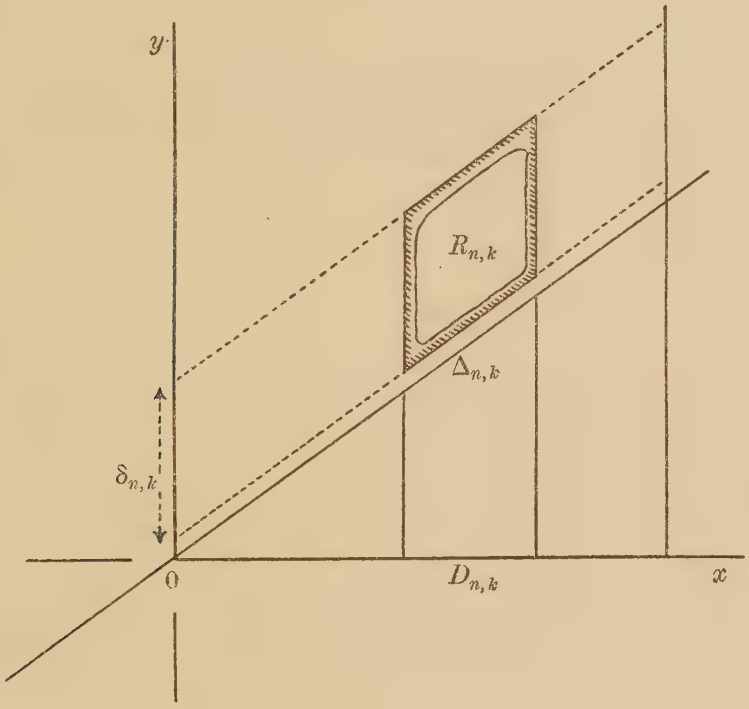
* In this case τ is certainly an S_r-translation number of $f(x)$ if $\tau\alpha$ and $\tau\beta$ are both sufficiently small, mod 2π. Conversely, if this is the case, then $f_N(x)$ has the same property and is the S-limit of a sequence of trigonometric polynomials whose exponents are all of the form $(m\alpha+n\beta)$. Since $f(x)$ is S_r-a.p. and therefore the S_r-limit of $f_N(x)$ as $N\to\infty$, it is also the S_r-limit of a sequence of trigonometric polynomials of the type specified.

† We no longer require the notation $f_c(x)$ in the sense introduced above.

we put $f(x, c+x) = 0$ except when x belongs to $D_{n,k}$, and here we make $f \leqslant \sqrt{n}$. Thus in δ_n we have

$$\int_{\Delta} f(x, c+x)\,dx \leqslant \frac{2\pi}{a\sqrt{n}}.$$

We shall, of course, make c_n decrease to zero as $n \to \infty$ and accordingly $f(x, y)$ will possess the property C.



When x varies in $D_{n,k}$, and c varies in $\delta_{n,k}$, the point $(x, c+x)$ varies in a parallelogram $R_{n,k}$; we shall make $f(x, y) = \sqrt{n}$ throughout $R_{n,k}$ except for a narrow band round the edge, across which $f(x, y)$ changes continuously from the central value $\sqrt{n}$ to the value 0 which it takes all round the boundary of $R_{n,k}$. Now, given $c_{n,k}$, it is clear that, if $c_{n,k+1}$ be chosen sufficiently small and if the boundary strip be sufficiently narrow, there will be an "advanced" Fejér sum of $f(x, y)$ which exceeds $\sqrt{n}/9$ throughout $\Delta_{n,k}$.

Choosing the numbers $c_{n,k}$ in their appropriate order in this manner, we obtain at the same time a sequence of Fejér sums of the resulting

$f(x, y)$ which converges formally to $f(x, y)$ but diverges at every point of Δ.

Minor alterations in the above construction will make the Fejér sums diverge only at the points of an arbitrary closed set of points on the segment Δ, and by addition of such functions we can make the Fejér sums diverge in an arbitrary closed set of points on the line $y = x$ (and converge elsewhere on this line).

The transition from periodic functions of two variables to S-a.p. functions possessing an integral basis of two members is made as described at the beginning of this section.

It is not difficult to see that all the results of §§ 2–4 concerning summability $(C, 1)$ of the Fourier series of an S-a.p. function may be extended to summability (C, k) and summability by Abel's method.

ON THE SUMMABILITY OF POWER SERIES

By A. C. OFFORD.

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1. *Introduction.* The Cesàro summation of the power series

$$f(z) = \sum_0^\infty a_n z^n, \tag{1}$$

where

$$a_n = o(n^\gamma) \quad (\gamma > -1), \tag{2}$$

on its circle of convergence has been investigated by Dienes*, Hardy and Littlewood†, M. Riesz‡, Zygmund§, and others. For simplicity I take the circle of convergence to have unit radius. In this paper I shall adopt the procedure used, for example, by Hardy and Littlewood, and by M. Riesz ‖ in the first proof of his extension of Fatou's theorem¶ that a power series is summable (C, γ) at every regular point of the circle of convergence. It is a "complex variable" method, and consists in obtaining a summation formula from Cauchy's theorem.

Let K denote a constant (*i.e.* always independent of the variables n, z), not necessarily the same in each connection, and write

$$\sum_0^\infty S_n^\gamma(\theta_0)\, z^n = (1-z)^{-\gamma-1} \sum_0^\infty a_n\, e^{in\theta_0} z^n = \frac{f(ze^{i\theta_0})}{(1-z)^{\gamma+1}} \tag{3}$$

and

$$(1-z)^{-\gamma-1} = \sum_0^\infty C_n^{\gamma+n} z^n.$$

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* Dienes, **3**, **4**.

† Hardy and Littlewood, **7**.

‡ Riesz, **13**, **14**, **15**, **16**.

§ Zygmund, **19**.

‖ Riesz, **14**, for $(\gamma > 0)$. Zygmund, **19**, 91, for $(\gamma > -1)$. Riesz's later proof is in **15**, **16**, and Landau, **11**, 73.

¶ Fatou, 5, 389.

Then, by definition*, the series

$$\sum_0^\infty a_n e^{in\theta_0} \tag{4}$$

is summable (C, γ) to $S(\theta_0)$ if

$$S_n^\gamma(\theta_0) \sim C_n^{\gamma+n} S(\theta_0).$$

The series is bounded (C, γ) if

$$|S_n^\gamma(\theta_0)| < K C_n^{\gamma+n}.$$

Let C denote the contour $|z| = r = 1-1/n$; then from (3) we have, by Cauchy's theorem,

$$S_n^\gamma(\theta_0) = \frac{1}{2\pi i}\int_C \frac{f(ze^{i\theta_0})}{(1-z)^{\gamma+1}z^{n+1}}\,dz.$$

If $f(z) = 1$ this expression becomes

$$C_n^{\gamma+n} = \frac{1}{2\pi i}\int_C \frac{dz}{(1-z)^{\gamma+1}z^{n+1}}.$$

Since $S(\theta_0)$ is a constant, we may multiply through by it and obtain

$$S_n^\gamma(\theta_0) - C_n^{\gamma+n} S(\theta_0) = \frac{1}{2\pi i}\int_C \frac{f(ze^{i\theta_0}) - S(\theta_0)}{(1-z)^{\gamma+1}z^{n+1}}\,dz,$$

i.e.

$$\frac{S_n^\gamma(\theta_0)}{C_n^{\gamma+n}} - S(\theta_0) = \frac{e^{i(\gamma+n+1)\theta_0}}{2\pi i\, C_n^{\gamma+n}}\int_C \frac{f(z) - S(\theta_0)}{(e^{i\theta_0}-z)^{\gamma+1}z^{n+1}}\,dz. \tag{5}$$

Now

$$C_n^{\gamma+n} \sim K n^\gamma, \tag{6}$$

and so the necessary and sufficient condition for (4) to be summable (C, γ) to $S(\theta_0)$ is that the integral on the right-hand side of (5) shall be $o(n^\gamma)$. It is obvious that the corresponding condition for summability $(C, \gamma+\delta)$ $\delta > 0$ is obtained by simply replacing γ by $\gamma+\delta$.

I propose to deduce from (5) a criterion requiring only a knowledge of the behaviour of $f(z)$ in the immediate neighbourhood of $e^{i\theta_0}$. First notice that by (1) and (2) we can choose n_0 so that

$$\begin{aligned}
|f(z)| &\leqslant K(n_0) + o\left\{\sum_{n_0}^\infty n^\gamma |z|^n\right\} \\
&\leqslant K + o\left\{\sum_{n_0}^\infty C_n^{\gamma+n}|z|^n\right\} \quad \text{by (6)} \\
&= o(1-|z|)^{-\gamma-1} \\
&= o(n^{\gamma+1}) \quad \text{on } C.
\end{aligned} \tag{7}$$

* Borel, **2**, 92. Hobson, **8**, 2, 70.

Now let η be any positive number, however small, which is independent of n. Divide the contour C into two parts, C_1, for which $|\arg z| \leqslant \eta$, and C_2, and the integral in (5) into two corresponding parts I_1 and I_2. Then, if it can be proved that I_2 is $o(n^\gamma)$, it follows that $I_1 = o(n^\gamma)$ is a necessary and sufficient condition for summability (C, γ). and constitutes a local convergence criterion. This can be accomplished as regards summability $(C, \gamma+1)$ by using the relation (7), in which case the appropriate part of the integral which takes the place of that in (5) gives

$$|I_2'| = \left| \int_{C_2} \frac{f(z) - S(\theta_0)}{(e^{i\theta_0} - z)^{\gamma+2} z^{n+1}} \, dz \right| = \int_{C_2} o(n^{\gamma+1}) \, |dz| = o(n^{\gamma+1}),$$

which is the required result. The foregoing proof I have taken from an unpublished paper by Bosanquet. M. Riesz*, in his extension of Fatou's theorem, makes a more direct application of the condition (2), and by following his line of argument, I prove in Theorem 1 of § 2 that an integral of very much the same form as I_2 is $o(n^\gamma)$, and so obtain a criterion for summability (C, γ) of the required form.

In § 3 I obtain three theorems as simple deductions from Theorem 1. Theorem 3 states that, if $f(z)$ is bounded within a sector $(\alpha \leqslant \theta \leqslant \beta)$, then at any point of (α, β) the boundary series is either summable (C, γ) or not summable (A). This has been proved by Andersen† for the case of a function bounded within the whole circle, when the series is summable (C, δ) $\delta > 0$ or not summable (A).

An alternative procedure‡ is to express the series (1) as the sum of two harmonic series of the form

$$u(r, \theta) = \tfrac{1}{2}a_0 + \sum_1^\infty (a_n \cos n\theta + \beta_n \sin n\theta) r^n, \tag{8}$$

where $$a_n = o(n^\gamma), \quad \beta_n = o(n^\gamma),$$

and the results are then deduced from theorems on trigonometrical series. The following theorem has been stated by M. Riesz§ as an extension of Fatou's theorem||.

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* Riesz, **14**.

† Andersen, **1**. See also Hardy and Littlewood, **7**, 83.

‡ See, for example, Young, **18**, and Zygmund, **19**.

§ Riesz, **13**.

|| Fatou, *loc. cit.*

THEOREM A. *If*

(i) $\lim_{r\to 1} u(r, \theta) = U(\theta)$ *exists at almost all points of the arc* (α, β),

(ii) $U(\theta)$ *is integrable in Lebesgue's sense in* (α, β),

(iii) *we have uniformly, for* $\alpha \leqslant \theta \leqslant \beta$,

$$\lim_{r\to 1} \int_\alpha^\theta u(r, \theta)\, d\theta = \int_\alpha^\theta U(\theta)\, d\theta,$$

then the series (8) *for* $r = 1$ *behaves* (C, γ) *in exactly the same way as the Fourier series of the function equal to* $U(\theta)$ *in* (α, β) *and zero outside* (α, β).

This theorem may be proved by using W. H. Young's theory of restricted Fourier series* or independently. After I had obtained the results of § 3, Profs. G. H. Hardy and A. Zygmund kindly pointed out that they can be deduced, even for harmonic series, from Theorem A. Since, however, no full proof of Theorem A has, so far as I know, been published, I have thought it worth while to give my more direct proofs.

Finally, I must express my thanks to Dr. L. S. Bosanquet for his valuable suggestions and criticisms.

2. THEOREM 1. *If* $f(z) = \Sigma a_n z^n$, *where* $a_n = o(n^\gamma)$, $\gamma > -1$ *is regular for* $|z| < 1$, *and if* η *is any positive number, however small, then the necessary and sufficient condition that the series* (4) *shall be summable* (C, γ) *to* $S(\theta_0)$ *is that, if* $z = (1-1/n)\, e^{i\theta}$, *then*

$$\int_{\theta=\theta_0-\eta}^{\theta_0+\eta} \{f(z) - S(\theta_0)\}\{(e^{i\theta_0} - z)^{-\gamma-1} + c_0 + c_1 z\} z^{-n-1}\, dz = o(n^\gamma), \qquad (9)$$

where c_0 *and* c_1 *are certain constants.*

If the symbol o be replaced by O throughout, we obtain the necessary and sufficient condition for boundedness (C, γ). The proof for summability (C, γ) requires only a trivial modification to obtain the corresponding proof for boundedness, but in this case there is no need to introduce the term $S(\theta_0)$.

It is hardly necessary to remark that we can obtain the necessary and sufficient condition for the series to be summable $(C, \gamma+\delta)$, $\delta > 0$, by

* Young, **17**. See Zygmund, **19**, foot-note [56].

simply replacing γ by $\gamma+\delta$ in (9), for it is not assumed in the hypothesis that $a_n = o(n^\gamma)$ is the best possible condition on a_n, and so if the theorem is true for γ it is true for any constant greater than γ.

In all that follows, K will be used to denote a constant depending on the parameters γ, θ_0, η but independent of n, z; it is not always the same constant.

Proof. By hypothesis η is fixed independently of n, however small it is taken. Hence, writing

$$Q(z) = (e^{i\theta_0}-z)^{-\gamma-1}+c_0+c_1 z, \tag{10}$$

we can choose c_0, c_1 so that

$$Q(e^{i(\theta_0+\eta)}) = Q(e^{i(\theta_0-\eta)}) = 0. \tag{11}$$

Then c_0, c_1 depend upon η but are independent of n. As in § 1, let C be the contour $|z| = r = 1-1/n$.

Then

$$\int_C \frac{f(z)-S}{z^{n+1}}(c_0+c_1 z)\,dz = c_0 a_n + c_1 a_{n-1} = o(n^\gamma), \tag{12}$$

and, therefore, combining (10) and (12) with (5), we obtain a necessary and sufficient condition for summability (C, γ), viz.

$$\int_C \frac{f(z)-S(\theta_0)}{z^{n+1}}\,Q(z)\,dz = o(n^\gamma). \tag{13}$$

Now divide the contour C into two parts, C_1, for which $|\arg z| < \eta$, and C_2. Then the integral along C_1 is the expression in (9). Hence to complete the proof we have to prove that the integral along C_2 is $o(n^\gamma)$.

Put $$z_1 = re^{i(\theta_0+\eta)}, \quad z_2 = re^{i(\theta_0-\eta)};$$

then, from (11),

$$\frac{Q(z_1)}{z_1-e^{i(\theta_0+\eta)}} \to Q'(e^{i(\theta_0+\eta)}) \quad \text{as} \quad r\to 1.$$

So therefore

$$\left.\begin{aligned} &|Q(z_1)| < K(1-r) = K/n \\ \text{and similarly} \quad &|Q(z_2)| < K/n \end{aligned}\right\}. \tag{14}$$

Let z_0 be some fixed point inside the circle, but not the origin, and then write

$$\left.\begin{aligned} F_1(z)-F_1(z_0) &= \int_{z_0}^{z} \frac{f(z)-S}{z^{n+1}}\,dz \\ F_2(z)-F_2(z_0) &= \int_{z_0}^{z} F_1(z)\,dz \end{aligned}\right\}. \tag{15}$$

Using (7) we get, if z is on C,

$$F_1(z) = o(n^{\gamma+1}). \tag{16}$$

Now, returning to the integral (13) taken along the contour C_2, and integrating it twice by parts, we get

$$\begin{aligned} I &= \left| \int_{z_1}^{z_2} \frac{f(z)-S}{z^{n+1}}\, Q(z)\,dz \right| \\ &\leqslant \left| \Big[F_1(z)\, Q(z) \Big] \right| + \left| \Big[F_2(z)\, Q'(z) \Big] \right| + \left| \int_{z_1}^{z_2} F_2(z)\, Q''(z)\,dz \right|. \end{aligned} \tag{17}$$

By (14) and (16)

$$\begin{aligned} F_1(z_1)\, Q(z_1) &= o(n^{\gamma+1}) \,.\, O(n^{-1}) \\ &= o(n^{\gamma}), \end{aligned}$$

and similarly $$F_1(z_2)\, Q(z_2) = o(n^{\gamma}).$$

Now $Q'(z)$ and $Q''(z)$ are bounded on C_2, so that, by (17), it only remains to prove that $F_2(z) = o(n^{\gamma})$ on C_2.

The method is to expand $F_2(z)$ as an infinite series and substitute $a_n = o(n^{\gamma})$, but the Laurent terms at the beginning of the series make the work longer than in the proof of (7).

If now a_0 is written in place of $a_0 - S$, then, from (15),

$$F_1(z)-F_1(z_0) = \sum_{\nu=0}^{\infty} a_\nu \int_{z_0}^{z} z^{\nu-n-1}\,dz;$$

so write

$$F_1(z) = \sum_{\substack{\nu=0 \\ \nu \neq n}}^{\infty} \frac{a_\nu}{\nu-n}\, z^{\nu-n} + a_n \log \frac{z}{z_0} = \sum_{\substack{\nu=0 \\ \nu \neq n}}^{\infty} \frac{a_\nu}{\nu-n}\, z^{\nu-n} + o(n^{\gamma}),$$

and $$F_2(z) = \sum_{\substack{\nu=0, \\ \nu \neq n, \\ n-1}}^{\infty} \frac{a_\nu}{(\nu-n)(\nu-n+1)}\, z^{\nu-n+1} + o(n^{\gamma}).$$

Now, given ϵ, we can find ν_0 so that, for all $\nu > \nu_0$, $|a_\nu| < \epsilon\nu^\gamma$, while, for $\nu \leqslant \nu_0$, $|a_\nu| < M$ say.

Let n be chosen so that $n-2 \geqslant \nu_0+1$. Then

$$|F_2(z)| \leqslant o(n^\gamma) + \left\{ \sum_{\nu=0}^{\nu_0} + \sum_{\nu_0+1}^{n-2} + \sum_{n+1}^{2n} + \sum_{2n}^{\infty} \right\} \frac{|a_\nu| r^{\nu-n+1}}{|\nu-n||\nu-n+1|}$$

$$= o(n^\gamma) + \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4.$$

We shall now make use of the facts that

$$r^{-n} = \left(1-\frac{1}{n}\right)^{-n} \to e,$$

and that

$$r^{-n} < e\frac{n}{n-1} < 2e.$$

Then

$$\Sigma_1 = \sum_{\nu=0}^{\nu_0} \frac{|a_\nu| r^{\nu-n+1}}{(n-\nu)(n-\nu-1)} < \frac{2eM(\nu_0+1)}{(n-\nu_0)(n-\nu_0-1)} = o\left(\frac{1}{n}\right),$$

and

$$\Sigma_2 = \sum_{\nu_0+1}^{n-2} \frac{|a_\nu| r^{\nu-n+1}}{(n-\nu)(n-\nu-1)}$$

$$< 2e \sum_{1}^{n-\nu_0-2} \frac{|a_{\nu+\nu_0}|}{(n-\nu-\nu_0)(n-\nu-\nu_0-1)}$$

$$= 2e \sum_{1}^{n-\nu_0-2} \frac{|a_{n-\nu-1}|}{\nu(\nu+1)}. \tag{18}$$

First consider the case in which $\gamma \geqslant 0$; then $|a_{n-\nu-1}| < \epsilon n^\gamma$, since $\nu_0 < n-\nu-1 < n$, so that

$$\Sigma_2 = o(n^\gamma) \sum_{1}^{\infty} \frac{1}{\nu(\nu+1)} = o(n^\gamma).$$

Again, when $-1 < \gamma < 0$, write $\gamma = -\delta$, so that $0 < \delta < 1$, and observe that, if $a \geqslant 2$ and $b \geqslant 2$, then $ab \geqslant a+b$, *i.e.* $a^{-\delta} \leqslant b^\delta (a+b)^{-\delta}$.

Now let $a = n-\nu-1$ and $b = \nu+1$; then, since $\nu_0 < n-\nu-1 < n-1$, we have

$$|a_{n-\nu-1}| < \epsilon(n-\nu-1)^{-\delta} \leqslant \epsilon n^{-\delta}(\nu+1)^\delta.$$

Therefore, continuing from (18),

$$\Sigma_2 = o(n^{-\delta}) \sum_{1}^{n-\nu_0-2} \frac{(\nu+1)^\delta}{\nu(\nu+1)} = o(n^{-\delta}) \sum_{1}^{\infty} \frac{1}{\nu^{2-\delta}} = o(n^\gamma).$$

Again, if $\gamma \geqslant 0$, $$\Sigma_3 = \sum_{n+1}^{2n} \frac{|a_\nu| r^{\nu-n+1}}{(\nu-n)(\nu-n+1)}$$

$$< \epsilon(2n)^\gamma 2e \sum_{n+1}^{\infty} \frac{1}{(\nu-n)(\nu-n+1)}$$

$$= o(n^\gamma) \sum_{1}^{\infty} \frac{1}{\nu(\nu+1)}$$

$$= o(n^\gamma).$$

In the case $\gamma < 0$ we have only to omit the factor 2^γ. Finally

$$\Sigma_4 = \sum_{2n+1}^{\infty} \frac{|a_\nu| r^{\nu-n+1}}{(\nu-n)(\nu-n+1)} = \sum_{n+1}^{\infty} \frac{|a_{\nu+n}| r^{\nu+1}}{\nu(\nu+1)},$$

and this is a Taylor series whose coefficients are $o(\nu^{\gamma-2})$. Therefore, by the proof of (7),

$$\Sigma_4 = o(n^{\gamma-1}).$$

Thus $F_2(z) = o(n^\gamma)$ on C_2, and the proof is complete.

3. The following results are deductions from the foregoing theorem; but here we shall make the hypothesis $\gamma > 0$.

THEOREM 2. *If $f(z) = \Sigma a_n z^n$, where $a_n = o(n^\gamma)$, $\gamma > 0$, and if $f(z)$ tends to S as z tends to 1 along any path from within the circle $|z| = 1$, then the series Σa_n is summable (C, γ) to S.*

THEOREM 3. *If $f(z) = \Sigma a_n z^n$, where $a_n = o(n^\gamma)$, $\gamma > 0$, and if $f(re^{i\theta})$ is bounded within the sector $(\alpha \leqslant \theta \leqslant \beta, 0 \leqslant r < 1)$, then the series $\Sigma a_n e^{in\theta}$ is summable (C, γ) whenever it is summable (A). The series will be summable (C, γ) for almost all θ in (α, β).*

THEOREM 4. *If $f(z) = \Sigma a_n z^n$, where $a_n = o(n^\gamma)$, $\gamma > 0$, and if*

$$|f(re^{i\theta})| < \chi(\theta) \quad (\alpha \leqslant \theta \leqslant \beta),$$

where $\chi(\theta)$ is integrable in Lebesgue's sense in (α, β), then the series $\Sigma a_n e^{in\theta}$ is bounded (C, γ) at all those points of (α, β) at which the four derivatives of the indefinite integral of $\chi(\theta)$ are finite. The series will be summable (C, γ) at a point of this set if it is summable (A), and it will be summable (C, γ) at almost all points of (α, β).

Theorem 2 has been proved by Dienes* for the case $\gamma \geqslant 1$. It is known, and easily shown, that, if $f(z)$ tends to S as z tends to 1 along any path from within the circle, there is a sector enclosing $z=1$ within which $f(z)$ is bounded†. Hence Theorem 2 is included in Theorem 3. The three following theorems are required, the first of which is Montel's‡, the second Privaloff's extension§ of Fatou's theorem‖, and the third is Hölder's extension¶ of Frobenius' theorem.

MONTEL'S THEOREM. *If $f(z)$ is regular and bounded within the angle AOB, and if $f(z)$ tends to a limit as z tends to O along any one line lying inside the angle, then $f(z)$ tends to a limit as z tends to O uniformly within any angle which is contained in AOB.*

PRIVALOFF'S THEOREM. *If $f(z)$ is regular within the circle $|z|<1$, and if E is a set of points of positive measure situated on the circumference $|z|=1$, then a necessary and sufficient condition that, for almost all points z_0 of E, $f(z)$ shall tend to a unique limit as z tends to z_0 along any path which does not touch the circumference, is that at every point z_0 there is an angle with vertex z_0, symmetrical about $(0, z_0)$, within which $f(z)$ is bounded.*

HÖLDER'S THEOREM. *If $f(z)=\sum_0^\infty a_n z^n$ is regular for $|z|<1$, and if $\Sigma a_n z_0^n$, $|z_0|=1$ is bounded (C, γ), then, for all values of z near $z=z_0$ and within an angle $(-\frac{1}{2}\pi+\delta, \frac{1}{2}\pi-\delta)$ $(\delta>0)$, with vertex $z=z_0$, symmetrical about $(0, z_0)$, the function $f(z)$ is bounded.*

Proof of Theorem 3. Since $f(z)$ is bounded and c_0, c_1 are independent of n, we have

$$\left|\int_{-\eta}^{\eta} \frac{f(z)-S}{z^{n+1}}(c_0+c_1 z)\,d\theta\right| < K.$$

Hence, by Theorem 1, it is sufficient to prove that, when $z=(1-1/n)e^{i\theta}$,

$$I=\left|\int_{\theta_0-\eta}^{\theta_0+\eta} \frac{f(z)-S(\theta_0)}{(e^{i\theta_0}-z)^{\gamma+1}z^n}\,d\theta\right| = o(n^\gamma),$$

* Dienes, **3**, **4**, 59, and **20**, 475. Ferenczi, **21**, has given Theorem 3 in the case $\gamma>1$.

† Dienes, **4**, 39, and **20**, 474.

‡ Julia, **10**, 97.

§ Privaloff, **12**, 92.

‖ Fatou, **5**, 337, and Landau, **11**, 40.

¶ Borel, **2**, 107, where the theorem is stated and proved for summability. The argument however, applies equally well to the case of boundedness. See also Dienes, **20**, 439.

on the hypothesis that $f(z) \to S(\theta_0)$ as $z \to e^{i\theta_0}$. For simplicity we take $\theta_0 = 0$ and we need only consider the range $(0, \eta)$.

Then
$$I < K \int_0^\eta \frac{|f(z)-S|}{(1-2r\cos\theta+r^2)^{\frac{1}{2}(\gamma+1)}}\, d\theta \quad (r = 1-1/n)$$
$$< K \left\{ \int_0^{A/n} + \int_{A/n}^{\eta} \frac{|f(z)-S|}{(n^{-2}+\theta^2)^{\frac{1}{2}(\gamma+1)}}\, d\theta \right\}$$
$$= I_1 + I_2.$$

A is a constant which has yet to be determined, and n is taken to be greater than A/η.

Now $f(z)$ is bounded within the sector, so that

$$I_2 < Kn^\gamma \int_{A/n}^{\eta} \frac{n\, d\theta}{(1+n^2\theta^2)^{\frac{1}{2}(\gamma+1)}} < Kn^\gamma \int_A^\infty \frac{d\theta}{(1+\theta^2)^{\frac{1}{2}(\gamma+1)}},$$

and, since $\gamma > 0$, the integral is convergent. Therefore, by taking A great enough, we can make the integral less than any given ϵ,

i.e. $$I_2 = o(n^\gamma).$$

Now write $$A = \tan(\tfrac{1}{2}\pi - \delta);$$

since A is fixed, δ will also be fixed and positive. The arc $(0, A/n)$ on $r = 1-1/n$ lies inside the angle, vertex $z = 1$, enclosed by the real axis and a line making an angle $\frac{1}{2}\pi - \delta$ with it. Thus, by Montel's theorem, $f(z)$ tends uniformly to S as z tends to 1 within this angle,

i.e. $$|f(re^{i\theta}) - S| < \epsilon, \quad \text{if} \quad \theta \leqslant A/n \quad \text{and} \quad r_0 < r < 1.$$

Therefore

$$I_1 < \epsilon \int_0^{A/n} \frac{d\theta}{(n^{-2}+\theta^2)^{\frac{1}{2}(\gamma+1)}} = o(n^\gamma) \int_0^\infty \frac{d\theta}{(1+\theta^2)^{\frac{1}{2}(\gamma+1)}} = o(n^\gamma).$$

This proves the first part of the theorem; the second part, that the series is summable (C, γ) at almost all points of (α, β), now follows from Privaloff's theorem. We notice that the condition that the series shall be summable (A), *i.e.* that $f(z)$ tends to a limit radially, may be replaced, in virtue of Montel's theorem, by the condition that $f(z)$ tends to a limit along a path which cuts the circumference. This remark applies also to Theorem 4.

Proof of Theorem 4. Let E be the set of those points of (α, β) at which the four derivatives of the indefinite integral of $\chi(\theta)$ are finite. I shall prove the theorem in two steps, viz. (i) the series is bounded (C, γ) at all points of E, (ii) the series is summable (C, γ) at a point of E if it is summable (A).

Now, by combining Hölder's theorem with Privaloff's theorem, we see that, if the series is bounded (C, γ) at all points of E, it will be summable (A) at almost all points of E, and hence by (ii) it will be summable (C, γ) at almost all points of E. It is known* that the measure of the set E is equal to the length of the arc (α, β), and so the series will be summable (C, γ) at almost all points of (α, β).

Let θ_0 be a point of E and, as in the proof of Theorem 3, for simplicity, we may suppose that $\theta_0 = 0$. Also, since the addition of a constant makes no difference to the behaviour of $\chi(\theta)$, we may take

$$|f(re^{i\theta}) - S| < \chi(\theta).$$

Write
$$\Phi(\theta) = \int_0^\theta \chi(\theta)\, d\theta; \tag{19}$$

then, by hypothesis, $\Phi(\theta)$ has all its derivatives finite at $\theta = 0$, and so, for $-\eta \leqslant \theta \leqslant \eta$, we have

$$\left|\frac{\Phi(\theta)}{\theta}\right| < M, \tag{20}$$

where M is a constant.

(i) The series Σa_n is bounded (C, γ). Now

$$\left|\int_{-\eta}^{\eta} \frac{f(z)-S}{z^{n+1}}(c_0+c_1 z)\, d\theta\right| < K\int_{-\eta}^{\eta} \chi(\theta)\, d\theta < K.$$

Again,

$$I = \left|\int_0^\eta \frac{f(z)-S}{z^{n+1}(1-z)^{1+\gamma}}\, d\theta\right|$$

$$< K\int_0^\eta \frac{\chi(\theta)}{(n^{-2}+\theta^2)^{\frac{1}{2}(\gamma+1)}}\, d\theta$$

$$< K\left[\Phi(\theta)(n^{-2}+\theta^2)^{-\frac{1}{2}(\gamma+1)}\right]_0^\eta + K\int_0^\eta \frac{\Phi(\theta)}{\theta}\,\frac{\theta^2}{(n^{-2}+\theta^2)^{\frac{1}{2}(\gamma+1)+1}}\, d\theta.$$

* Hobson, **9**, 1, § 405.

By (19) $\Phi(0) = 0$ and η is fixed; therefore, using (20),

$$I < K + KM \int_0^\eta \frac{d\vartheta}{(n^{-2}+\theta^2)^{\frac{1}{2}(\gamma+1)}} < Kn^\gamma,$$

which is the required result.

(ii) The series Σa_n is summable (C, γ) if it is summable (A).

Since Σa_n is bounded (C, γ), it follows from Hölder's theorem that $f(z)$ is bounded within the angle $(-\frac{1}{2}\pi+\delta, \frac{1}{2}\pi-\delta)$ $(\delta > 0)$, vertex the point 1, and so by Montel's theorem, if $f(z)$ tends to S along one path it tends uniformly to S within this angle.

Write $A = \tan(\frac{1}{2}\pi-\delta)$, and, as before, take $n > A/\eta$, so that

$$I = \int_0^\eta \frac{|f(z)-S|}{(n^{-2}+\theta^2)^{\frac{1}{2}(\gamma+1)}}\, d\theta$$

$$< \int_0^{A/n} \frac{|f(z)-S|}{(n^{-2}+\theta^2)^{\frac{1}{2}(\gamma+1)}}\, d\theta + \left[\Phi(\theta)(n^{-2}+\theta^2)^{-\frac{1}{2}(\gamma+1)}\right]_{A/n}^{\eta} + \int_{A/n}^{\eta} \frac{\Phi(\theta)}{\theta} \frac{d\vartheta}{(n^{-2}+\theta^2)^{\frac{1}{2}(\gamma+1}}$$

$$= I_1 + I_2 + I_3.$$

Then

$$I_2 < K + \frac{\Phi(A/n)}{(A/n)} \frac{n^\gamma}{(1+A^2)^{\frac{1}{2}\gamma}} < K + \frac{M}{A^\gamma} n^\gamma \quad \text{by (20)},$$

$$= o(n^\gamma),$$

if A is taken sufficiently large. The remaining terms, I_3 and I_1, are $o(n^\gamma)$ by the argument used in the proof of Theorem 3, and so the theorem is proved.

The following example, which has been given by Dienes*, shows that, if θ_0 is not a point of the set E, it is possible for the series $\Sigma a_n e^{in\theta_0}$ to be summable (A) without being summable (C, γ). In fact, it may be that the function $f(z)$ tends to a limit, as z tends to $e^{i\theta_0}$, along any path lying within an angle of the form $(-\frac{1}{2}\pi+\delta, \frac{1}{2}\pi-\delta)$. For simplicity let $\theta_0 = 0$, then consider

$$f(z) = \frac{e^{1/(z-1)}}{(1-z)^k} = \sum_0^\infty a_n z^n,$$

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* Dienes, 3.

where $0<k<1$. Now it is clear that $f(z)$ tends to zero as z tends to 1 along any path from within the circle such that $\arg(1-z)|<\frac{1}{2}\pi-\delta$.

Fejér* has proved that

$$a_n \sim \frac{1}{\sqrt{(\pi e)}} \frac{\sin[2\sqrt{n}+(\frac{3}{4}-\frac{1}{2}k)\pi]}{n^{\frac{1}{4}-\frac{1}{2}k}} \quad \text{for any } k,$$

$$= o(1), \quad \text{if} \quad 0<k<1.$$

Let $0<\gamma<\frac{1}{2}$, and choose $k>\frac{1}{2}+\gamma$, so that

$$\sum_0^\infty S_n^\gamma z^n = (1-z)^{-\gamma-1} \sum_0^\infty a_n z^n = (1-z)^{-\gamma-k-1} e^{1/(z-1)}.$$

Therefore, by Fejér's formula,

$$S_n^\gamma \sim \frac{\sin[2\sqrt{n}+(\frac{1}{4}-\frac{1}{2}\gamma+\frac{1}{2}k)\pi]}{\sqrt{(\pi e)}} n^{\frac{1}{2}k+\frac{1}{2}\gamma-\frac{1}{4}},$$

and so, if $k>\frac{1}{2}+\gamma$, S_n^γ is not $o(n^\gamma)$, and hence the series Σa_n is not summable (C, γ), although $a_n = o(n^\gamma)$.

Now let $F(\theta)$ be the boundary function. Then

$$F(\theta) = (1-e^{i\theta})^{-k} e^{-(1-e^{i\theta})^{-1}},$$

but $$|1-e^{i\theta}| = 2\sin\tfrac{1}{2}\theta \sim \theta, \quad \text{near} \quad \theta = 0,$$

and $$(1-e^{i\theta})^{-1} = \frac{1-e^{-i\theta}}{2(1-\cos\theta)} = \frac{1-\cos\theta+i\sin\theta}{2(1-\cos\theta)}.$$

Hence $$\mathfrak{R}(1-e^{i\theta})^{-1} = \tfrac{1}{2},$$

and therefore $$|F(\theta)| \sim e^{-\frac{1}{2}}\theta^{-k}, \quad \text{near } \theta = 0.$$

It follows that $|F(\theta)|$ is integrable L in $(-\pi, \pi)$, and hence, by a theorem of Hardy and Littlewood†, that $|f(re^{i\theta})|<\chi(\theta)$ in $(-\pi, \pi)$, where $\chi(\theta)$ is integrable L. The point $z=1$, however, is not a point of the set E, for the derivative of the integral of $|F(\theta)|$ tends to infinity as θ tends to zero.

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* Fejér, **6**.

† Hardy and Littlewood, **8**, 108.

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THE GENERALIZATION OF CERTAIN INEQUALITY THEOREMS INVOLVING POWERS

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1. *Introduction.*

The starting point of the work in the present paper is a group of inequality theorems due to Hardy and Littlewood, of which the following are the principal†:

$$(1.1) \qquad \int_0^1 A^r(x)(1-x)^{(r/q)-1}\,dx \leqslant K\left(\sum_1^\infty a_n{}^p\, n^{-(p+q-pq)/q}\right)^{r/p},$$

where
$$A(x) = \sum_1^\infty a_n x^n \quad (a_n \geqslant 0);$$

$$(1.2) \qquad \sum_1^\infty A_n{}^r\, n^{(r/q)-1} \leqslant K\left(\int_0^1 a^p(x)(1-x)^{-(p+q-pq)/q}\,dx\right)^{r/p},$$

where
$$A_n = \int_0^1 a(x)\, x^n\,dx \quad [a(x) \geqslant 0].$$

These two theorems hold subject to the conditions

$$r \geqslant p \geqslant 1, \quad q > 0;$$

and K depends only on p, q, r. Moreover, it is understood here (and also in similar inequalities throughout the paper) that the convergence of the right-hand side implies that of the left.

There is clearly a close formal correspondence between (1.1) and (1.2), which extends indeed to the proofs. This is due to the fact that they may both be deduced from the theorem

$$(1.3) \qquad \int_0^\infty F^r(x)\, x^{(r/q)-1}\,dx \leqslant K\left(\int_0^\infty f^p(x)\, x^{-(p+q-pq)/q}\,dx\right)^{r/p},$$

where
$$F(x) = \int_0^\infty f(t)\, e^{-tx}\,dt.$$

† Hardy and Littlewood, **5**, especially Theorems 3 and 4.

For instance, to obtain (1.1) we take

$$f(x) = a_n \quad (n-1 < x \leqslant n),$$

so that $F(x) \geqslant A(e^{-x})$. Putting $e^{-x} = \xi$, $x = \log \xi^{-1} \geqslant 1-\xi$, we see that the integral on the left of (1.3) is not less than that in (1.1); while the integral on the right of (1.3) is less than a constant multiple of the series in (1.1).

Theorems such as (1.1) are open to generalization in two different ways. In the first place, we may replace the power series $A(x)$ that occurs by series of other or more general kinds. I have established in a previous paper the following analogue† of (1.1):

$$(1.4) \qquad \int_0^\infty A^r(x)\, x^{(r/q)-1}\, dx \leqslant K\left(\sum_2^\infty a_n{}^p\, n^{-1}(\log n)^{-(p+q-pq)/q}\right)^{r/p},$$

where

$$A(x) = \sum_2^\infty a_n\, n^{-1-x}.$$

Further, for a Dirichlet series of general type we have:

$$(1.5) \qquad \int_0^\infty A^r(x)\, x^{(r/q)-1}\, dx \leqslant K\left(\sum_1^\infty a_n{}^p\, \lambda_n^{-(p+q-pq)/q}(\lambda_n-\lambda_{n-1})^{1-p}\right)^{r/p},$$

where

$$A(x) = \sum_1^\infty a_n\, e^{-\lambda_n x} \quad (\lambda_1 < \lambda_2 < \ldots).$$

This theorem has been stated by Owen‡: for $\lambda_n = n$, it is equivalent to (1.1), for $\lambda_n = \log n$ to (1.4), and for $\lambda_n = \log\log n$, $\lambda_n = \log\log\log n$, ... to a result which I gave at the same time as (1.4)§. There are, of course, companion theorems similar to (1.2), with the moment constants replaced by the numbers

$$A_n = \int_0^\infty a(x)\, n^{-1-x}\, dx \quad \text{or} \quad \int_0^\infty a(x)\, e^{-\lambda_n x}\, dx.$$

In the second place, instead of generalizing the *contenis* of inequalities such as (1.1), we may seek to generalize their *form*. The problem then is to substitute for the powers appearing some more general functions of the variables; and it is with this problem that the present paper is concerned. As a first instance, let us take the case $r = p$ of (1.1)—in which, incidentally, the best possible value‖ of K is $[\Gamma(1/q)]^p$—and replace the

† Mulholland, **7**, Theorem 1.

‡ Owen, **9**, Theorem 2. Owen has not so far published a proof; but I observe that the methods of my previous paper are sufficient, *cf.* § 9 below.

§ *Loc. cit.*, Theorem 9.

‖ Mulholland, **7**, § 6.

p-th powers by a function $\Phi(x)$. The theorem suggested is

$$(1.6) \qquad \int_0^1 \Phi[(1-x)^{1/q} A(x)](1-x)^{-1}\,dx \leqslant \sum_1^\infty \Phi[\Gamma(1/q)\, n^{1-(1/q)} a_n]\, n^{-1}.$$

It is shown in Theorem 2 below that this holds whenever $\Phi(x)$ is increasing and convex, with $\Phi(0) = 0$; and $\Gamma(1/q)$ is still the best possible constant. Again, if in (1.1) we replace the $(1/q)$-th powers by a function $\theta(x)$ and the r-th powers by a function $\mathrm{X}(x)$, we get

$$(1.7) \qquad \int_0^1 \mathrm{X}[\theta(1-x)\, A(x)](1-x)^{-1}\,dx \leqslant \mathrm{X}\Phi^{-1}\Big(B_1 \sum_1^\infty \Phi\,[B_2\, n\theta(1/n)]\, n^{-1}\Big),$$

where B_1 and B_2 are constants. Theorem 6 below asserts this inequality under the hypothesis that $\Phi(x)/x$ and $\mathrm{X}(x)/\Phi(x)$ are increasing (in the broad sense), and that $\theta(x)$ satisfies the conditions

$$(1.8) \qquad K\theta(x) \leqslant \theta(2x) \leqslant K'\theta(x) \quad (1 < K < K').$$

On the same lines there is no difficulty in obtaining a generalization of (1.2), and of other theorems in the same group, which were not quoted above. Moreover, the extension to Dirichlet series in the manner of (1.4) and (1.5) does not meet with any serious obstacles.

As regards the method of proof employed in this paper, it is, essentially, an adaptation of the line of argument that was developed in the first instance in connection with (1.4) and its companion theorems, and which supplies an alternative to Hardy and Littlewood's proof of (1.1), etc. That method consisted in a reduction to the wellnigh trivial case when $r = p = 1$ by means of Hölder's inequality. However, this important inequality has no very satisfactory generalization† apart from Jensen's relation

$$(1.9) \qquad \Phi\Big(\frac{\Sigma ux}{\Sigma u}\Big) \leqslant \frac{\Sigma u\, \Phi(x)}{\Sigma u},$$

which holds for convex functions $\Phi(x)$. Fortunately, with the aid of one or two devices, this can be made to serve our purpose.

The next question to be discussed is whether the restrictions laid on $\Phi(x)$, $\mathrm{X}(x)$, and $\theta(x)$ are necessary—as well as sufficient—for the truth of the results: this question is answered in § 8. To start with, suppose that (1.6) holds for some positive q; then it appears that $\Phi(x)$, if continuous, must also be increasing and convex. As regards the restrictions on (1.7), the position is not so simple. This is because the constants B_1

† *Cf.* Cooper, **3**; Hardy, **4**. But see (2.191) below.

and B_2 in that inequality rob it of its sharpness, being unspecified and not, like $\Gamma(1/q)$ in (1.6), best possible constants.

In a general way, if any inequality formed with functions like $\Phi(x)$, $X(x)$, etc., has in it an unspecified factor K, we may expect that the restrictions necessary will not be of a "precise" character, but will themselves involve an unspecified K. Accordingly, in dealing with the properties of the functions concerned which are necessary for (1.7)—and also for similar generalizations of inequalities of other kinds—we are led to consider the results of relaxing certain precise properties, such as monotony and convexity, by the introduction of a constant factor K into their defining inequalities. For instance, if a function $\phi(x)$ satisfies

$$\phi(x) \leqslant K\phi(y) \quad (x \leqslant y),$$

we shall say that $\phi(x)$ is *quasi-increasing*. Again, by introducing constants into (1.9) we obtain a property which may be called *quasi-convexity*. This property is investigated in 2.3 below, and appears to be of considerable interest; it secures the inequality properties of convexity (apart from the presence of constants K) without requiring any special local properties, and, in fact, is really a property of the function in its whole range and not in any finite interval interior to the range.

Returning to the theorem (1.7), we find that it will still hold when the hypothesis is merely that $\Phi(x)/x$ and $X(x)/\Phi(x)$ are quasi-increasing. Incidentally, (1.8) implies that $\theta(x)x^{-h}$ is quasi-increasing and $\theta(x)x^{-k}$ quasi-decreasing for some $k > h > 0$. It is shown that these conditions on $\Phi(x)$, $X(x)$, and $\theta(x)$ are necessary to the truth of (1.7), provided that we exclude from consideration functions that are not quasi-monotonic: moreover, the necessity of the condition for each function is established more or less independently of assumptions about the other functions.

It should be remarked that all quantities occurring are non-negative unless otherwise stated; further, that K is used indiscriminately for any positive constant, not necessarily the same from one occurrence to the next. A, B, C, etc., are also used for positive constants and will usually refer to the same numbers throughout a given argument. When used with a suffix they represent numbers permanently associated with the functions $\theta(x)$, $\Phi(x)$, $X(x)$, etc.; otherwise, each is associated with a certain type of inequality. Generally their precise value is of no consequence, and it is only for the sake of clearness in the argument that they are distinguished.

Lastly, I should like to express my thanks to Prof. Littlewood for suggesting the problem of this paper to me in the first place, and for his encouragement since.

2. *Restrictive assumptions. Convexity and quasi-convexity.*

2.1. When we replace powers of x by functions $\phi(x)$, it is necessary to impose on the latter restrictions of various kinds: I enumerate below such of these restrictions as will be required later.

In the first place it will be supposed throughout that

$$\phi(0) = 0; \quad \phi(x) \geqslant 0 \quad (x \geqslant 0); \tag{2.11}$$

and that

$$\phi(x) \text{ is bounded in any finite interval.} \tag{2.12}$$

Secondly, to generalize positive powers of x it is natural to take functions which are increasing—at least in the broad sense. Thus we have

$$\phi(x) \leqslant \phi(y) \quad (x \leqslant y) \tag{2.13}$$

as a possible restriction.

Next we consider what may be assumed about $\phi(x)$ to make it correspond to x^k with $k \geqslant 1$. The simplest assumption is that

$$\phi(x)/x \text{ is increasing (in the broad sense).} \tag{2.141}$$

This may also be stated as

$$\phi(\alpha x) \leqslant \alpha\phi(x) \quad (\alpha \leqslant 1). \tag{2.142}$$

We might also suppose that

$$\phi(x) \text{ is convex;} \tag{2.151}$$

that is

$$\phi[\tfrac{1}{2}(x+y)] \leqslant \tfrac{1}{2}[\phi(x)+\phi(y)] \tag{2.152}$$

for all x and y. This, together with (2.12), implies† that $\phi(x)$ is continuous, and that it satisfies Jensen's inequality

$$\phi\left(\frac{\Sigma ux}{\Sigma u}\right) \leqslant \frac{\Sigma u\phi(x)}{\Sigma u}. \tag{2.161}$$

It may easily be shown that, when $\phi(0) = 0$, (2.161) implies (2.141).

As a deduction from (2.161) we have Jensen's inequality for integrals

$$\phi\left(\frac{\int ux\,dt}{\int u\,dt}\right) \leqslant \frac{\int u\phi(x)\,dt}{\int u\,dt}, \tag{2.162}$$

† Jensen, **6**.

where u and x are functions of t and the integrals all have the same range. We note also that, when the sums in (2.161) contain only two terms, the inequality may be written in the form

$$\phi(x) \leqslant \frac{v-x}{v-u}\,\phi(u)+\frac{x-u}{v-u}\,\phi(v) \quad (u<x<v). \tag{2.163}$$

Geometrically, this states that between the points $t=u$ and $t=v$ the curve $y=\phi(t)$ lies on or below the chord joining these points. Again, by keeping x fixed in (2.163) and letting u and v vary we can easily deduce that, if $\phi(x)$ is convex, there is for each x a number m_x such that

$$\phi(t) \geqslant \phi(x)+m_x(t-x); \tag{2.164}$$

and conversely, if such a number exists for all x, $\phi(x)$ is convex. Geometrically, (2.164) states that there is a line through the point $\{x, \phi(x)\}$, with slope m_x, such that no point of the curve $y=\phi(t)$ lies below it.

Sometimes it is necessary to restrict the order of magnitude of the functions $\phi(x)$ as $x\to\infty$ or 0. We may do this by supposing that

$$\phi(x)/x^k \text{ is decreasing}, \tag{2.171}$$

or that

$$\phi(x)/x^h \text{ is increasing}. \tag{2.172}$$

If $\phi(x)$ satisfies both (2.171) and (2.172) we shall have

$$\text{Min}\,[t^h, t^k] \leqslant \phi(tx)/\phi(x) \leqslant \text{Max}\,[t^k, t^h]. \tag{2.173}$$

The information about the ratio $\phi(tx)/\phi(x)$ that (2.173) supplies is not always sufficient, however, and for some purposes we shall require one of the assumptions

$$\phi(xy) \leqslant \phi(x)\,\phi(y) \tag{2.181}$$

and

$$\phi(xy) \geqslant \phi(x)\,\phi(y). \tag{2.182}$$

We note that if $\phi(x)$ satisfies one of these inequalities its inverse will satisfy the other. These properties have been investigated by Cooper†; he shows, among other things, that any function satisfying (2.181) is such that

$$\log\phi(x)/\log x \text{ tends to a finite limit as } x\to 0 \text{ or } \infty. \tag{2.183}$$

Nevertheless, there is no need for $\phi(x)$ to be asymptotically equivalent to

† Cooper, **2**, especially §§ 4, 5, 6.

a power of x, since the following functions†, for instance, satisfy (2.181) when $\alpha > 0$ and (2.182) when $\alpha < 0$:

$$x^k[1+a \overset{+}{\log} x]^\alpha, \quad x^k[1+a \overset{+}{\log}(1/x)]^\alpha; \tag{2.184}$$

$$x^k[1+a \log(1+x)]^\alpha, \quad x^k[1+a \log\{1+(1/x)\}]^\alpha. \tag{2.185}$$

We may remark that, for $k > 1$, the first of the functions (2.184) will be convex when $\alpha \geqslant 0$, and the second when $\alpha \leqslant 0$, provided that a is suitably chosen; but that neither of the functions is convex otherwise, owing to the discontinuity in the derivative at $x = 1$. The functions (2.185) can be made convex when $k > 1$ and a is suitably chosen, whatever the sign of α. More complicated expressions also, such as

$$x^k[1+a \overset{+}{\log} x]^\alpha [1+b \log(1+a \overset{+}{\log} x)]^\beta \ldots, \tag{2.186}$$

may be constructed so as to have properties like those of the functions (2.184) and (2.185).

The question whether functions satisfying (2.181) and (2.182) can also be convex is of interest in view of the inequalities that are obtainable by assuming convexity in addition to these properties. We may instance the following generalization of Hölder's inequality, which is due to Cooper‡.

If $\phi(x)$ is convex and satisfies (2.182), *then*

$$\phi[\Sigma uab] \leqslant \frac{\phi^{-1}[\Sigma u\phi(a)]}{\psi^{-1}[1/\Sigma u\psi(b)]}, \tag{2.191}$$

$$\phi\left[\int uab\,dt\right] \leqslant \frac{\phi^{-1}\left[\int u\phi(a)\,dt\right]}{\psi^{-1}\left[1\Big/\int u\psi(b)\,dt\right]}, \tag{2.192}$$

where

$$\phi^{-1}(x)\psi^{-1}(x) = x, \tag{2.193}$$

and u, a, b are functions of n or t, as the case may be.

2.2. We come now to the conditions that are obtained when we relax the various restrictions enumerated above by introducing a constant factor into the inequalities.

In the first place, (2.13) is replaced by

$$\phi(x) \leqslant C\phi(y) \quad (x \leqslant y;\ C \geqslant 1). \tag{2.211}$$

† For the properties of these functions see Mulholland, **7**, § 5.

‡ *Loc. cit.*, § 9. The result is stated there in the integral form (2.192).

When $C = 1$, this reduces to (2.13) and $\phi(x)$ is increasing (in the broad sense); in general, when $\phi(x)$ satisfies (2.211) we say that it is *quasi-increasing*. On the other hand, if $\phi(x)$ satisfies

$$\phi(x) \geqslant c\phi(y) \quad (x \leqslant y;\ c \leqslant 1), \tag{2.212}$$

it will be *quasi-decreasing*; in either case $\phi(x)$ is *quasi-monotonic*.

We can now write down the corresponding extensions of the restrictions (2.141), (2.171), and (2.172). In place of (2.141) we have

$$\phi(x)/x \text{ is quasi-increasing}; \tag{2.22}$$

and in place of (2.171) and (2.172) we have

$$\phi(x)/x^k \text{ is quasi-decreasing}, \tag{2.231}$$

$$\phi(x)/x^h \text{ is quasi-increasing}. \tag{2.232}$$

These may also be written

$$\phi(tx) \leqslant Et^k \phi(x) \quad (t \geqslant 1), \tag{2.241}$$

$$\phi(\tau x) \leqslant D\tau^h \phi(x) \quad (\tau \leqslant 1). \tag{2.242}$$

When both conditions hold we have, instead of (2.173),

$$C^{-1} \operatorname{Min}[t^h, t^k] \leqslant \phi(tx)/\phi(x) \leqslant C \operatorname{Max}[t^h, t^k], \tag{2.25}$$

where $C = \operatorname{Max}(D, E)$. It may easily be shown that, if $\phi(x)$ is quasi-increasing, an inequality of type (2.241) is deducible from the simple assumption†

$$\phi(2x) \leqslant E'\phi(x), \tag{2.261}$$

which yields (2.241) with $k = \log E'/\log 2$, $E = CE'$. Similarly, for any quasi-increasing $\phi(x)$,

$$\phi(\tfrac{1}{2}x) \leqslant D'\phi(x) \quad (0 < D' < 1) \tag{2.262}$$

implies (2.242) with $h = \log(1/D')/\log 2$, $D = CD'$.

It may be mentioned that quasi-increasing functions which satisfy either of the conditions (2.181), (2.182) will have both the above properties, as we may see at once by putting $y = 2$ and $y = \frac{1}{2}$ in the said conditions.

2.3. If we treat the property of convexity on the above lines, we are led to the following.

† For increasing functions this was pointed out by Burkill in **1**, Lemma 2. To get (2.241) it is only necessary to approximate to t by a power of 2 and apply (2.261) a sufficient number of times.

DEFINITION. *Let $\phi(x)$ be defined and non-negative in $(0, \infty)$, and let it satisfy the condition*

$$\psi(x) \leqslant \phi(x) \leqslant A\psi(Bx) \tag{2.311}$$

for some convex function $\psi(x)$. Then we shall say that $\phi(x)$ is quasi-convex.

It is convenient to replace the function $\psi(x)$ by $\phi^*(x)$, the greatest convex function that never exceeds $\phi(x)$; we may call $\phi^*(x)$ the "convex envelope" of the function $\phi(x)$. We shall have thus, since $\psi(x) \leqslant \phi^*(x)$,

$$\phi^*(x) \leqslant \phi(x) \leqslant A\phi^*(Bx). \tag{2.312}$$

In order to define $\phi^*(x)$ analytically, we observe that, for any of the convex functions $\psi(x)$, (2.161) gives

$$\begin{aligned}\psi(x) &\leqslant \lambda\psi(a)+(1-\lambda)\psi(a') \quad [x = \lambda a+(1-\lambda)a'] \\ &\leqslant \lambda\phi(a)+(1-\lambda)\phi(a').\end{aligned}$$

This suggests the definition

$$\phi^*(x) = \underline{\text{bound}}\,[\lambda\phi(a)+\lambda'\phi(a')], \tag{2.321}$$

where

$$\lambda+\lambda' = 1, \quad \lambda a+\lambda' a' = x. \tag{2.322}$$

Clearly $\psi(x) \leqslant \phi^*(x) \leqslant \phi(x)$; moreover, $\phi^*(x)$ is bounded in any finite interval (a, a') since it certainly cannot exceed $\phi(a)+\phi(a')$. It remains to be proved† that

$$\phi^*(x) \text{ is convex and continuous.} \tag{2.323}$$

Now (2.321) gives us

$$\phi^*(x) \leqslant \frac{v-x}{v-u}\,\phi(u)+\frac{x-u}{v-u}\,\phi(v) \quad (u < x < v),$$

and from this it follows, as for (2.164), that there is some number m_x such that

$$\phi(t) \geqslant \phi^*(x)+m_x(t-x).$$

But

$$\begin{aligned}\phi^*(t) &= \underline{\text{bound}}\,[\lambda\phi(a)+\lambda'\phi(a')] \quad (\lambda a+\lambda' a' = t) \\ &\geqslant \underline{\text{bound}}\,[\lambda\{\phi^*(x)+m_x(a-x)\}+\lambda'\{\phi^*(x)+m_x(a'-x)\}] \\ &= \underline{\text{bound}}\,[\phi^*(x)+m_x(\lambda a+\lambda' a'-x)] \\ &= \phi^*(x)+m_x(t-x).\end{aligned}$$

† The result is quite elementary, but I have not succeeded in finding a proof in the literature.

Thus $\phi^*(x)$ satisfies (2.164) and hence must be convex. Lastly, $\phi^*(x)$ is bounded in any finite interval and therefore, being convex, is continuous.

We are now in a position to state the extension of Jensen's inequality (2.161).

THEOREM 1. *The necessary and sufficient condition that the inequalities*

$$\phi\left(\frac{\Sigma ux}{\Sigma u}\right) \leqslant A\,\frac{\Sigma u\phi(Bx)}{\Sigma u}, \tag{2.331}$$

$$\phi\left(\frac{\int ux\,dt}{\int u\,dt}\right) \leqslant A\,\frac{\int u\phi(Bx)\,dt}{\int u\,dt} \tag{2.332}$$

should hold for all functions u and x of n (or t) is that $\phi(x)$ should be quasi-convex with factors A and B.

Clearly, it will be sufficient to deal with (2.331). In the first place, let us assume that $\phi(x)$ is quasi-convex with factors A and B; *i.e.* that it satisfies (2.312). Then

$$\phi\left(\frac{\Sigma ux}{\Sigma u}\right) \leqslant A\phi^*\left(\frac{B\Sigma ux}{\Sigma u}\right) \leqslant A\,\frac{\Sigma u\phi^*(Bx)}{\Sigma u} \leqslant A\,\frac{\Sigma u\phi(Bx)}{\Sigma u},$$

by applying Jensen's inequality (2.161) to the convex function $\phi^*(x)$. This proves the sufficiency part of the theorem. For the other part, let us suppose that $\phi(x)$ satisfies (2.331), and let $\phi^*(x)$ be its convex envelope; in particular, we have

$$\phi(\lambda a+\lambda' a') \leqslant A\{\lambda\phi(Ba)+\lambda'\phi(Ba')\};$$

and so, writing $\lambda a+\lambda' a' = x/B$ and taking the lower bounds for fixed x of both sides,

$$\phi(x/B) \leqslant A\phi^*(x),$$

or

$$\phi(x) \leqslant A\phi^*(Bx).$$

Thus (2.312) is satisfied and $\phi(x)$ is quasi-convex†.

COROLLARY. *For $\phi(x)$ to be quasi-convex it is sufficient that*

$$\phi(\lambda a+\lambda' a') \leqslant A\{\lambda\phi(Ba)+\lambda'\phi(Ba')\} \tag{2.333}$$

for all rational λ, provided that $\phi(x)$ is quasi-monotonic.

† For any non-negative function $\phi(x)$, $\phi^*(x)$ will exist, though it may be identically 0; in that case the argument above shows that $\phi(x) \equiv 0$.

Suppose that $\phi(x)$ is quasi-increasing, and let κ be any number between 0 and 1. Take λ rational and less than κ; then, if $y > x$,

$$\kappa x + (1-\kappa) y < \lambda x + (1-\lambda) y,$$

and hence

$$\begin{aligned}\phi[\kappa x + (1-\kappa) y] &\leqslant C\phi[\lambda x + (1-\lambda) y] \\ &\leqslant CA\{\lambda\phi(Bx) + (1-\lambda)\phi(By)\}.\end{aligned}$$

Letting $\lambda \to \kappa$ it follows that (2.333) is satisfied for all λ with $A' = CA$ in place of A. But we have seen above that (2.333), when it holds for all λ, implies that $\phi(x)$ is quasi-convex.

It should be noticed that, even when the factors A and B are specified, quasi-convexity allows considerable latitude for local irregularities. While if we say merely that a function is quasi-convex for *some* A and B, we are making an assertion that only has meaning when applied to the behaviour of the function in its whole range; for clearly the function may have arbitrary values in a finite interval interior to its range and yet be quasi-convex if suitably defined outside this interval. The latitude allowed by quasi-convexity is in some ways an advantage, as, for instance, when we have to deal with functions like those in (2.184), which have a discontinuity in their derivative; when the discontinuity is in the wrong sense, such functions cannot be convex, but may still be quasi-convex. The functions in (2.184) actually are quasi-convex, as may be readily verified.

[*Added 27 February*, 1932. Further, we may notice that quasi-convexity with $B = 1$ is sharply distinguished from the less stringent property with $B > 1$. This will be clear from the following.

2.34. *If $\phi(x)/x$ is increasing, then $\phi(x)$ is quasi-convex with $A = 1$, $B = 2$. More generally, if $\phi(x)/x$ is quasi-increasing, $\phi(x)$ is quasi-convex with $A = C/(t-C)$, $B = t$, for any $t > C$; and conversely, provided that $\phi(0) = 0$.*

Supposing $\phi_1(x) \equiv \phi(x)/x$ to be increasing, we have, for any a and b,

$$a\phi_1(b) \leqslant a\phi_1(a) + b\phi_1(b) = \phi(a) + \phi(b).$$

Therefore, when $\Sigma u = 1$,

$$\begin{aligned}t\phi(\Sigma ux) &= \Sigma utx \,.\, \phi_1(\Sigma ux) \quad (t > 1) \\ &\leqslant \Sigma u\,[\phi(tx) + \phi(\Sigma ux)] = \Sigma u\phi(tx) + \phi(\Sigma ux);\end{aligned}$$

and this involves quasi-convexity with factors $A = 1/(t-1)$, $B = t$. For $t = 2$ we get the first part of 2.34; and the next part follows with the aid

of an obvious modification. While for the converse we have, putting a/B for a, and 0 for β in (2.333),

$$\phi(\lambda a/B) \leqslant A\lambda\phi(a) = AB(\lambda/B)\,\phi(a);$$

whence the result, by (2.242) with $\tau = \lambda/B$.]

3. *Preliminary lemmas.*

LEMMA 1. *When $c \geqslant 1$, we have*

$$(3.11) \qquad a_n \equiv \binom{-c}{n}(-1)^n \equiv \frac{1}{n}\,\frac{\Gamma(c+n)}{\Gamma(c)\,\Gamma(n)} \geqslant \frac{n^{c-1}}{\Gamma(c)},$$

and, when $0 < c \leqslant 1$,

$$(3.12) \qquad a_{n-1} \equiv \binom{-c}{n-1}(-1)^{n-1} \equiv \frac{\Gamma(c+n-1)}{\Gamma(c)\,\Gamma(n)} \geqslant \frac{n^{c-1}}{\Gamma(c)} \geqslant a_n.$$

Suppose first that $c \geqslant 1$, and let

$$\frac{1}{v_n} \equiv \frac{a_n}{n^{c-1}} = n^{-c}\,c(1+c)\left(1+\frac{c}{2}\right)\ldots\left(1+\frac{c}{n-1}\right).$$

Then

$$\frac{v_n}{v_{n+1}} = \left(\frac{n}{n+1}\right)^c\left(1+\frac{c}{n}\right) = \left(1+\frac{c}{n}\right)\left(1+\frac{1}{n}\right)^{-c} \leqslant 1,$$

since $c \geqslant 1$†. Hence v_n increases with u, and therefore

$$v_n \leqslant \lim_{n\to\infty} v_n = \lim_{n\to\infty} n^c \Big/ \left[c(1+c)\left(1+\frac{c}{2}\right)\ldots\left(1+\frac{c}{n-1}\right)\right] = \Gamma(c).$$

Thus $n^{c-1} \leqslant \Gamma(c)\,a_n$. For $c \leqslant 1$, we get the inequality reversed; so that we have now established (3.11) and the right-hand side of (3.12). The left-hand side of the latter can be proved similarly.

LEMMA 2. *If $c > 0$, and $0 \leqslant x < 1$, then*

$$(3.13) \qquad Kx(1-x)^{-c} \leqslant \sum_1^\infty n^{c-1}x^n \leqslant \Gamma(c)(1-x)^{-c}.$$

When $c \leqslant 1$, $t^{c-1}x^t$ is decreasing and hence

$$\sum_1^\infty n^{c-1}x^n \leqslant \int_0^\infty t^{c-1}x^t\,dt.$$

† We appeal to the elementary fact that $(1+x)^a - (1+ax)$ is positive for positive x if $a \geqslant 1$ or $a \leqslant 0$, and that otherwise it is negative.

The integral on the right is equal to

$$\int_0^\infty t^{c-1} \exp\left[-t \log(1/x)\right] dt = [\log(1/x)]^{-c} \int_0^\infty u^{c-1} e^{-u}\, du,$$

which is not greater than the right side of (3.13). When $c > 1$, we compare the given series with the binomial expansion

$$(1-x)^{-c} = \sum_0^\infty c(c+1)\ldots(c+n-1)\, x^n/n! = \sum_0^\infty a_n x^n.$$

In virtue of (3.11), we have

$$\sum_1^\infty n^{c-1} x^n \leqslant \Gamma(c) \sum_0^\infty a_n x^n = \Gamma(c)(1-x)^{-c}.$$

This completes the proof of the right-hand side of (3.13). For the left-hand side, when $c \leqslant 1$, we have, by (3.12),

$$\sum_1^\infty n^{c-1} x^n \geqslant \Gamma(c) \sum_1^\infty a_n x^n = \Gamma(c)\left[(1-x)^{-c} - 1\right]$$

$$\geqslant c\Gamma(c)(1-x)^{-c} x.$$

When $c \geqslant 1$ we have

$$\sum_1^\infty n^{c-1} x^n \geqslant \int_0^\infty t^{c-1} x^{t+1}\, dt$$

$$\sim \Gamma(c)\, x(1-x)^{-c},$$

as $x \to 1$; whence the result stated.

LEMMA 3. *For all positive c we have*

$$K n^{-c} \leqslant \int_0^1 (1-x)^{c-1} x^n\, dx \leqslant \Gamma(c)\, n^{-c}. \tag{3.14}$$

For, if we put $u = n(1-x)$, the integral becomes

$$n^{-c} \int_0^n u^{c-1}(1-un^{-1})^n\, du \leqslant n^{-c} \int_0^n u^{c-1} e^{-u}\, du \leqslant n^{-c}\, \Gamma(c),$$

since $(1-un^{-1})^n$ increases to its limit e^{-u} as $n \to \infty$. Moreover, the integral on the left is not less than its value for $n = 1$; we may take this value as K.

3.2. We turn to lemmas involving functions $\theta(x)$, which we suppose to have properties of the type considered in (2.231) and (2.232). That is, *we suppose that for some $k \geqslant h > 0$ we have $\theta(x)/x^h$ quasi-increasing and*

$\theta(x)/x^k$ *quasi-decreasing, or*

$$C^{-1}\,\mathrm{Min}\,[t^h, t^k] \leqslant \theta(tx)/\theta(x) \leqslant C\,\mathrm{Max}\,[t^h, t^k] \quad (\textit{all } x, t \geqslant 0). \tag{3.21}$$

As already mentioned at (2.261) and (2.262), this condition may be secured by the assumption

$$\theta(x)/D' \leqslant \theta(2x) \leqslant E'\theta(x) \quad (\text{all } x). \tag{3.211}$$

Under these conditions we may obtain the results below.

LEMMA 4. *If* $\theta(x)$ *satisfies* (3.21), *then*

$$1+B_4 x \leqslant 1+\theta(1-x)\sum_1^\infty \frac{x^n}{n\theta(1/n)} \leqslant B_2 \quad (0 \leqslant x \leqslant 1), \tag{3.22}$$

where B_2, B_4 *are constants depending only on* $\theta(x)$.

Let us prove first the right-hand inequality. Now

$$\begin{aligned}\theta(1-x) &= \theta\,[n^{-1}.n(1-x)]\\ &\leqslant \theta(n^{-1})\,C\,\mathrm{Max}\,[n^h(1-x)^h,\ n^k(1-x)^k]\\ &= \theta(1/n)\,\chi\,[n(1-x)],\end{aligned}$$

say; hence

$$1+\theta(1-x)\sum_1^\infty x^n\,[n\theta(1/n)]^{-1} \leqslant 1+\sum_1^\infty \chi\,[n(1-x)]\,n^{-1}x^n.$$

But $\chi(u) \leqslant C(u^h+u^k)$, and thus the latter series does not exceed

$$C(1-x)^h\sum_1^\infty n^{h-1}x^n + C(1-x)^k\sum_1^\infty n^{k-1}x^n.$$

Lemma 2 shows that both terms here are bounded above by constants, and the right-hand side of (3.22) follows. On the other hand

$$\theta(1-x) \geqslant \theta(1/n)\,C^{-1}\,\mathrm{Min}\,[n^h(1-x)^h,\ n^k(1-x)^k],$$

and hence, writing $\chi_1\,[n(1-x)]$ for the second factor on the right, we have

$$\begin{aligned}\frac{\theta(1-x)}{x}\sum_1^\infty \frac{x^n}{n\theta(1/n)} &\geqslant \sum_1^\infty \frac{\chi_1\,[n(1-x)]}{n}\,x^{n-1}\\ &\geqslant \chi_1(1-x)+\int_2^\infty \chi_1\,[(t-1)(1-x)]\,x^{t-1}t^{-1}\,dt.\end{aligned}$$

But

$$\begin{aligned}\chi_1\,[(t-1)(1-x)] &= \chi_1\,[x(t-1)(x^{-1}-1)]\\ &\geqslant \chi_1\,[x(t-1)\log(1/x)];\end{aligned}$$

and hence, putting $(t-1)\log(1/x) = u$, we have

$$x^{-1}\theta(1-x)\sum_1^\infty x^n\,[n\theta(1/n)]^{-1} \geqslant \chi_1(1-x)+\int_{\log(1/x)}^\infty \chi_1(xu)\,e^{-u}\,[u+\log(1/x)]^{-1}\,du.$$

Now, in $(0, \frac{1}{2})$ the first term on the right is not less than $\chi_1(\frac{1}{2})$ and in $(\frac{1}{2}, 1)$ the second term is not less than

$$\int_{\log 2}^\infty \chi_1(\tfrac{1}{2}u)\,e^{-u}(u+\log 2)^{-1}\,du.$$

We may therefore take the smaller of this and $\chi_1(\frac{1}{2})$ as the B_4 in (3.22).

COROLLARY. *Under the same conditions we have*

$$V_n \equiv \theta(1-x)\,x^n\,[\theta(1/n)]^{-1} \leqslant B_2'. \tag{3.221}$$

For $V_n \leqslant \chi[n(1-x)]x^n$, and, as before, it is sufficient to consider the case $\chi(x) = x^\alpha$ $(\alpha > 0)$. But

$$n^\alpha(1-x)^\alpha x^n \leqslant n^\alpha\,[\log(1/x)]^\alpha \exp\,[-n\log(1/x)] \leqslant \operatorname*{Max}_{u\geqslant 0}\,[u^\alpha e^{-u}],$$

and the last expression is a constant which may be taken as B_2'.

LEMMA 5. *If $\theta(x)$ satisfies* (3.21), *then*

$$B_3 \leqslant \int_0^1 \frac{\theta(1-x)\,x^n\,dx}{\theta(1/n)(1-x)} \leqslant B_1. \tag{3.23}$$

We have, as before,

$$\theta(1-x) \leqslant \theta(1/n)\,\chi\,[n(1-x)].$$

Hence

$$\int_0^1 \frac{\theta(1-x)\,x^n}{\theta(1/n)(1-x)}\,dx \leqslant \int_0^1 \chi\,[n(1-x)]\,\frac{x^n\,dx}{1-x} = \int_0^n \chi(u)(1-un^{-1})^n\,u^{-1}\,du \leqslant \int_0^\infty \chi(u)\,e^{-u}\,u^{-1}\,du,$$

as in Lemma 3. But the integral last written is certainly finite and may be taken as the B_1 in (3.23). Coming now to the left-hand side of (3.23), we have

$$\theta(1-x) \geqslant \theta(1/n)\,\chi_1\,[n(1-x)],$$

and consequently it is sufficient to find a lower bound for the following integral:

$$\int_0^1 \chi_1[n(1-x)]\,x^n(1-x)^{-1}\,dx = \int_0^n \chi_1(u)(1-un^{-1})^n\,u^{-1}\,du$$

$$> \int_0^1 \chi_1(u)(1-u)u^{-1}\,du,$$

and the last integral may be taken as B_3.

4. *The special case where* $X(x) = \Phi(x)$.

4.1. When $r = p \geqslant 1$, and we write κ for $1/q$, (1.1) takes the form

$$\int_0^1 (1-x)^{p\kappa-1} A^p(x)\,dx \leqslant [\Gamma(\kappa)]^p \sum_1^\infty a_n^p\, n^{p-p\kappa-1};$$

and, as already mentioned, $[\Gamma(\kappa)]^p$ is the best possible constant. As a generalization of this we have

THEOREM 2. *Let* $\kappa > 0$ *and* $\Phi(x)$ *be increasing and convex, with* $\Phi(0) = 0$. *Then, if*

$$A(x) = \sum_1^\infty a_n x^n, \tag{4.11}$$

we have

$$\int_0^1 \Phi[(1-x)^\kappa A(x)](1-x)^{-1}\,dx \leqslant \sum_1^\infty \Phi[\Gamma(\kappa)\,n^{1-\kappa}a_n]\,n^{-1}; \tag{4.12}$$

and $\Gamma(\kappa)$ *on the right cannot be replaced by any smaller constant.*

Case I. $\kappa \geqslant 1$. Write $u_0 = 0$,

$$\sum_1^\infty u_n \equiv \sum_1^\infty \Phi[\Gamma(\kappa)\,n^{1-\kappa}a_n]\,n^{-1},$$

and

$$\sum_0^\infty a_n x^n \equiv \sum_0^\infty \binom{-\kappa}{n}(-x)^n = (1-x)^{-\kappa};$$

so that, in virtue of (3.11),

$$a_n = \Phi^{-1}(nu_n)\,n^{\kappa-1}/\Gamma(\kappa)$$

$$\leqslant \Phi^{-1}(nu_n)\,a_n.$$

Hence
$$(1-x)^\kappa A(x) = (1-x)^\kappa \sum_1^\infty a_n x^n$$
$$\leqslant \frac{\sum_0^\infty \Phi^{-1}(nu_n)\, a_n x^n}{\sum_0^\infty a_n x^n},$$

and therefore, applying Jensen's inequality (2.161),

$$\Phi\,[(1-x)^\kappa A(x)] \leqslant \frac{\Sigma\, nu_n\, a_n x^n}{\Sigma\, a_n x^n}$$
$$= (1-x)^\kappa \sum_1^\infty n a_n u_n x^n.$$

Thus the integral in (4.12) is not greater than†

$$\int_0^1 (1-x)^{\kappa-1} \sum_1^\infty n a_n u_n x^n\, dx = \sum_1^\infty n a_n u_n \int_0^1 (1-x)^{\kappa-1} x^n\, dx.$$

But
$$n a_n \int_0^1 (1-x)^{\kappa-1} x^n\, dx < n a_n \int_0^1 (1-x)^{\kappa-1} x^{n-1} dx$$
$$= n a_n\, \Gamma(\kappa)\, \Gamma(n)/\Gamma(\kappa+n) = 1;$$

(4.12) now follows.

Case II. $\kappa < 1$. It is not now true that $a_n \geqslant n^{\kappa-1}/\Gamma(\kappa)$; accordingly we use (3.12), namely

$$\beta_n \equiv n^{\kappa-1}/\Gamma(\kappa)\, a_{n-1} \leqslant 1,$$

which is valid for $\kappa < 1$, and employ the series

$$\sum_1^\infty a_{n-1} x^{n-1} = (1-x)^{-\kappa}$$

in place of $\Sigma\, a_n x^n$ $(n \geqslant 0)$. Thus

$$(1-x)^\kappa A(x) = (1-x)^\kappa \sum_1^\infty \Phi^{-1}(nu_n)\, n^{\kappa-1} x^n\, [\Gamma(\kappa)]^{-1}$$
$$= \frac{\Sigma\, \beta_n x\, \Phi^{-1}(nu_n)\, a_{n-1} x^{n-1}}{\Sigma\, a_{n-1} x^{n-1}}.$$

† Integration term-by-term is legitimate, since everything is positive and the series is uniformly convergent in any interval $(\epsilon, 1-\delta)$: similarly for all the subsequent occurrences of this argument.

Hence

$$\Phi\,[(1-x)^{\kappa}A(x)] \leqslant \frac{\Sigma\Phi[\beta_n x\Phi^{-1}(nu_n)]a_{n-1}x^{n-1}}{\Sigma a_{n-1}x^{n-1}};$$

and, since $\Phi(x)$ is convex, with $\Phi(0)=0$, and $\beta_n x \leqslant 1$, we have†

$$\Phi[\beta_n x\Phi^{-1}(nu_n)] \leqslant \beta_n x n u_n.$$

This leads to

$$\Phi[(1-x)^{\kappa}A(x)] \leqslant [\Gamma(\kappa)]^{-1}(1-x)^{\kappa}\sum_1^{\infty} u_n n^{\kappa}x^n.$$

The integral in (4.12) is therefore not greater than

$$[\Gamma(\kappa)]^{-1}\sum_1^{\infty} u_n n^{\kappa}\int_0^1 (1-x)^{\kappa-1}x^n\,dx,$$

and, by Lemma 3, the last integral is less than $\Gamma(\kappa)n^{-\kappa}$; whence (4.12).

It only remains to be shown that the $\Gamma(\kappa)$ in (4.12) cannot be replaced by any smaller number without the inequality being made false. Since $\Phi(x)/x$ is increasing, and so $\Phi(x)$ itself is strictly increasing, it will be sufficient to show that the ratio of the two sides of (4.12) can be made arbitrarily near 1 by choice of an appropriate example. For this purpose we take

(4.131)
$$\left.\begin{aligned} u_n &= \Phi(1)\,n^{-1} \\ a_n &= n^{\kappa-1}/\Gamma(\kappa) \end{aligned}\right\} \quad (n \leqslant N),$$
$$u_n = a_n = 0 \qquad (n > N).$$

Then, as $N\to\infty$,
$$\Sigma u_n \sim \Phi(1)\log N;$$

and, if $\kappa \leqslant 1$,

$$A(x) = [\Gamma(\kappa)]^{-1}\sum_1^N n^{\kappa-1}x^n > [\Gamma(\kappa)]^{-1}\int_1^N t^{\kappa-1}x^t\,dt$$
$$= [\Gamma(\kappa)]^{-1}[\log(1/x)]^{-\kappa}\int_{\log(1/x)}^{N\log(1/x)} u^{\kappa-1}e^{-u}\,du$$

(4.132)
$$> [\log(1/x)]^{-\kappa}(1-\delta),$$

where δ is arbitrary, provided that

$$x > 1-\omega_\delta, \quad N\log(1/x) > X = X_\delta.$$

Since $\log(1/x) > 1-x$ in $(0,\ 1)$, it is sufficient to suppose that $N(1-x) > X$. Thus (4.132) holds good in $(1-\omega,\ 1-XN^{-1})$ when $\kappa \leqslant 1$. If $\kappa > 1$, it

† See (2.142) and the remark following (2.161).

is necessary to subtract the maximum term of $A(x)$ from the above, but it may easily be shown that this does not affect the validity of (4.132). Hence, in $(1-\omega, 1-XN^{-1})$,

$$\begin{aligned}(1-x)^\kappa A(x) &> [(1-x)/\log(1/x)]^\kappa (1-\delta)\\ &> (1-2\delta),\end{aligned}$$

provided that $x > 1-\omega'$, say, since $(1-x)/\log(1/x) \to 1$ with x. Hence

$$\begin{aligned}\int_0^1 \Phi[(1-x)^\kappa A(x)](1-x)^{-1}dx &> \int_{1-\omega'}^{1-XN^{-1}} > \Phi[1-2\delta]\int_{1-\omega'}^{1-XN^{-1}} (1-x)^{-1}dx\\ &= \Phi[1-2\delta]\{\log N - \log(X/\omega')\}\\ &> \Phi[1-2\delta](1-\delta)\log N,\end{aligned}$$

for sufficiently large N. Thus we have an example in which the ratio of the left-hand side of (4.12) to the right-hand side is greater than

$$\Phi[1-2\delta](1-\delta)/\Phi(1).$$

Since $\Phi(x)$, being increasing and convex, is continuous, this expression can be made as near to 1 as we please. The last part of Theorem 2 has thus been established.

4.2. The result for moment constants corresponding to Theorem 2 is

THEOREM 3. *Let κ and $\Phi(x)$ be as in Theorem 2. Then, if*

$$A_n = \int_0^1 a(x)\,x^n\,dx, \tag{4.21}$$

we have

$$\sum_1^\infty \Phi[n^\kappa A_n]\,n^{-1} \leqslant \int_0^1 \Phi[\Gamma(\kappa)(1-x)^{1-\kappa}a(x)](1-x)^{-1}dx, \tag{4.22}$$

and $\Gamma(\kappa)$ on the right cannot be replaced by any smaller constant.

The proof is closely analogous to that of Theorem 2, except that it is not necessary to divide the proof into two cases. The rôle of $\Sigma a_{n-1}x^{n-1}$ in Case II of Theorem 2 is now taken by the integral

$$\int_0^1 [\log(1/x)]^{\kappa-1}x^{n-1}dx;$$

while, corresponding to the factor $\beta_n x$ which had to be introduced there, we use the factor

$$\beta(x) = x[(1-x)/\log(1/x)]^{\kappa-1}.$$

This never exceeds 1, since we have

$$(1-x)/\log(1/x) \leqslant 1,$$

$$\frac{x\log(1/x)}{1-x} = \frac{\log(1/x)}{(1/x)-1} \leqslant 1.$$

Further, the integral form (2.162) of Jensen's inequality is required in place of the series form. Otherwise there is no essential change. And lastly, to show that $\Gamma(\kappa)$ on the right-hand side of (4.22) cannot be replaced by any smaller constant, we take the example

$$\begin{aligned} u(x) &\equiv \Phi[\Gamma(\kappa)(1-x)^{1-\kappa}a(x)](1-x)^{-1} \\ &= (1-x)^{-1} \quad (0 \leqslant x \leqslant 1-\delta), \\ &= 0 \qquad\quad (1-\delta < x \leqslant 1), \end{aligned}$$

and proceed as in Theorem 2.

4.3. There follow next the versions of the above theorems in which the function x^κ is replaced by a more general function $\theta(x)$. It will be supposed throughout that $\theta(x)$ is positive and satisfies (3.21). We may remark that results in a slightly different form can be obtained if we substitute for $\theta(x)$ the function $\vartheta(x) = 1/\theta(1/x)$, which clearly satisfies the prescribed conditions at the same time as $\theta(x)$; also that no attempt is made to determine the best possible constants in the theorems that follow.

THEOREM 4. *Let $\Phi(x)$ be increasing and convex, with $\Phi(0) = 0$; and let $\theta(x)$ be positive and satisfy* (3.21). *Then we have*

$$(4.31) \quad \int_0^1 \Phi[\theta(1-x)A(x)](1-x)^{-1}dx \leqslant B_1 \sum_1^\infty \Phi[B_2 n\theta(1/n)a_n]n^{-1},$$

where B_1, B_2 are constants depending only on $\theta(x)$.

Write $$u_n \equiv \Phi[B_2 n\theta(1/n)a_n]n^{-1}.$$

Then $$A(x) = B_2^{-1}\sum_1^\infty \Phi^{-1}(nu_n)x^n\{n\theta(1/n)\}^{-1}.$$

But, by Lemma 4,

$$\theta(1-x)\Big[v_0+\sum_1^\infty v_n\Big] \equiv \theta(1-x)\Big[\{\theta(1-x)\}^{-1}+\sum_1^\infty x^n\{n\theta(1/n)\}^{-1}\Big] \leqslant B_2.$$

Therefore, with the summations still taken from 1 to ∞,

$$\theta(1-x)A(x) \leqslant \frac{v_0.0+\Sigma v_n\Phi^{-1}(nu_n)}{v_0+\Sigma v_n},$$

and so, remembering that $\Phi(x)$ is convex, we have

$$\Phi[\theta(1-x)A(x)] \leqslant \frac{v_0\Phi(0)+\Sigma v_n nu_n}{v_0+\Sigma v_n}$$

$$\leqslant \theta(1-x)\Sigma u_n x^n \{\theta(1/n)\}^{-1},$$

since $\Phi(0)=0$, and $v_0=\{\theta(1-x)\}^{-1}$. Hence the integral on the left of (4.31) is not greater than

$$\int_0^1 \left(\Sigma \frac{u_n x^n}{\theta(1/n)}\right) \frac{\theta(1-x)\,dx}{1-x} = \Sigma \frac{u_n}{\theta(1/n)} \int_0^1 \frac{\theta(1-x)\,x^n\,dx}{1-x}$$

$$\leqslant B_1 \Sigma u_n,$$

by Lemma 5.

THEOREM 5. *With $\Phi(x)$ and $\theta(x)$ as in Theorem 4, we have*

$$(4.32)\quad \sum_1^\infty \Phi[\theta(n)A_n]n^{-1} \leqslant B_2 \int_0^1 \Phi\left(B_1(1-x)\theta\{1/(1-x)\}a(x)\right)(1-x)^{-1}\,dx,$$

where B_1, B_2 depend only on $\theta(x)$.

The proof of this is similar to that of Theorem 4.

COROLLARY. *Inequalities of the form* (4.31) *and* (4.32) *can still be established when we assume merely that $\Phi(x)/x$ is quasi-increasing.*

For the modification of Jensen's inequality† given in Theorem 1 only differs from the original by the presence of the factors A and B on the right, and the latter clearly will not affect the form of (4.31) and (4.32); though, of course, B_1 and B_2 will be replaced by CAB_1 and BB_2, which are constants depending on $\Phi(x)$ as well as on $\theta(x)$. It is clear also that a possible breakdown in the definition of $\Phi^{-1}(x)$ is of no consequence, since u_n and $\Phi^{-1}(nu_n)$ are only introduced for convenience of presentation.

5. *The principal theorems.*

5.1. We give here the theorems in the more general case, which involve a function $\mathrm{X}(x)$ as well as $\Phi(x)$. The result may be stated thus.

THEOREM 6. *Let $\Phi(x)$, $\mathrm{X}(x)$, and $\theta(x)$ be non-negative functions with the following properties:*

(i) *$\Phi(x)/x$ is increasing;*

(ii) *$\mathrm{X}(x)/\Phi(x)$ is increasing;*

(iii) *$\theta(x)$ satisfies the condition* (3.21).

† $\Phi(x)$ satisfies this modified inequality, in virtue of (2.34).

Then we have

$$(5.11)\quad \int_0^1 \mathrm{X}[\theta(1-x)A(x)](1-x)^{-1}dx \leqslant \mathrm{X}\Phi^{-1}\Big(B_1'\sum_1^\infty \Phi[2B_2 n\theta(1/n)a_n]n^{-1}\Big),$$

where B_1', B_2 are constants depending only on $\theta(x)$ and $\mathrm{X}(x)/\Phi(x)$.

We write $\Lambda(x) \equiv x\lambda(x) \equiv \mathrm{X}\Phi^{-1}(x)$; then, since $\Phi(x)$ is increasing, (ii) implies that $\lambda(x)$ is increasing. Now, putting

$$u_n \equiv \Phi[2B_2 n\theta(1/n)a_n]n^{-1},$$

we have, as in Theorem 4, since $\Phi(x)$ is quasi-convex with $A=1$, $B=2$,

$$\Phi[\theta(1-x)A(x)] \leqslant \theta(1-x)\Sigma u_n x^n\{\theta(1/n)\}^{-1} = \Sigma u_n V_n,$$

say; and $V_n \leqslant B_2'$, by (3.221). Writing $B_1' = \mathrm{Max}[B_1, B_2']$, we have $V_n \leqslant B_1'$, and hence

$$\Lambda\Phi[\theta(1-x)A(x)] \leqslant \Lambda[\Sigma u_n V_n] = \Sigma u_n V_n . \lambda[\Sigma u_n V_n] \leqslant \lambda[B_1'\Sigma u_n]\Sigma u_n V_n,$$

since $\lambda(x)$ is increasing. But the left-hand side is $\mathrm{X}[\theta(1-x)A(x)]$, and therefore the integral in (5.11) is not greater than

$$\lambda[B_1'\Sigma u_n]\int_0^1 \Sigma u_n V_n(1-x)^{-1}dx = \lambda[B_1'\Sigma u_n]\Sigma u_n \int_0^1 V_n(1-x)^{-1}dx.$$

Now, by Lemma 5, the last integral is bounded above by B_1, and therefore also by B_1'. Hence the last expression is less than or equal to $\Lambda[B_1'\Sigma u_n]$, as required.

For moment constants there is the following result to correspond to the theorem just proved.

THEOREM 7. *With $\Phi(x)$, $\mathrm{X}(x)$, and $\theta(x)$ as in Theorem 6, we have*

$$(5.12)\quad \sum_1^\infty \mathrm{X}[\theta(n)A_n]n^{-1} \leqslant \mathrm{X}\Phi^{-1}\Big(B_2''\int_0^1 \Phi[B_1(1-x)\theta\{1/(1-x)\}a(x)](1-x)^{-1}dx\Big).$$

The proof of this theorem is like that of Theorem 6 and will therefore be omitted.

COROLLARY. *Inequalities of the forms* (5.11) *and* (5.12) *can still be obtained when we assume merely that* $\Phi(x)/x$ *and* $\mathrm{X}\Phi^{-1}(x)/x$ *are quasi-increasing.*

For the relaxation of the conditions on $\Phi(x)$ and $\mathrm{X}\Phi^{-1}(x)/x$ will only lead to the introduction of extra constant factors K at various stages of the proof, and these can be absorbed into the constants already present. $\Phi^{-1}(x)$ may possibly be many-valued, but the results will still hold good if stated in terms of $\Lambda(x) \equiv \mathrm{X}\Phi^{-1}(x)$ instead of in terms of $\mathrm{X}(x)$.

5.2. The theorems of this paragraph, like the original theorems given by Hardy and Littlewood, can be presented in rather simpler forms in certain special cases. To do this, however, it is necessary to restrict the functions $\Phi(x)$ and $\mathrm{X}(x)$ somewhat more than in the general case. It will be sufficient to give some of the special cases of Theorem 6 (on power series).

Suppose therefore that $\mathrm{X}(x)$ has the property (2.181), namely

$$\mathrm{X}(xy) \leqslant \mathrm{X}(x)\,\mathrm{X}(y) \quad (\text{all } x, y). \tag{5.201}$$

Then
$$\mathrm{X}[\theta(1-x)\,A(x)] \geqslant \mathrm{X}[A(x)]/\mathrm{X}[1/\theta(1-x)].$$

We now arrange that the denominator on the right shall cancel out with the factor $(1-x)^{-1}$ of the integral in Theorem 6. This requires

$$\theta(x) = 1/\mathrm{X}^{-1}(1/x).$$

$\theta(x)$ then has the property (2.181) at the same time as $\mathrm{X}(x)$ and therefore, in virtue of the remark below (2.262), satisfies the conditions (3.21). The inequality (5.11) thus leads to the following

THEOREM 8. *If* $\Phi(x)$ *and* $\mathrm{X}(x)$ *are as in Theorem* 6, *and in addition* $\mathrm{X}(x)$ *has the property* (2.181), *we have*

$$\int_0^1 \mathrm{X}[A(x)]\,dx \leqslant K\mathrm{X}\Phi^{-1}\Big(\sum_1^\infty \Phi[B_2 a_n n/\mathrm{X}^{-1}(n)]n^{-1}\Big). \tag{5.21}$$

A similar argument gives

THEOREM 9. *If* $\Phi(x)$ *and* $\mathrm{X}(x)$ *are as in Theorem* 6, *and in addition* $\Phi(x)$ *has the property* (2.181), *we have*

$$\int_0^1 \mathrm{X}\Big((1-x)\,\Phi^{-1}\{1/(1-x)\}\,A(x)\Big)(1-x)^{-1}\,dx \leqslant K\mathrm{X}\Phi^{-1}\Big(\sum_1^\infty \Phi(a_n)\Big). \tag{5.22}$$

There remains the question whether it is possible to simplify both sides of the inequality (5.11) and so obtain a generalization of the theorem†

$$\int_0^1 A^{p'}(x)\,dx \leqslant K\left(\sum_1^\infty a_n{}^p\right)^{p'-1}, \tag{5.23}$$

where $1<p\leqslant 2$, $p'=p/(p-1)$. It appears, however, that we can get only the following result.

THEOREM 10. *Suppose that $\Phi(x)/x$ is increasing and that it has the property* (2.181): *let $\Psi(x)$ be defined by*

$$\Psi^{-1}(x)=x\Phi^{-1}(1/x),$$

and suppose that $\Psi(x)/\Phi(x)$ is increasing. Then we have

$$\int_0^1 \Psi\,[\Psi^{-1}(1-x)\,A(x)](1-x)^{-1}\,dx \leqslant K\Psi\Phi^{-1}\left(\sum_1^\infty \Phi(a_n)\right), \tag{5.24}$$

where K depends only on $\Phi(x)$.

It can easily be shown that, under the conditions stated, $\Phi(x)/x^2$, if monotonic, must be decreasing; this corresponds to the condition $p\leqslant 2$ attached to (5.23). The latter is the analogue for power series of one part of the Young-Hausdorff theorem for Fourier series; it is therefore of interest to note that (5.23) appears to have no direct generalization which is conveniently obtainable‡.

6. *The case when $\Phi(x)/x$ is decreasing.*

6.1. So far we have supposed $\Phi(x)/x$ to be increasing. If, however, it is decreasing we get the inequalities of the previous theorems with their signs reversed§. As in the case when $\Phi(x)=x^p$, it does not seem possible to obtain inequalities with best possible constants. Accordingly we give below, as specimens, the results corresponding to Theorems 6 and 7.

† Hardy and Littlewood, **5**, Theorem 9.

‡ It is possible to get generalizations of the kind considered for the Young-Hausdorff theorem, but the discussion of these must be left to a subsequent paper.

§ See Hardy and Littlewood, **5**, § 4, where the case $0<p\leqslant 1$ is considered.

THEOREM 11. *Let* $\Phi(x)$, $\mathrm{X}(x)$, *and* $\theta(x)$ *be as in Theorem* 6, *except that* $\Phi(x)/x$ *and* $\mathrm{X}(x)/\Phi(x)$ *are to be decreasing. Then we have*

$$(6.11)\quad B_5\,\mathrm{X}\Phi^{-1}\Big(\sum_1^\infty \Phi\,[n\theta(1/n)\,a_n]\,n^{-1}\Big) \leqslant \int_0^1 \mathrm{X}\,[\theta(1-x)\,A(x)](1-x)^{-1}\,dx.$$

THEOREM 12. *Under the same conditions*

$$(6.12)\quad B_6\,\mathrm{X}\Phi^{-1}\Big(\int_0^1 \Phi\,[B_3(1-x)\,\theta\{1/(1-x)\}\,a(x)](1-x)^{-1}\,dx\Big) \leqslant \mathrm{X}(A_0)+\sum_1^\infty \mathrm{X}\,[\theta(n)\,A_n]\,n^{-1}.$$

The proofs of these theorems are very like those of Theorems 6 and 7, and will therefore be omitted. However, in the inequality of Theorem 12 there appears a term $\mathrm{X}(A_0)$ which was not needed before†. This enters into the proof at the end. It can be shown after the manner of the proof of Theorem 6 that, if

$$u(x) \equiv \Phi\,[B_3(1-x)\,\theta\{1/(1-x)\}\,a(x)](1-x)^{-1}$$

and $n \geqslant 1$, we have, writing $1/\vartheta(x)$ for $\theta(1/x)$,

$$\mathrm{X}\,[\theta(n)\,A_n] \geqslant K\lambda\Big(\int_0^1 u(x)\,dx\Big)\int_0^1 u(x)\,x^n\,\vartheta(1-x)\,[\vartheta(1/n)]^{-1}\,dx;$$

while, in the same way,

$$\mathrm{X}\,(A_0) \geqslant K\lambda\Big(\int_0^1 u(x)\,dx\Big)\int_0^1 u(x)\,\vartheta(1-x)\,dx.$$

Hence

$$\Big(\mathrm{X}(A_0)+\sum_1^\infty \mathrm{X}\,[\theta(n)\,A_n]\,n^{-1}\Big)\Big/K\lambda\Big(\int_0^1 u(x)\,dx\Big) \geqslant \int_0^1 u(x)\Big(\vartheta(1-x)+\sum_1^\infty \vartheta(1-x)\,x^n\,[n\vartheta(1/n)]^{-1}\Big)\,dx,$$

and, by Lemma 4, the series on the right is not less than $B_4\,x$, and hence, when taken together with the extra term, is greater than a positive non-zero constant. Theorem 12 now follows.

† For the necessity of this term, *cf. ibid.*, 4.2.

6.2. It is clear that from the above theorems certain special cases may be obtained with simpler forms, as in 5.2. There will be results corresponding to Theorem 8, but not to Theorem 9†.

7. *Derivative theorems.*

From Theorem 8 may be deduced a theorem involving two power series‡:

$$A(x) = \sum_1^\infty a_n x^n, \quad B(x) = \sum_1^\infty b_n x^n.$$

THEOREM 13. *Suppose that* $\Phi(x)$ *and* $\Psi(x)$ *are increasing, convex, and vanishing at* 0, *and that they both satisfy* (2.181); *let* $\mathrm{X}(x)$ *be defined by*

$$\mathrm{X}^{-1}(x)\,\Psi^{-1}(x) = x, \tag{7.1}$$

and suppose that $\mathrm{X}(x)$ *is convex, and* $\mathrm{X}(x)/\Phi(x)$ *increasing. Then*

$$\int_0^1 (1-x)\,\Phi^{-1}[1/(1-x)]\,\Psi^{-1}[1/(1-x)]\,A(x)\,B(x)\,dx \leqslant K\frac{\Phi^{-1}[\Sigma\,\Phi(a_n)]}{\Psi^{-1}[1/\Sigma\,\Psi(b_n)]}. \tag{7.2}$$

It follows from (7.1) that $\mathrm{X}^{-1}(x)$ has the property (2.181) at the same time as $\Psi(x)$; thus $\mathrm{X}(x)$ will have the opposite property (2.182), namely

$$\mathrm{X}(xy) \geqslant \mathrm{X}(x)\,\mathrm{X}(y) \quad (\text{all } x, y).$$

Since $\mathrm{X}(x)$ is also convex we may apply (2.192) and get

$$\int_0^1 (1-x)\,\Phi^{-1}\{1/(1-x)\}\,A(x)\,.\,(1-x)\,\Psi^{-1}\{1/(1-x)\}\,B(x)\,.\,(1-x)^{-1}\,dx \leqslant \frac{\mathrm{X}^{-1}\left(\int_0^1 \mathrm{X}\left\{(1-x)\,\Phi^{-1}\{1/(1-x)\}\,A(x)\right\}(1-x)^{-1}\,dx\right)}{\Psi^{-1}\left(1\Big/\int_0^1 \Psi\left\{(1-x)\,\Psi^{-1}\{1/(1-x)\}\,B(x)\right\}(1-x)^{-1}\,dx\right)}.$$

But, by Theorem 9, the integral in the numerator is not greater than

$$K\mathrm{X}\Phi^{-1}[\Sigma\,\Phi(a_n)];$$

while that in the denominator does not exceed

$$K\,\Sigma\,\Psi(b_n).$$

† Compare Hardy and Littlewood, **5**, 4.3.

‡ *Cf. ibid.*, Theorem 16.

Therefore the integral in (7.2) is not greater than

$$\frac{X^{-1}[KX\Phi^{-1}\{\Sigma\Phi(a_n)\}]}{\Psi^{-1}[1/K\Sigma\Psi(b_n)]} \leqslant K\frac{\Phi^{-1}[\Sigma\Phi(a_n)]}{\Psi^{-1}[1/\Sigma\Psi(b_n)]}.$$

In a similar way we can obtain the parallel result for moment constants.

THEOREM 14. *Under the same conditions*

$$(7.3) \qquad \sum_1^\infty n\Phi^{-1}(1/n)\Psi^{-1}(1/n)A_nB_n \leqslant \frac{\Phi^{-1}\left(\int_0^1 \Phi[a(x)]\,dx\right)}{\Psi^{-1}\left(1\Big/\int_0^1 \Psi[b(x)]\,dx\right)}.$$

Like its prototype in Hardy and Littlewood's paper, Theorem 13 may be transformed into a theorem on double series. For

$$A(x)B(x) = \sum_1^\infty\sum_1^\infty a_m b_n x^{m+n},$$

and hence the left-hand side of (7.2) is equal to

$$\sum_1^\infty\sum_1^\infty a_m b_n \int_0^1 (1-x)^2\Phi^{-1}[1/(1-x)]\Psi^{-1}[1/(1-x)]x^{m+n}(1-x)^{-1}dx.$$

If we use the fact that $\Phi(x)$ and $\Psi(x)$ satisfy (2.181), and apply an argument like that of Lemma 5, we can show that this integral is not less than

$$\frac{K}{(m+n)^2\Phi^{-1}[1/(m+n)]\Psi^{-1}[1/(m+n)]}.$$

We now have

THEOREM 15. *Under the same conditions as for Theorem* 13 *we have*

$$(7.4) \qquad \sum_1^\infty\sum_1^\infty \frac{a_m b_n}{(m+n)^2\Phi^{-1}[1/(m+n)]\Psi^{-1}[1/(m+n)]} \leqslant K\frac{\Phi^{-1}[\Sigma\Phi(a_n)]}{\Psi^{-1}[1/\Sigma\Psi(b_n)]}.$$

At first sight it would seem as if we might get a generalization of Hilbert's theorem on double series by taking $\Phi^{-1}(x)\Psi^{-1}(x) = x$ in (7.4). However, $\Phi(x)$ and $\Psi(x)$ could not both satisfy (2.181) if this held, so that the conditions attached to (7.4) would be violated†.

Lastly, we may remark that Theorem 12 can be transformed in a similar manner.

† A generalization of Hilbert's theorem can be established directly; see **7**, Theorem 1.

8. *Restrictions on the functions* $\Phi(x)$, $X(x)$, *and* $\theta(x)$.

8.1. The theorems above have been proved with the aid of various assumptions concerning the nature of the functions $\Phi(x)$, $X(x)$, and $\theta(x)$; we proceed to inquire how far these assumptions are necessary to the truth of the theorems. In what follows we shall consider only the conditions attached to Theorems 2, 4, and 6; there are similar results for the conditions belonging to the other theorems.

Some information on this matter may be obtained by considering the case when $\Phi(x) = x^p$, $X(x) = x^r$, and $\theta(x) = x^{1/q}$. We then have the inequality (1.1) subject to the conditions $r \geqslant p \geqslant 1$, $q > 0$. It is known† that these conditions are necessary. This suggests that for Theorem 6 to be true $\Phi(x)/x$, $X(x)/\Phi(x)$, and $\theta(x)$ must be increasing (or at least quasi-increasing). However, rather more than this was assumed; it was supposed that $\Phi(x)$ in Theorem 2 was quasi-convex and vanishing for $x = 0$, and that $\theta(x)$ satisfied (3.21); we shall have to examine the necessity of these conditions also.

To start with, it is clear on inspection that the left-hand side of the inequality in Theorem 6 will be infinite unless $X[\theta(x)] \to 0$ with x, and that the right will in general be infinite unless $\Phi(x) \to 0$ with x and $\Phi(0) = 0$. Since Theorem 6 should hold, in particular, when $X(x) = \Phi(x)$ (the case that has been stated separately as Theorem 4), we must have $\theta(x) \to 0$, and therefore $X(x) \to 0$, with x; but $\Phi(x) \to 0$ is incompatible with the supposition that $\Phi(x)$ should be quasi-decreasing, and similarly for the other functions. Thus, as far as quasi-monotonic functions are concerned, *we may suppose without excluding any cases of real interest that* $X(x)$, $\Phi(x)$, *and* $\theta(x)$ *are quasi-increasing, vanish at* $x = 0$, *and tend to* 0 *with* x. Incidentally, the condition that $\theta(x)$ should be quasi-increasing is the generalization of the condition $q > 0$ which is necessary for (1.1).

8.2. We take first Theorem 2, to prove which we assumed that $\Phi(x)$ was increasing, vanishing at $x = 0$, and convex. Concerning the necessity of these conditions we have the following result.

THEOREM 16. *If* $\Phi(x)$ *is continuous*‡ *and such that the inequality of Theorem* 2 *holds for all power series admissible, then* $\Phi(x)$ *must also be increasing and convex.*

It is sufficient to show that $\Phi(x)$ is convex, since any convex function which vanishes at $x = 0$ is necessarily increasing. For the sake of

† Hardy and Littlewood, **5**, 4.4.

‡ The same result can also be obtained if we merely assume that $\Phi(x)$ is increasing; but the proof then requires a more complicated example, such as that used below in Theorem 17.

simplicity we shall consider the case $q = \kappa = 1$. Thus we have

$$\int_0^1 \Phi[(1-x)A(x)](1-x)^{-1}dx \leqslant \sum_1^\infty \Phi(a_n)n^{-1} \tag{8.21}$$

for all power series $$A(x) = \sum_1^\infty a_n x^n \quad (a_n \geqslant 0).$$

We take an adaptation of the example used to show that $\Gamma(\kappa)$ is the best possible constant in Theorem 2; let

$$\left.\begin{aligned} a_{2n-1} &= \alpha \\ a_{2n} &= \beta \end{aligned}\right\} \quad (1 \leqslant n \leqslant N),$$

$$a_n = 0 \qquad (n > 2N).$$

Then

$$\sum_1^\infty \Phi(a_n)n^{-1} = \Phi(\alpha)\sum_1^N (2n-1)^{-1} + \Phi(\beta)\sum_1^N (2n)^{-1} \tag{8.22}$$

$$\leqslant (\tfrac{1}{2}\log N + 2)\{\Phi(\alpha)+\Phi(\beta)\},$$

while $$A(x) = \alpha\sum_1^N x^{2n-1} + \beta\sum_1^N x^{2n},$$

$$= (\alpha x + \beta x^2)(1-x^{2N})/(1-x^2),$$

and $$(1-x)A(x) = x(\alpha+\beta x)(1-x^{2N})/(1+x).$$

Hence, if $1-\epsilon < x < \epsilon^{1/2N} = \eta$, we have

$$\tfrac{1}{2}(\alpha+\beta) > (1-x)A(x) > \tfrac{1}{2}(\alpha+\beta)(1-\epsilon)^3,$$

and therefore, $\Phi(x)$ being continuous, the integral in (8.21) is greater than

$$\int_{1-\epsilon}^{\eta} \Phi[\tfrac{1}{2}(\alpha+\beta)(1-\vartheta)](1-x)^{-1}dx,$$

where $0 \leqslant \vartheta \leqslant 1-(1-\epsilon)^3$; and it may easily be shown, as in the similar argument in the proof of Theorem 2, that this integral is asymptotically equal to

$$\Phi[\tfrac{1}{2}(\alpha+\beta)(1-\vartheta)]\log N$$

when N is large. Combining this with (8.21) and (8.22), we have, after letting N tend to ∞,

$$\Phi[\tfrac{1}{2}(\alpha+\beta)(1-\vartheta)] \leqslant \tfrac{1}{2}[\Phi(\alpha)+\Phi(\beta)].$$

Since ϑ may be arbitrarily small and $\Phi(x)$ is continuous, this implies that

$$\Phi[\tfrac{1}{2}(\alpha+\beta)] \leqslant \tfrac{1}{2}[\Phi(\alpha)+\Phi(\beta)].$$

In other words $\Phi(x)$ is convex.

For the conditions on Theorem 4 we can establish a corresponding result.

THEOREM 17. *If $\Phi(x)$ is quasi-increasing and the inequality of Theorem 4 holds, for all power series $A(x)$, with some function $\theta(x)$ which satisfies* (3.21), *then $\Phi(x)/x$ is quasi-increasing.*

We are given now

$$(8.23) \qquad \int_0^1 \Phi[\theta(1-x)A(x)](1-x)^{-1}dx \leqslant K_1 \sum_1^\infty \Phi[K_2 n\theta(1/n)a_n]n^{-1}$$

and, in virtue of the Corollary to Theorem 1, it will be sufficient† if we prove that

$$(8.24) \qquad \Phi\left(\frac{\mu x+\nu y}{\mu+\nu}\right) \leqslant A\,\frac{\mu\Phi(Bx)+\nu\Phi(By)}{\mu+\nu}$$

for all positive integers μ and ν. To do this we employ an elaboration of the example used in the last theorem. Writing $v_n = K_2 n\theta(1/n)a_n/B$, we take v_n equal to α in the first μ terms of the series, equal to β in the next ν terms, and so on, alternately, as far as $n = (\mu+\nu)N$; after which we put $v_n = 0$. For $\mu = \nu = 1$ we get the case which corresponds to the example used above; in general, the right-hand side of (8.23) has the asymptotic value

$$K_1 \log N\{\mu\Phi(B\alpha)+\nu\Phi(B\beta)\}/(\mu+\nu),$$

while for the left-hand side we can show, somewhat as before and by employing the method used in the second part of the proof of Lemma 4, that

$$\theta(1-x)A(x) \geqslant BK_2^{-1}K(\mu\alpha+\nu\beta)/(\mu+\nu) \quad (1-\omega \leqslant x \leqslant 1-XN^{-1}).$$

Choosing $B = K_2/K$ and completing the proof as in the last theorem, we get

$$C\Phi\left(\frac{\mu\alpha+\nu\beta}{\mu+\nu}\right) \leqslant K_1\,\frac{\mu\Phi(B\alpha)+\nu\Phi(B\beta)}{\mu+\nu},$$

which is the same as (8.24) provided that we take $K_1 = AC$.

† For (8.24) implies that $\Phi(x)$ is quasi-convex, and hence, by (2.34), that $\Phi(x)/x$ is quasi-increasing.

8.3. Turning now to the conditions required for the proof of Theorem 6, we note in the first place that when $X(x) = \Phi(x)$ this theorem reduces to Theorem 4. Thus the condition that $\Phi(x)/x$ should be quasi-increasing is in a certain sense necessary to the truth of Theorem 6 also. Next we have to consider the condition that $X(x)/\Phi(x)$ should be quasi-increasing. In the case of powers, to show that the condition $r \geqslant p$ is necessary to the truth of (1.1), Hardy and Littlewood employ the example† $a_n = n^{(1/q)-1}(\log n)^{-(1/p)-\delta}$, $n \geqslant 2$; but it appears that a simpler example of the type used above will suffice to deal with the matter. It is convenient to state the result in terms of the function $\Lambda(x) = X\Phi^{-1}(x)$.

THEOREM 18. *Let the inequality of Theorem* 6, *namely*

$$(8.311) \qquad \int_0^1 \Lambda\Phi[\theta(1-x)A(x)](1-x)^{-1}dx \leqslant \Lambda\left(K_1 \sum_1^\infty \Phi[K_2 n\theta(1/n)a_n]n^{-1}\right),$$

hold, for all power series $A(x) = \Sigma a_n x^n$, *with some* $\Phi(x)$ *which vanishes at* 0, *is quasi-increasing, and satisfies the condition*

$$(8.312) \qquad \Phi(2x) \leqslant A\Phi(x)$$

for some A; *and with some* $\theta(x)$ *satisfying a condition of type* (3.21). *Then*

$$(8.313) \qquad \Lambda(x)/x \leqslant K\Lambda(y)/y \quad (x \leqslant y),$$

i.e. $\Lambda(x)/x$ *is quasi-increasing.*

For simplicity I shall take $\Phi(x)$ and $\Lambda(x)$ to be increasing, and not merely quasi-increasing; this involves no loss of generality. To establish (8.313) we consider the example

$$(8.32) \qquad K_2 a_n n\theta(1/n) = u \quad (n \leqslant N),$$
$$= 0 \quad (n > N).$$

The right-hand side of (8.311) is then

$$(8.331) \qquad \Lambda\left(K_1\Phi(u)\sum_1^N n^{-1}\right) = \Lambda\{K_1\Phi(u)(1+\epsilon_N)\log N\},$$

where $\epsilon_N \to 0$ as $N \to \infty$, while we may show as before that the left-hand side of (8.311) is greater than

$$(8.332) \qquad \Lambda\Phi(K'u)\int_{\frac{1}{2}}^{e^{-1/N}} (1-x)^{-1}dx = \Lambda\Phi(K'u)(1+\epsilon_N')\log N,$$

† Hardy and Littlewood, **5**, 4.4. I take this opportunity to remark that the corresponding example for (1.4) is $a_n = (\log n)^{(1/q)-1}(\log\log n)^{-(1/p)-\delta}$, $n \geqslant 3$; and that the example given in § 3.4 of my previous paper is insufficient.

where $\epsilon_N' \to 0$ as $N \to \infty$. Next, as at (2.261), it follows from (8.312) that

$$\Phi(\tau x) \geqslant A^{-1}\tau^k \Phi(x) \quad (\tau \leqslant 1),$$

where $k = \log A/\log 2$. Hence, in particular, if $K' \leqslant 1$ we have

$$\Phi(K'u) \geqslant K''\Phi(u),$$

and this is true trivially when $K' \geqslant 1$. It follows now from (8.311), (8.331), and (8.332) that

$$(8.333) \qquad \Lambda[K''\Phi(u)] \log N(1+\epsilon_N') \leqslant \Lambda[K_1\Phi(u) \log N(1+\epsilon_N)].$$

But nothing in our argument has involved any restriction on u, so that we may take

$$\Phi(u) = t/\{K_1 \log N(1+\epsilon_N)\} = v_N t/K'',$$

say. This leads to

$$\Lambda(v_N t)/\{v_N \Lambda(t)\} \leqslant K \quad (N \to \infty).$$

Since $v_N \to 0$ and $\Lambda(x)$ is increasing, we have

$$\Lambda(xt)/\{x\Lambda(t)\} \leqslant K \quad (x \to 0).$$

Hence

$$\Lambda(xt) \leqslant Kx\Lambda(t) \quad (0 \leqslant x \leqslant 1)$$

or

$$\Lambda(xt)/(xt) \leqslant K\Lambda(t)/t,$$

i.e. $\Lambda(x)/x$ is quasi-increasing.

8.4. When we come to the necessity of the conditions imposed on $\theta(x)$ in Theorems 4–7 the results are as follows.

THEOREM 19. *Let the inequality of Theorem* 6 *hold with some* $\mathrm{X}(x)$ *and some* $\Lambda(x) \equiv \mathrm{X}\Phi^{-1}(x)$ *that are quasi-increasing. Then the function* $\theta(x)$, *supposed quasi-increasing, must satisfy inequalities of the type* (3.21).

Let us bear in mind that $\mathrm{X}(x)$, $\Lambda(x)$, and $\theta(x)$ are all three quasi-increasing; and let us take the example

$$(8.41) \qquad u_n \equiv \Phi[K_2 n\theta(1/n)\, a_{n]} n^{-1} = n^{-1} \quad (N \leqslant n < 2N),$$

$$= 0 \qquad (n < N, \ n \geqslant 2N).$$

Then, by hypothesis,

$$(8.42) \qquad \int_0^1 \mathrm{X}[\theta(1-x)\, A(x)](1-x)^{-1}\, dx \leqslant \Lambda\Big[K_1 \sum_N^{2N-1} n^{-1}\Big] \leqslant K.$$

The series in fact does not exceed $\log 2$. Now, for the left-hand side, replacing each term in $A(x)$ by its lower bound, we get

$$\begin{aligned} A(x) &= K_2^{-1} \sum_{N}^{2N-1} \Phi^{-1}(nu_n)\, x^n [n\theta(1/n)]^{-1} \\ &\geqslant \Phi^{-1}(1)\, K_2^{-1}\, N x^{2N-1}/(2N-1)\, K\theta(1/N) \\ &\geqslant K x^{2N}/\theta(N^{-1}). \end{aligned}$$

Thus (8.42) implies

$$\int_0^1 \mathrm{X}\Big(Kx^{2N}\, \theta(1-x)/\theta(N^{-1}) \Big)(1-x)^{-1}\, dx \leqslant K.$$

Discarding the portion of the integral outside the range $(1-aN^{-1},\ 1-bN^{-1})$, we have

$$\mathrm{X}\Big(K(1-aN^{-1})^{2N}\, \theta(bN^{-1})/\theta(N^{-1}) \Big) \int_{1-aN^{-1}}^{1-bN^{-1}} (1-x)^{-1} dx \leqslant K.$$

But this integral is equal to $\log(a/b)$; therefore

$$\mathrm{X}[Ke^{-2a}\, \theta(bN^{-1})/\theta(N^{-1})] \log(a/b) \leqslant K. \tag{8.43}$$

Take first $b = 4$, $a = 4\frac{1}{2}$; then

$$\theta(4N^{-1})/\theta(N^{-1}) \leqslant K \quad (N \geqslant 5 > a).$$

Hence, if $N^{-1} \leqslant x \leqslant 2N^{-1} \leqslant \frac{2}{5}$,

$$\theta(2x)/\theta(x) \leqslant K\theta(4N^{-1})/\theta(N^{-1}) \leqslant K;$$

and, therefore, since $\theta(2x)/\theta(x)$ is clearly bounded in $(\frac{2}{5},\ \frac{1}{2})$,

$$\theta(2x) \leqslant K\theta(x) \quad (0 < 2x \leqslant 1) \tag{8.441}$$

But this is of the form (2.261) and therefore implies that

$$\theta(tx) \leqslant Kt^k\, \theta(x) \quad (t \geqslant 1), \tag{8.442}$$

where $k > 0$. Secondly, take $a = 1$ in (8.43); then

$$\mathrm{X}\,[Ke^{-2}\, \theta(bN^{-1})/\theta(N^{-1})] \to 0 \text{ as } b \to 0.$$

Hence, for an ϵ, we may choose b so small that

$$\theta(bN^{-1})/\theta(N^{-1}) \leqslant \epsilon \quad (N \geqslant 2 > a). \tag{8.451}$$

From this we may deduce, somewhat as before,

$$\theta(\tau x) \leqslant \eta\tau^{\kappa}\, \theta(x) \quad (\tau \leqslant 1), \tag{8.452}$$

where $\eta = K\epsilon$ is less than 1 by choice of ϵ, and $\kappa = \log(1/\eta)/\log 2$. But (8.452) and (8.442) together constitute a condition of type (3.21), so that the proof is complete.

9. *Analogous results for Dirichlet series.*

We have to consider now the generalizations of the theorems on Dirichlet series and on the related integrals. I shall content myself with the extensions of the theorem (1.4) on ordinary Dirichlet series and the theorem (1.5) on general Dirichlet series. There are naturally similar theorems for the related integrals

$$A_n = \int_0^\infty a(x)\, n^{-x}\, dx \quad \text{or} \quad \int_0^\infty a(x)\, e^{-\lambda_n x} dx;$$

but they may be written down quite easily on the model of the theorems for series, and need not be stated separately.

THEOREM 20. *If* $\Phi(x)$, $\mathrm{X}(x)$, *and* $\theta(x)$ *are as in Theorem* 6, *and*

$$A(x) = \sum_2^\infty a_n\, n^{-1-x}, \tag{9.1}$$

then

$$\int_0^\infty \mathrm{X}[\theta(x)\, A(x)]\, x^{-1}\, dx \leqslant \mathrm{X}\Phi^{-1}\Big(C_1 \sum_2^\infty \Phi\,[C_2 \log n\, \theta(1/\log n)\, a_n]\,(n \log n)^{-1}\Big). \tag{9.2}$$

THEOREM 21. *Under the same conditions, if* $\{\lambda_n\}$ *is a strictly increasing sequence of positive numbers and*

$$A(x) = \sum_1^\infty a_n\, e^{-\lambda_n x}, \quad \delta_n = \lambda_n - \lambda_{n-1}, \tag{9.3}$$

then

$$\int_0^\infty \mathrm{X}[\theta(x)\, A(x)]\, x^{-1}\, dx \leqslant \mathrm{X}\Phi^{-1}\Big(D_1 \sum_1^\infty \Phi[D_2\, \delta_n^{-1} \lambda_n\, \theta(1/\lambda_n)\, a_n]\, \delta_n\, \lambda_n^{-1}\Big). \tag{9.4}$$

Theorem 20 is equivalent to the particular case of the latter theorem for which $\lambda_n = \log n$ and $a_1 = 0$. The proof of either theorem follows exactly the lines of the proof of Theorem 6, the only essential difference being that Lemmas 4 and 5 are no longer applicable: it will suffice, therefore, to give the new lemmas needed for Theorem 21.

LEMMA 6. *If $\theta(x)$ satisfies the condition* (3.21), *then*

$$1+D_4 e^{-\lambda_1 x} \leqslant 1+\theta(x)\sum_1^\infty \delta_n e^{-\lambda_n x}[\lambda_n \theta(1/\lambda_n)]^{-1} \leqslant D_2. \tag{9.5}$$

For the right-hand side, as in Lemma 4, it is sufficient to prove that

$$\sum_1^\infty \delta_n (\lambda_n x)^\alpha e^{-\lambda_n x}\lambda_n^{-1} \quad (\alpha > 0)$$

is bounded above by a constant. Now, if $\alpha \leqslant 1$, $(tx)^\alpha e^{-tx}t^{-1}$ decreases as t increases, and hence the general term of the series does not exceed the integral of this function over $(\lambda_{n-1}, \lambda_n)$, an interval of length δ_n. Thus we have as an upper bound for the series the expression

$$\int_0^\infty (tx)^\alpha e^{-tx}t^{-1}\,dt = \int_0^\infty u^{\alpha-1}e^{-u}\,du = K.$$

When $\alpha > 1$, we note that $u^{\alpha-1}e^{-u}$ takes its maximum value at the point $u = Tx \equiv \alpha-1$, and is monotonic on either side of T. Accordingly, the part of the series for which $\lambda_n > T$ may be treated as before; while for the terms with $\lambda_n \leqslant T$ we have

$$\begin{aligned}\Sigma\,\delta_n(\lambda_n x)^\alpha e^{-\lambda_n x}\lambda_n^{-1} &= \Sigma\,\delta_n(\lambda_{n-1}x)^\alpha e^{-\lambda_n x}\lambda_n^{-1}{}_1 + \Sigma\,\delta_n x^\alpha[\lambda_n^{\alpha-1}-\lambda_{n-1}^{\alpha-1}]e^{-\lambda_n x}\\ &\leqslant \int_0^T (tx)^\alpha e^{-tx}t^{-1}\,dt + \operatorname*{Max}_{\lambda_n \leqslant T}\delta_n \cdot x^\alpha e^{-\lambda_1 x}\lambda_T^{\alpha-1}\\ &\leqslant \int_0^\infty u^{\alpha-1}e^{-u}\,du + (\lambda_T x)^\alpha \leqslant \Gamma(\alpha)+(Tx)^\alpha = K,\end{aligned}$$

where λ_T denotes the largest λ_n which does not exceed T. The right-hand inequality in (9.5) has thus been proved, and the left-hand inequality can be established by a similar adaptation of the methods of Lemma 4.

LEMMA 7. *If $\theta(x)$ satisfies the condition* (3.21), *then*

$$D_3 \leqslant \int_0^\infty \frac{\theta(x)\,e^{-\lambda x}\,dx}{\theta(1/\lambda)\,x} \leqslant D_4. \tag{9.6}$$

As in Lemma 5, the integral does not exceed

$$\int_0^\infty \chi(\lambda x)\,e^{-\lambda x}x^{-1}\,dx = \int_0^\infty \chi(u)\,e^{-\lambda u}u^{-1}\,du$$

which we may take as D_4: similarly also for the left-hand side.

The proof of the lemmas, and with them the theorems of this paragraph, is now complete.

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ELIMINANTS OF CHARACTERISTIC EQUATIONS

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1. *Introduction.* In a complementary paper† I have considered the solution in descriptive terms of the equation $px = xq$ in a linear associative algebra. A discussion of the same problem, analytically, leads to some interesting results on the eliminants of characteristic equations. A brief survey of the properties of the original solution discloses the nature of the problem.

There is no solution of the equation $px = xq$ other than $x = 0$, unless the reduced equations of p and q have at least one zero in common, in which case there are solutions which are divisors of zero. Further, if the reduced equations are identical, there are additional solutions which are neither zero nor divisors of zero.

The general solution, given in § 7*,

$$x = (p-\rho_{s+1})\dots(p-\rho_l)\{\Sigma(p-\lambda_2)^{t_2}\dots(p-\lambda_s)^{t_s}$$
$$\times[(p-\lambda_1)^{t_1-1}C_1+(p-\lambda_1)^{t_2-2}C_1(q-\lambda_1)+\dots+(p-\lambda_1)^{t_1-t_1'}C_1(q-\lambda_1)^{t_1'-1}]$$
$$\times(q-\lambda_2)^{t_2'}\dots(q-\lambda_s)^{t_s'}\}(q-\sigma_{s+1})\dots(q-\sigma_m) \quad (t_1 > t_1'), \tag{1}$$

contains as many arbitrary constants as there are different common roots in the reduced equations. These constants, however, are not scalars, and so the number of independent solutions is not given explicitly by (1), although determinable in any given case. It is clear that this number cannot exceed the order n of the algebra (§ 6*), nor be less than the number of different common roots of the reduced equations (§ 3*). In fact, it is easy to show that the number of independent solutions cannot be less than the number of common roots of the reduced equations of

† R. Wilson, "The equation $px = xq$ in a linear associative algebra", *Proc. London Math. Soc.* (2), 30 (1930), 359–366. References to this paper will bear an asterisk.

p and q counted according to multiplicity†, and it will now be identified with the rank of the eliminant of the characteristic equations of p (left) and q (right).

2. *Analytical solution.* Let

$$p = p_1 e_1 + p_2 e_2 + \ldots + p_n e_n,$$

$$q = q_1 e_1 + q_2 e_2 + \ldots + q_n e_n,$$

where

$$e_r e_s = \sum_{t=1}^{n} \gamma_{rst} e_t,$$

and the p's, q's, γ's are taken over a field.

If we assume a solution

$$x = x_1 e_1 + x_2 e_2 + \ldots + x_n e_n,$$

the identity

$$\left(\sum_{r=1}^{n} p_r e_r\right)\left(\sum_{s=1}^{n} x_s e_s\right) = \left(\sum_{r=1}^{n} x_r e_r\right)\left(\sum_{s=1}^{n} q_s e_s\right)$$

becomes

$$\sum_{r,s,t=1}^{n} p_r x_s \gamma_{rst} e_t = \sum_{r,s,t=1}^{n} x_r q_s \gamma_{rst} e_t,$$

giving rise to the n linear equations

$$\sum_{r=1}^{n} \left[\sum_{s=1}^{n} (p_s \gamma_{srt} - q_s \gamma_{rst})\right] x_r = 0 \quad (t = 1, 2, \ldots, n).$$

When we write a_{tr} for $\sum_{s=1}^{n} (p_s \gamma_{srt} - q_s \lambda_{rst})$, these become

$$\sum_{r=1}^{n} a_{tr} x_r = 0 \quad (t = 1, 2, \ldots, n).$$

It follows that, unless the determinant $|a_{tr}|$ of order n is zero, there is no solution other than $x_1 = x_2 = \ldots = x_n = 0$, or $x = 0$.

If $|a_{tr}| = |a_{rt}| = 0$, consider the set of identical relations

$$pe_1 - e_1 q = a_{11} e_1 + a_{21} e_2 + \ldots + a_{n1} e_n,$$

$$pe_2 - e_2 q = a_{12} e_1 + a_{22} e_2 + \ldots + a_{n2} e_n,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$pe_n - e_n q = a_{1n} e_1 + a_{2n} e_2 + \ldots + a_{nn} e_n.$$

† J. B. Shaw, *Synopsis of linear algebras* (1907), § 47 in conjunction with § 2* gives the desired result.

If $|a_{rt}|$ be of rank $n-1$, then, applying Cramér's rule,

$$(pe_1-e_1q)A_{1s}+(pe_2-e_2q)A_{2s}+\ldots+(pe_n-e_nq)A_{ns}=0,$$

or
$$p(A_{1s}e_1+A_{2s}e_2+\ldots+A_{ns}e_n)=(A_{1s}e_1+A_{2s}e_2+\ldots+A_{ns}e_n)q,$$

giving the solution

$$x=c(A_{1s}e_1+A_{2s}e_2+\ldots+A_{ns}e_n),$$

where c is an arbitrary scalar and A_{rt} the cofactor of a_{tr}.

There are no other solutions, since

$$A_{1s}:A_{2s}:\ldots:A_{ns}=A_{1l}:A_{2l}:\ldots:A_{nl}.$$

Similarly, if $|a_{rt}|$ is of rank $n-k$, the solution involves k scalar arbitraries and there are k independent solutions.

Hence the number of independent solutions of the equation $px=xq$ is exactly that of the rank of $|a_{rt}|$, and this determinant is an eliminant of the reduced equations of p and q, since its vanishing is the necessary and sufficient condition that they have a common root (§1). It is also the eliminant of the characteristic equations of p (left) and q (right), since every root of the reduced equation is a root of either characteristic equation, and conversely†.

3. *The second eliminant.* The eliminant of the characteristic equations of p (right) and q (left) is obtained similarly from the equation $qx=xp$. The condition that either eliminant should vanish is the same, viz. that the reduced equations of p and q have a common root. We shall prove that in this case both eliminants have the same rank.

Since, from (1), the solutions of $px=xq$ are all of the form

$$x=p^hf_0(q)+p^{h-1}f_1(q)+\ldots+pf_{h-1}(q)+f_h(q),$$

to every such corresponds uniquely the solution

$$x=q^hf_0(p)+q^{h-1}f_1(p)+\ldots+qf_{h-1}(p)+f_h(p),$$

of $qx=xp$, and *conversely*.

Now consider a complete set of linearly independent solutions of $px=xq$. Any other solution is expressible as a linear sum of scalar multiples of these, and the corresponding solution of $qx=xp$ has the same form. Consequently to every linearly dependent solution of $px=xq$ corresponds a linearly dependent solution of $qx=xp$, and conversely. It

† L. E. Dickson, *Algebras and their arithmetics* (1923), § 68.

follows that the number of linearly independent solutions must be the same for both equations and hence that the two eliminants have the same rank.

4. *Eliminants of characteristic equations.* These results are of special interest in regard to matrices. It will be noted that in this case the left and right-hand characteristic equations are the same and that, if the matrix is regarded as belonging to a linear associative algebra of order n^2, such equations are the n-th powers of the so-called characteristic equation. Moreover, they differ in form, that of the left-hand for p consisting, in fact, of a determinant of order n^2 arranged as follows, equated to zero:

$p_{rr}-\lambda$ occurs in the places $(nr+1-t,\ nr+1-t)\quad (r, t=1, 2, \ldots, n)$;

p_{rs} ,, ,, ,, $(nr+1-t,\ ns+1-t)\quad (r, s, t=1, 2, \ldots, n;\ r \neq s)$,

and the remaining elements are zeros; while that of the right-hand for q consists of a determinant of order n^2 arranged as follows and equated to zero:

$$\begin{array}{ccccc} q_{11}-\lambda, & q_{12}, & \cdots, & & q_{1n} \\ q_{21}, & q_{22}-\lambda, & \cdots, & & q_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ q_{n1}, & q_{n2}, & \cdots, & & q_{nn}-\lambda \end{array}$$

occupies the n squares with vertices

$$(nr+1,\ nr+1),\quad \{nr+1,\ n(r+1)\},\quad \{n(r+1),\ nr+1\},\quad \{n(r+1),\ n(r+1)\}$$
$$(r=0, 1, 2, \ldots, n-1),$$

the remaining elements being zeros.

Hence the first eliminant of the so-called characteristic equations of p and q is the determinant subtract of these two (§ 2), while the second eliminant is obtained by interchanging the p's and q's in the resulting determinant (§ 3). These are also eliminants of the reduced equations of p and q, and they are of the same rank, while the defect of this rank from n^2 is at least as great as the number of common roots of the reduced equations counted according to multiplicity.

5. *Further eliminants.* The equations $px=qx$, $xp=xq$ have solutions other than $x=0$ only if $p-q$ is a left- or right-hand divisor of zero

respectively. Obtaining these solutions analytically, as in the first part of § 2, the number of scalar arbitraries arising is the defect from n in the rank of the determinant subtract of the left- and right-hand characteristic equations of p and q. In the case of n^2-matrices these reduce to the n-th powers of the so-called characteristic determinant of $p-q$ and the scalar arbitraries occur n, $2n$, ..., n^2 at a time, as the case may be, in conformity with the usual result.

6. *Common solutions other than* $x=0$. These exist for $px=qx$, $xp=xq$ only if $N(p-q)=0$, giving the solution $CR(p-q)$, where $(p-q)R(p-q)=0$ is the reduced equation of $p-q$ and C an arbitrary permutable with p and q. The equations $px=xq$, $qx=xp$, have common solutions only if $N(p-q)=0$ and if q is a function of p, conditions which are necessary also in the case of $px=xq$, $xp=xq$. When these hold it follows that the solution $CR(p-q)$ satisfies all four equations and also $px=xp$, $qx=xq$.

7. *Applications*†. It is of interest to discuss the examples of § 7* from the point of view of the eliminants.

Example 1. Consider

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x = x \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where the characteristic equations have only one root in common.

As in § 7* the solution is

$$x = \begin{pmatrix} (a-\lambda_2')c_1', & \beta c_1' \\ (a-\lambda_2')c_2', & \beta c_2' \end{pmatrix},$$

where, since $(a-\lambda_2')c_1'-bc_2'=0$, there is only one arbitrary constant involved.

If there are two roots λ_1, λ_2 common, the solution is

$$x = \begin{pmatrix} (a-\lambda_2)c_1', & \beta c_1' \\ (a-\lambda_2)c_2', & \beta c_2' \end{pmatrix} + \begin{pmatrix} (a-\lambda_1)c_1'', & \beta c_1'' \\ (a-\lambda_1)c_2'', & \beta c_2'' \end{pmatrix},$$

† I use this opportunity to correct certain comments on the constants occurring in the examples of the previous paper. Some of these are erroneous, namely Examples 1*, 2*, and others misleading, namely Examples 3*, 4*.

involving two arbitrary constants, while the characteristic equations

$$\begin{pmatrix} a-\lambda, & b \\ c, & d-\lambda \end{pmatrix} = 0, \quad \begin{pmatrix} a-\lambda, & \beta \\ \lambda, & \delta-\lambda \end{pmatrix} = 0$$

have the eliminants

$$\begin{vmatrix} a-a, & -\gamma, & b, & 0 \\ -\beta, & a-\delta, & 0, & b \\ c, & 0, & d-a, & -\gamma \\ 0, & c, & -\beta, & d-\delta \end{vmatrix}, \quad \begin{vmatrix} a-a, & -c, & \beta, & 0 \\ -b, & a-d, & 0, & \beta \\ \gamma, & 0, & \delta-a, & -c \\ 0, & \gamma, & -b, & \delta-d \end{vmatrix}$$

of ranks three and two according as there are one or two common roots†,

Example 2. Consider the equation $px = xq$ in complex quaternions, where the reduced equations

$$\lambda^2 - 2p_0\lambda + N(p) = 0, \quad \lambda^2 - 2q_0\lambda + N(q) = 0$$

have only one root λ_1 in common.

The solution $(p-\lambda_2)\,C_1(q-\lambda_2')$, where C_1 is an arbitrary quaternion, involves only *one* scalar arbitrary‡, while, if both roots λ_1, λ_2 are common, the solution

$$x = (p-\lambda_2)\,C_1(q-\lambda_2) + (p-\lambda_1)\,C_2(q-\lambda_1)$$

involves two scalar arbitraries.

The eliminants, obtained from the characteristic equations,

$$\begin{vmatrix} d_0, & -d_1, & -d_2, & -d_3 \\ d_1, & d_0, & -s_3, & s_2 \\ d_2, & s_3, & d_0, & -s_1 \\ d_3, & -s_2, & s_1, & d_0 \end{vmatrix}, \quad \begin{vmatrix} -d_0, & d_1, & d_2, & d_3 \\ -d_1, & -d_0, & -s_3, & s_2 \\ -d_2, & s_3, & -d_0, & -s_1 \\ -d_3, & -s_2, & s_1, & -d_0 \end{vmatrix},$$

where $$d_r = p_r - q_r, \quad s_r = p_r + q_r \quad (r = 0, 1, 2, 3),$$

are of ranks three and two according as there are one or two common roots§.

Example 3. It follows from §4 that the eliminants can be of rank less than n only if p and q are equal scalars, in which case the rank is 0.

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† Unless special conditions hold. See Example 3.

‡ See § 8 *infra*.

§ Except when p and q are equal scalars, as in Example 3.

Example 4. Consider the system for which

$$p = p_0+p_1 i+p_2 j, \quad q = q_0+q_1 i+q_2 j \quad (i^2 = j,\ j^2 = i,\ ij = 1 = ji),$$

where multiplication is commutative but divisors of zero exist.

The eliminants of

$$\lambda^3-3p_0\lambda^2+3\lambda(p_0{}^2-p_1 p_2)+N(p) = 0,$$

$$\lambda^3-3q_0\lambda^2+3\lambda(q_0{}^2-q_1 q_2)+N(q) = 0,$$

are each equal to

$$\begin{vmatrix} d_0 & d_2 & d_1 \\ d_1 & d_0 & d_2 \\ d_2 & d_1 & d_0 \end{vmatrix},$$

where

$$d_r = p_r-q_r \quad (r = 0,\ 1,\ 2).$$

Hence the equations have one common root if $d_0+d_1+d_2 = 0$, and two if $d_0 : d_1 : d_2 = 1 : 1 : 1$, $1 : w : w^2$ or $1 : w^2 : w$.

Similar results hold for the more general system with n cyclic units.

8. In conclusion, I give a proof of the result (assumed in example 2, *supra*) that, if the reduced equations of the complex quaternions p and q have only one root λ in common, then the solution $(p-\lambda_1)C(q-\lambda_1)$ of $px = xq$ contains only one scalar arbitrary.

Writing $p' = p-\lambda_1$, $q' = q-\lambda_1$, $C = c_0+c_1 i+c_2 j+c_3 k$, we have

$$\begin{aligned} p'Cq' = {} & [c_0(p_0'+p_1 i+p_2 j+p_3 k)+c_1(-p_1+p_0' i+p_3 j-p_2 k) \\ & \quad +c_2(-p_2-p_3 i+p_0' j+p_1 k) \\ & \quad +c_3(-p_3+p_2 i-p_1 j+p_0' k)](q_0'+q_1 i+q_2 j+q_2 k) \\ = {} & [c_0(p_0' q_0'-p_1 q_1-p_2 q_2-p_3 q_3)+c_1(-p_1 q_0'-p_0' q_1-p_3 q_2+p_2 q_3) \\ & \quad +c_2(-p_2 q_0'+p_3 q_1-p_0' q_2-p_1 q_3)+c_3(-p_3 q_0'-p_2 q_1+p_1 q_2-p_0' q_3)] \\ & +[c_0(p_0' q_1+p_1 q_0'+p_2 q_3-p_3 q_2)+c_1(p_0' q_0'-p_1 q_1+p_2 q_2+p_3 q_3) \\ & \quad +c_2(-p_3 q_0'-p_2 q_1+p_0' q_3-p_1 q_2)+c_3(p_2 q_0'-p_3 q_1-p_1 q_3-p_0' q_2)] i \\ & +[c_0(p_0' q_2+p_2 q_0'+p_3 q_1-p_1 q_3)+c_1(p_3 q_0'-p_1 q_2-p_0' q_3-p_2 q_1) \\ & \quad +c_2(p_0' q_0'-p_2 q_2+p_1 q_1+p_3 q_3)+c_3(-p_1 q_0'-p_3 q_2-p_2 q_3+p_0' q_1)] j \\ & +[c_0(p_0' q_3+p_3 q_0'+p_1 q_2-p_2 q_1)+c_1(-p_2 q_0'-p_1 q_3+p_0' q_2-p_3 q_1) \\ & \quad +c_2(p_1 q_0'-p_2 q_3-p_3 q_2-p_0' q_1)+c_3(p_0' q_0'-p_3 q_3+p_1 q_1+p_2 q_2)] k \\ = {} & c_0 P+c_1 Q+c_2 R+c_3 S. \end{aligned}$$

Consider the ratio of the coefficients of 1 and i in P and Q. These are equal if

$$\frac{p_0'q_0'-p_1q_1-p_2q_2-p_3q_3}{-p_1q_0'-p_0'q_1-p_3q_2+p_2q_3} = \frac{p_0'q_1+p_1q_0'+p_2q_3-p_3q_2}{p_0'q_0'-p_1q_1+p_2q_2+p_3q_3},$$

i.e. if $$(p_0'^2+p_1^2)(q_0'^2+q_1^2) = (p_2^2+p_3^2)(q_2^2+q_3^2),$$

i.e. if $$p_0'^2+p_1^2+p_2^2+p_3^2 = q_0'^2+q_1^2+q_2^2+q_3^2,$$

which is so. Similarly for the others.

Hence P, Q, R, S are scalar multiples of one another, which proves the statement.

THE ELECTRICAL FIELD OF A SPHERE LYING BETWEEN TWO INFINITE CONDUCTING PLANES

By C. M. RIGBY.

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1. *Introduction.*

The problem is to determine the electrical field produced by a distribution of charge on a conducting sphere lying midway between two parallel earthed planes. The capacity of the sphere is also to be determined. The difficulty of the problem is that the potential is given on two distinct and geometrically unrelated boundaries, and that harmonic functions appropriate to the treatment of one boundary require considerable adaptation before they are appropriate for the treatment of conditions on the other boundary.

The method adopted is to construct a class of harmonic functions which vanish over the two planes. A suitable fundamental harmonic function of this class is that given by a point charge placed between the planes with its infinite train of images. Additional harmonic functions may then be constructed by successive differentiation of the fundamental function with respect to the position of the point charge. These functions are then transformed by means of Whittaker's integral into a form appropriate to the spherical boundary. The problem of finding the potential is thus reduced to that of determining the constants multiplying the harmonic functions, and for this purpose an infinite system of linear equations is obtained. The solution of these equations is of difficulty and calls for special methods.

This method is obviously capable of generalisation in other distributions of charge or potential on the sphere.

2. *Fundamental potential functions.*

We first determine a series of potential functions which vanish on the planes $z = \pm c$ and are regular at all points between the planes with the

exception of the origin. The field due to a unit charge at $(0, 0, \zeta)$ in the presence of the conducting planes is known to be the field due to positive unit charges at the points

$$z = \pm 4pc + \zeta \quad (p = 0, 1, 2, \ldots) \tag{2.01}$$

and negative unit charges at the points

$$z = \pm 2(2p+1)c - \zeta \quad (p = 0, 1, 2, \ldots). \tag{2.02}$$

The field due to a positive unit charge at $(0, 0, \zeta)$ may be written in the form of a Whittaker* integral

$$\pm \frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{z - \zeta + i\varpi \cos t}, \tag{2.03}$$

where $\varpi^2 = x^2 + y^2$; and the upper or lower sign is taken according as z is greater or less than ζ. Writing $\rho = z + i\varpi \cos t$ we have the field due to a unit charge at $(0, 0, \zeta)$, in the presence of the conducting planes $z = \pm c$,

$$V_0(\zeta) = \frac{1}{2\pi} \int_0^{2\pi} dt \left[\frac{1}{\rho - \zeta} - \sum_{p=1}^{\infty} \left\{ \left(\frac{1}{\rho - \zeta - 4pc} - \frac{1}{\rho - \zeta + 4pc} \right) + \left(\frac{1}{\rho + \zeta - 2(2p-1)c} - \frac{1}{\rho + \zeta + 2(2p-1)c} \right) \right\} \right], \tag{2.04}$$

provided that $\zeta < z < c$; while, if $\zeta > z > -c$, the same potential is given with the sign of the first term changed.

If $|\rho - \zeta| < 4c$ and $|\rho + \zeta| < 2c$, each term may be expanded in negative powers of c, and we obtain

$$V_0(\zeta) = \frac{1}{2\pi} \int_0^{2\pi} dt \left[\frac{1}{\rho - \zeta} - \sum_{q=0}^{\infty} \{ A_q (\rho - \zeta)^q - B_q (\rho + \zeta)^q \} \right], \tag{2.05}$$

where the coefficients are given by

$$\begin{cases} A_q = \dfrac{1 + (-1)^q}{(4c)^{q+1}} S_{q+1} & (q \geqslant 1), \\ B_q = \dfrac{1 + (-1)^q}{(2c)^{q+1}} \left(1 - \dfrac{1}{2^{q+1}} \right) S_{q+1} & (q \geqslant 1), \end{cases} \tag{2.06}$$

$$A_0 - B_0 = -\frac{1}{c} (1 - \tfrac{1}{2} + \tfrac{1}{3} - \ldots),$$

* Whittaker, *Math. Annalen*, 57 (1903), 332.

and S_q is the zeta function for an integral argument, so that*

$$S_q = 1 + \frac{1}{2^q} + \frac{1}{3^q} + \ldots . \tag{2.07}$$

Since $V_0(\zeta)$ vanishes identically for $z = \pm c$, for all values of ζ, $|\zeta| < c$, the expression obtained by differentiating it any number of times with respect to ζ will have the same property. In particular, putting $\zeta = 0$ after differentiating, we have the following system of potentials which vanish on $z = \pm c$ and are regular between these planes except at the origin:

$$V_n(0) = \frac{1}{2\pi}\int_0^{2\pi} dt \left[\frac{n!}{\rho^{n+1}} + \sum_{q=n}^{\infty} \frac{q!}{(q-n)!}\rho^{q-n}\{(-1)^n A_q - B_q\}\right], \tag{2.08}$$

if $0 < z < c$; if $0 > z > -c$, the sign of the first term is to be changed. The condition for the validity of the expansion performed above has now become $|\rho| < 2c$. We now transform the expression for $V_n(0)$ to spherical harmonics; and, from considerations of symmetry, it is sufficient to consider the state of affairs in the region $z > 0$ or, in polar coordinates (r, θ, ϕ), $0 < \theta < \frac{1}{2}\pi$. Now

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{dt}{\rho^{n+1}} = \frac{1}{r^{n+1}} P_n(\cos\theta),$$

for $\cos\theta > 0$, where $P_n(\cos\theta)$ denotes the usual Legendre function. Further†

$$\frac{1}{2\pi}\int_0^{2\pi} \rho^n\, dt = r^n P_n(\cos\theta),$$

for all real θ. Hence

$$V_n(0) = n!\left[\frac{1}{r^{n+1}} P_n(\cos\theta) + \sum_{p=0}^{\infty} \frac{(n+p)!}{n!\,p!}\{(-1)^n A_{n+p} - B_{n+p}\} r^p P_p(\cos\theta)\right]. \tag{2.09}$$

This expression has been obtained on the assumption that, for all relevant values of t, $|\rho| < 2c$, *i.e.*

$$z^2 + \varpi^2 \cos^2 t < 4c^2;$$

and this condition is satisfied if $r < 2c$.

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* The notation here and in (5.01) is that of J. W. L. Glaisher in the *Quart. J. of Math.*, 45 (1914), 141, where these functions are tabulated.

† Whittaker, *loc. cit.*

Now, if n is even, $V_n(0)$ is symmetrical about the plane $z=0$, and further A_q and B_q vanish identically if q is odd. Hence, by (2.06), a set of potential functions symmetrical about $z=0$ and vanishing on $z=\pm c$ is given by

$$(2.10)\quad V_{2n}(0)=(2n)!\left[\frac{1}{r^{2n+1}}P_{2n}(\cos\theta)+\sum_{p=0}^{\infty}\frac{(2n+2p)!}{(2n)!\,(2p)!}r^{2p}P_{2p}(\cos\theta)\{A_{2n+2p}-B_{2n+2p}\}\right];$$

and, when $2n+2p\geqslant 0$,

$$(2.11)\qquad A_{2n+2p}-B_{2n+2p}=\frac{2}{(2c)^{2n+2p+1}}\mu_{2n+2p+1},$$

where

$$(2.12)\qquad \mu_n=\sum_{m=1}^{\infty}\frac{(-1)^m}{m^n}.$$

So we consider the potential function V, symmetrical about $z=0$, vanishing on $z=\pm c$, regular between these planes except at the origin, given by

$$(2.13)\qquad V=\sum_{n=0}^{\infty}\frac{C_n}{(2n)!}V_{2n}(0),$$

where C_n are arbitrary constants, or, when V is expressed in spherical harmonics,

$$(2.14)\quad V=\sum_{n=0}^{\infty}C_n\left[\frac{1}{r^{2n+1}}P_{2n}(\cos\theta)+2\sum_{p=0}^{\infty}\frac{(2n+2p)!}{(2n)!\,(2p)!}\frac{r^{2p}P_{2p}(\cos\theta)}{(2c)^{2n+2p+1}}\mu_{2n+2p+1}\right].$$

3. *Determination of potential.*

We now have to determine the constants C_n so that V is equal to unity on the sphere $r=a$, $a<c$. By equating coefficients in (2.14), we have

$$(3.01)\qquad \frac{C_0}{a}+2\sum_{p=0}^{\infty}C_p\frac{\mu_{2p+1}}{(2c)^{2p+1}}=1;$$

and, for $n\geqslant 1$,

$$\frac{C_n}{a^{2n+1}}+2\sum_{p=0}^{\infty}\frac{(2n+2p)!}{(2n)!\,(2p)!}\frac{a^{2n}}{(2c)^{2n+2p+1}}\mu_{2n+2p+1}C_p=0.$$

The equations (3.01) can be solved by the usual method and C_n is thus found as the quotient of two infinite determinants; in particular

$$C_0 = a \begin{vmatrix} 1-a_{11}, & -a_{12}, & -a_{13}, & \dots \\ -a_{21}, & 1-a_{22}, & -a_{23}, & \dots \\ -a_{31}, & -a_{32}, & 1-a_{33}, & \dots \\ \vdots & \vdots & \vdots & \end{vmatrix} \div \begin{vmatrix} 1-a_{00}, & -a_{01}, & -a_{02}, & \dots \\ -a_{10}, & 1-a_{11}, & -a_{12}, & \dots \\ -a_{20}, & -a_{21}, & 1-a_{22}, & \dots \\ \vdots & \vdots & \vdots & \end{vmatrix},$$

where $$a_{np} = -\frac{(2n+2p)!}{(2n)!\,(2p)!}\,\mu_{2n+2p+1}(2\lambda)^{4p+1}\,\frac{1}{2^{2n+2p}}.$$

But in this form the operation of finding C_0 for given values of λ is unwieldy. Writing

$$C_0 = a(1+D_0) \quad \text{and} \quad C_n = aD_n, \tag{3.02}$$

where $n \geqslant 1$, and

$$\lambda = a/(2c), \tag{3.03}$$

we have

$$\frac{D_n}{a^{2n}} + 2\lambda^{2n+1}\mu_{2n+1} + 2\sum_{p=1}^{\infty}\frac{(2n+2p)!}{(2n)!\,(2p)!}\,\lambda^{2n+2p+1}\mu_{2n+2p+1}\,\frac{D_p}{a^{2p}} = 0. \tag{3.04}$$

Then we assume

$$\frac{D_n}{2a^{2n}\lambda^{2n+1}} = \Sigma\,\frac{F_{m_1}}{\chi_{m_1}^{2n+1}} + \lambda\Sigma\Sigma\,\frac{F_{m_1,m_2}}{\chi_{m_1,m_2}^{2n+1}} + \dots, \tag{3.05}$$

where F and χ may depend on λ. The summations refer to the integers m_1, m_2, ..., covered by the summation signs in each term, but the range is defined later. Term by term this method gives the same results as the method of successive approximations, but the relation between successive F's and χ's can be determined explicitly*.

Substituting in (3.04), we have

$$\Sigma\,\frac{F_{m_1}}{\chi_{m_1}^{2n+1}} + \lambda\Sigma\Sigma\,\frac{F_{m_1,m_2}}{\chi_{m_1,m_2}^{2n+1}} + \dots + \mu_{2n+1} + 2\sum_{p=0}^{\infty}\frac{(2n+2p)!}{(2n)!\,(2p)!}\,\lambda^{4p+1}\mu_{2n+2p+1}\left\{\Sigma\,\frac{F_{m_1}}{\chi_{m_1}^{2p+1}} + \lambda\Sigma\Sigma\,\frac{F_{m_1,m_2}}{\chi_{m_1,m_2}^{2n+1}} + \dots\right\} = 0. \tag{3.06}$$

This is a formal substitution and the order of the summations in each term is altered without discussion. Inserting the values of the μ

* This method is due to Prof. G. B. Jeffery.

functions from (2.12) and summing with respect to p, we have

$$(3.07)\quad \Sigma \frac{F_{m_1}}{\chi_{m_1}^{2n+1}} + \lambda \Sigma\Sigma \frac{F_{m_1,m_2}}{\chi_{m_1,m_2}^{2n+1}} + \ldots + \sum_{q=1}^{\infty} \frac{(-1)^q}{q^{2n+1}}$$
$$+ \Sigma \sum_{q=1}^{\infty} \frac{\lambda F_{m_1}}{\chi_{m_1}} (-1)^q \left\{ \frac{1}{\{q+(\lambda^2/\chi_{m_1})\}^{2n+1}} + \frac{1}{\{q-(\lambda^2/\chi_{m_1})\}^{2n+1}} \right\}$$
$$+ \Sigma\Sigma \sum_{q=1}^{\infty} \frac{\lambda^2 F_{m_1,m_2}}{\chi_{m_1,m_2}} (-1)^q \left\{ \frac{1}{\{q+(\lambda^2/\chi_{m_1,m_2})\}^{2n+1}} + \frac{1}{\{q-(\lambda^2/\chi_{m_1,m_2})\}^{2n+1}} \right\}$$
$$+ \ldots = 0.$$

This will be identically satisfied if for all positive integral values of p the two expressions, the one concerning $m_1, m_2, \ldots, m_p$, and the other concerning $m_1, m_2, \ldots, m_{p-1}$, and q, vanish; for associated with each we have λ^{p-1}. So we must have

$$(3.08)\qquad \Sigma \frac{F_{m_1}}{\chi_{m_1}^{2n+1}} + \sum_{q=1}^{\infty} \frac{(-1)^q}{q^{2n+1}} = 0.$$

and so on.

This suggests that the first summation Σ with respect to m_1 shall be defined as $\sum_{m_1=0}^{\infty}$. Considering the fact that we have a summation over one term for p different m, and a summation over two terms for $p-1$ different m and one q, we define

$$(3.09)\qquad \lambda^{p-1} \Sigma \ldots \Sigma \frac{F_{m_1,\ldots,m_p}}{\chi_{m_1,\ldots,m_p}^{2n+1}}$$

as
$$\lambda^{p-1} \sum_{m_1, m_2, \ldots, m_{p-1}} \sum_{m_p=1}^{\infty} F_{m_1,\ldots,m_p},$$

where, unless otherwise expressed, the summations over $m_1, \ldots, m_p$ are all non-zero integral values, positive and negative.

Hence the general equation, the simplest case being given in (3.08), is

$$(3.10)\quad \sum_{m_1,\ldots,m_{r-1}} \sum_{m_r=1}^{\infty} \frac{F_{m_1,\ldots,m_r}}{\chi_{m_1,\ldots,m_r}^{2n+1}}$$
$$+ \sum_{m_1,\ldots,m_{r-2}} \sum_{m_{r-1}=1}^{\infty} \sum_{m_r=1}^{\infty} \frac{F_{m_1,\ldots,m_{r-1}}}{\chi_{m_1,\ldots,m_r}} (-1)^{m_r}$$
$$\times \left[\frac{1}{\{m_r+(\lambda^2/\chi_{m_1,\ldots,m_{r-1}})\}^{2n+1}} + \frac{1}{\{m_r-(\lambda^2/\chi_{m_1,\ldots,m_{r-1}})\}^{2n+1}} \right] = 0.$$

In the second term m_r has been written for q. From (3.08) we have

$$F_{m_1} = (-1)^{m_1-1} \quad \text{and} \quad \chi_{m_1} = m_1.$$

We suppose that all the $\chi_{m_1,\ldots,m_r}$ change sign with m_r and that

$$F_{m_1,\ldots,m_r} = F_{m_1,\ldots,-m_r}.$$

This is true for $r=1$, and we can show that it is true for $r=s+1$ if it is true for $r \leqslant s$. For, on this hypothesis,

$$\sum_{m_1,\ldots,m_{r-1}} \sum_{m_{r-1}=0}^{\infty} \sum_{m_r=1}^{\infty} \frac{F_{m_1,\ldots,m_{r-1}}}{\chi_{m_1,\ldots,m_{r-1}}} (-1)^{m_r} \times \left[\frac{1}{\{m_r + (\lambda^2/\chi_{m_1,\ldots,m_{r-1}})\}^{2n+1}} + \frac{1}{\{m_r - (\lambda^2/\chi_{m_1,\ldots,m_{r-1}})\}^{2n+1}} \right]$$

becomes

$$\sum_{m_1,\ldots,m_{r-1}} \sum_{m_r=1}^{\infty} \frac{F_{m_1,\ldots,m_{r-1}}}{|\chi_{m_1,\ldots,m_{r-1}}|} (-1)^{m_r} \frac{1}{\{m_r + (\lambda^2/\chi_{m_1,\ldots,m_{r-1}})\}^{2n+1}}.$$

Now equation (3.10) is satisfied if

$$(3.11) \qquad \frac{F_{m_1,\ldots,m_r}}{\chi^{2n+1}_{m_1,\ldots,m_r}} + \frac{F_{m_1,\ldots,m_{r-1}}}{|\chi_{m_1,\ldots,m_{r-1}}|} (-1)^{m_r} \frac{1}{\{m_r + (\lambda^2/\chi_{m_1,\ldots,m_{r-1}})\}^{2n+1}} = 0,$$

and this is satisfied identically if

$$(3.12) \qquad F_{m_1,\ldots,m_r} = (-1)^{m_{r-1}} \frac{F_{m_1,\ldots,m_{r-1}}}{|\chi_{m_1,\ldots,m_{r-1}}|}$$

and

$$\chi_{m_1,\ldots,m_r} = m_r + \frac{\lambda^2}{\chi_{m_1,\ldots,m_{r-1}}},$$

functions satisfying the hypothesis which we have used above. Hence

$$(3.13) \qquad F_{m_1,\ldots,m_r} = \frac{(-1)^{r+m_1+m_2+\ldots+m_r}}{|\chi_{m_1}\chi_{m_1,m_2}\cdots\chi_{m_1,\ldots,m_{r-1}}|}.$$

Denoting $\chi_{m_1,\ldots,m_r}$ by ϕ_r, we have

$$(3.14) \qquad \frac{F_{m_1,\ldots,m_r}}{\chi^{2n+1}_{m_1,\ldots,m_r}} = (-1)^r \frac{(-1)^{m_1+m_2+\ldots+m_r}}{|\phi_1\phi_2\cdots\phi_{r-1}|\phi_r^{2n+1}},$$

then, recalling (3.03),

$$(3.15) \quad D_n = -2a^{2n}\lambda^{2n+1}\left[\sum_{m_1=1}^{\infty} \frac{(-1)^{m_1}}{\phi_1^{2n+1}} + \ldots + (-\lambda)^{r-1} \sum_{m_1\ldots m_{r-1}} \sum_{m_r=1}^{\infty} \frac{(-1)^{m_1+\ldots+m_r}}{|\phi_1\cdots\phi_{r-1}|} \frac{1}{\phi_r^{2n+1}} \right] + \ldots$$

and

$$\phi_r = m_r + \frac{\lambda^2}{\phi_{r-1}}, \qquad \phi_1 = m_1.$$

Hence the potential V for $r < 2c$ is given by

$$V = \sum_{n=0}^{\infty} C_n \left[\frac{1}{r^{2n+1}} P_{2n}(\cos\theta) + 2 \sum_{p=0}^{\infty} \frac{(2n+2p)!}{(2n)!\,(2p)!} \frac{r^{2p} P_{2p}(\cos\theta)}{(2c)^{2n+2p+1}} \mu_{2n+2p+1} \right],$$

where $C_0 = a(1+D_0)$, $C_n = aD_n$, $n \geqslant 1$, and D_n is given by (3.15).

4. *The capacity.*

The surface density on the sphere is

$$-\frac{1}{4\pi}\left[\frac{\partial V}{\partial r}\right]_{r=a},$$

and the total charge is

$$\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} -\frac{1}{4\pi}\left[\frac{\partial V}{\partial r}\right]_{r=a} a^2 \sin\theta\, d\theta\, d\phi = \tfrac{1}{2}a^2 \int_0^{\pi} -\left[\frac{\partial V}{\partial r}\right]_{r=a} \sin\theta\, d\theta.$$

From (2.14) we see that

$$\int_0^{\pi} -\left[\frac{\partial V}{\partial r}\right]_{r=a} \sin\theta\, d\theta$$

$$= \int_0^{\pi} \sin\theta\, d\theta \sum_{n=0}^{\infty} C_n \left[\frac{2n+1}{a^{2n+2}} P_{2n}(\cos\theta) \right.$$

$$\left. + 2 \sum_{p=0}^{\infty} \frac{(2n+2p)!}{(2n)!\,(2p)!} \frac{2pa^{2p-1} P_{2p}(\cos\theta)}{(2c)^{2n+2p+1}} \mu_{2n+2p+1} \right].$$

But

$$\int_0^{2\pi} P_{2n}(\cos\theta) \sin\theta\, d\theta = 0 \quad (n \neq 0)$$

and

$$\int_0^{\pi} \sin\theta\, d\theta = 2.$$

Therefore the total charge on the sphere is

$$\tfrac{1}{2}a^2 C_0 \frac{1}{a^2} 2 = C_0.$$

The potential is unity, therefore the capacity of the sphere is equal to KC_0, where K is the specific inductive capacity of the medium between the sphere and the planes.

$$C_0 = a(1+D_0)$$

$$= a\left(1 - 2\lambda \sum_{m_1=1}^{\infty} \frac{(-1)^{m_1}}{\phi_1} + 2\lambda^2 \sum_{m_1} \sum_{m_2=1}^{\infty} \frac{(-1)^{m_1=m_2}}{|\phi_1|\phi_2} - \ldots\right)$$

$$= a\{1 - 2\lambda L_1 + 2\lambda^2 L_2 + \ldots + 2(-\lambda)^r L_r + \ldots\}, \tag{4.01}$$

where

$$L_r = \sum_{m_1 \ldots m_{r-1}} \sum_{m_r=1}^{\infty} \frac{(-1)^{m_1+\ldots+m_r}}{|\phi_1 \cdots \phi_{r-1}|} \frac{1}{\phi_r}.$$

By expanding $1/\phi_r = 1/\{m_r + (\lambda^2/\phi_{r-1})\}$ as a power series in λ, we have

$$\sum_{m_r=1}^{\infty} \frac{(-1)^{m_r}}{\phi_r} = \sum_{n_1=0}^{\infty} \frac{(-1)^{n_1}\lambda^{2n_1}}{\phi_{r-1}^{n_1}} \mu_{n_1+1}.$$

We then consider

$$\sum_{m_{r-1}} \sum_{m_r=1}^{\infty} \frac{(-1)^{m_{r-1}+m_r}}{|\phi_{r-1}|\phi_r} = \sum_{n_1=0}^{\infty} (-1)^{n_1}\lambda^{2n_1}\mu_{n_1+1} \sum \frac{(-1)^{m_{r-1}}}{|\phi_{r-1}|} \frac{1}{\phi_{r-1}^{n_1}}.$$

To transform $\sum_{m_r} \frac{(-1)^{m_{r-1}}}{|\phi_{r-1}|} \frac{1}{\phi_{r-1}^{n_1}}$, we take the cases of n_1 odd and n_1 even separately.

$$(4.02) \qquad \sum_{m_{r-1}} \frac{(-1)^{m_{r-1}}}{|\phi_{r-1}|} \frac{1}{\phi_{r-1}^{2n_1}} = 2 \sum_{n=0}^{\infty} \binom{2n_1+2n_2}{2n_1} \lambda^{4n_2} \mu_{2n_1+2n_2+1} \frac{1}{\phi_{r-2}^{2n_2}},$$

where $\binom{2n_1+2n_2}{2n_1}$ is the binomial coefficient $\frac{(2n_1+2n_2)!}{(2n_1)!\,(2n_2)!}$.

Thus

$$\sum_{m_{r-1}} \frac{(-1)^{m_{r-1}}}{|\phi_{r-1}|} \frac{1}{\phi_{r-1}^{2n_1+1}} = -2 \sum_{n_2=0}^{\infty} \binom{2n_1+2n_2+1}{2n_1} \lambda^{4n_2+2} \mu_{2n_1+2n_2+3} \frac{1}{\phi_{r-2}^{2n_2+1}}.$$

So we can reduce

$$\sum_{m_1 \ldots m_{r-1}} \frac{(-1)^{m_1+\ldots+m_{r-1}}}{|\phi_1 \cdots \phi_{r-1}|} \frac{1}{\phi_{r-1}^{2n_1+1}}$$

to

$$-2 \sum_{n_2=0}^{\infty} \lambda^{4n_2+2} \binom{2n_1+2n_2+1}{2n_1} \mu_{2n_1+2n_2+3} \sum_{m_1, \ldots, m_{r-2}} \frac{(-1)^{m_1+\ldots+m_{r-2}}}{|\phi_1 \cdots \phi_{r-2}|\, \phi_{r-2}^{2n_2+1}}.$$

Then, continuing step by step, we obtain the expression

$$(-2)^{r-2} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} \lambda^{4n_2+2+4n_3+2+\ldots+4n_r+2} \binom{2n_1+2n_2+1}{2n_1} \cdots \binom{2n_{r-1}+2n_r+1}{2n_{r-1}}$$

$$\times \mu_{2n_1+2n_2+3} \cdots \mu_{2n_{r-1}+2n_r+3} \sum_{m_1} \frac{(-1)^{m_1}}{|\phi_1|\,\phi_1^{2n_r+1}};$$

but

$$\sum_{m_1} \frac{(-1)^{m_1}}{|\phi_1|\,\phi_1^{2n_r+1}} = \sum_{m_1} \frac{(-1)^{m_1}}{|m_1|\,m_1^{2n_r+1}} = \sum_{m_1=1}^{\infty} \frac{(-1)^{m_1}}{m_1^{2n_r+2}} - \sum_{m_1=1}^{\infty} \frac{(-1)^{m_1}}{m_1^{2n_r+1}} = 0.$$

Therefore

$$(4.03)\quad \sum_{m_1,\ldots,m_{r-1}} \sum_{m_r=1}^{\infty} \frac{(-1)^{m_1+\ldots+m_r}}{|\phi_1 \cdots \phi_{r-1}|\phi_r}$$

$$= \sum_{m_1,\ldots,m_{r-1}} \frac{(-1)^{m_1+\ldots+m_{r-1}}}{|\phi_1 \cdots \phi_{r-1}|} \sum_{m_1=0}^{\infty} \frac{\lambda^{4n_1}}{\phi_{r-1}^{2n_1}} \mu_{2n_1+1}$$

$$= 2^{r-2} \sum_{n_1,\ldots,n_{r-1}=0}^{\infty} \lambda^{4n_1+4n_2+\ldots+4n_{r-1}} \binom{2n_1+2n_2}{2n_1} \cdots \binom{2n_{r-2}+2n_{r-1}}{2n_{r-2}}$$

$$\times \mu_{2n_1+1}\mu_{2n_1+2n_2+1} \cdots \mu_{2n_{r-2}+2n_{r-1}+1}.$$

i.e.

$$(4.04)\quad L_r = 2^{r-1} \sum_{n_1,\ldots,n_{r-1}=0}^{\infty} \lambda^{4(n_1+\ldots+n_{r-1})} \binom{2n_1+2n_2}{2n_1} \cdots \binom{2n_{r-2}+2n_{r-1}}{2n_{r-2}}$$

$$\times \mu_{2n_1+1}\mu_{2n_1+2n_2+1} \cdots \mu_{2n_{r-2}+2n_{r-1}+1}\mu_{2n_{r-1}+1}.$$

The summation $\sum_{n_1,\ldots,n_{r-1}=0}^{\infty}$ is equivalent to

$$\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{r-1}=0}^{k_{r-2}},$$

writing

$$(4.05)\quad \begin{cases} n_1+\ldots+n_{r-1} = k_1, \\ n_1+\ldots n_{r-2} = k_2, \\ n_1+n_2 = k_{r-2}, \\ n_1 = k_{r-1}; \end{cases}$$

so we have the capacity expressed as

$$(4.06)\quad ak\Big[1-2\lambda\mu_1+\ldots$$

$$+(-2\lambda)^r \sum_{k_1=0}^{\infty} \cdots \sum_{k_{r-1}=0}^{\infty} \lambda^{4k_1} \binom{2k_1-2k_3}{2k_1-2k_2} \cdots \binom{2k_{r-3}-2k_{r-1}}{2k_{r-3}-2k_{r-2}} \binom{2k_{r-2}}{2k_{r-1}}$$

$$\times \mu_{2k_{r-1}+1}\mu_{2k_{r-2}+1}\mu_{2k_{r-3}-2k_{r-1}+1} \cdots \mu_{2k_1-2k_3+1}\mu_{2k_1-2k_2+1}+\ldots\Big].$$

5. *The convergence of the series representing the capacity.*

From physical considerations it is to be expected that the series (4.06) is convergent for $\lambda < 0{\cdot}5$, that is to say so long as the diameter of the sphere is less than the distance between the planes. We can only succeed in showing that it is convergent when $\lambda < 0{\cdot}334$. It is possible that

rearranging the series has further restricted the convergence, but it seems more likely that a suitable majorising series has not been found.

Writing

$$s_n = -\mu_n = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m^n}, \tag{5.01}$$

we have

$$1-2\lambda\mu_1+\sum_{r=2}^{\infty}(-2\lambda)^r \sum_{k_1=0}^{\infty}\cdots\sum_{k_{r-1}=0}^{k_{r-2}} \lambda^{4k_1}\binom{2k_1-2k_3}{2k_1-2k_2}\cdots\binom{2k_{r-2}}{2k_{r-1}} \times \mu_{2k_{r-1}+1}\cdots\mu_{2k_1-2k_3+1}\mu_{2k_1-2k_2+1}+\cdots$$

$$= 1+2\lambda s_1+\sum_{r=2}^{\infty}(2\lambda)^r \sum_{k_1=0}^{\infty}\lambda^{4k_1} T_{k_1,\ldots,k_{r-1}}; \tag{5.02}$$

whence

$$T_{k_1,\ldots,k_{r-1}} = \sum_{k_2=0}^{k_1}\cdots\sum_{k_{r-1}=0}^{k_{r-2}}\binom{2k_1-2k_3}{2k_1-2k_2}\cdots\binom{2k_{r-2}}{2k_{r-1}}\mu_{2k_{r-1}+1}\cdots\mu_{2k_1-2k_2+1},$$

a positive number.

We have $$T_{k_1,\ldots,k_{r-1}} < \binom{r+k-1}{k} 2^{4k_1},$$

using the simple inequality $\binom{m+n}{n} < 2^{m+n}$ for the binomial coefficients. The coefficient of $(2\lambda)^r$ is a terminating series of n terms; it can be shown to be less than 5^m, where m is a positive integer such that $r \leqslant 4m+5 < r+4$, hence the series is convergent for $\lambda < 0{\cdot}334$.

6. *Calculation of the capacity.*

By equation (5.03) the capacity

$$k C_0 = ak\,[1+b_1(2\lambda)+b_2(2\lambda)^2+\ldots],$$

where b_r, consisting of sums of powers and products of s_n, can be determined from (5.03) and (5.02).

$$b_1 = s_1, \quad b_2 = s_1^2, \quad b_3 = s_1^3, \quad b_4 = s_1^4, \quad b_5 = s_1^5,$$

$$b_6 = s_1^6+\frac{s_3^2}{24}, \quad b_7 = s_1^7+\frac{2s_1s_3^2}{24},$$

$$b_8 = s_1^8+\frac{3s_1^2s_3^2}{24}, \quad b_9 = s_1^9+\frac{4s_1^3s_3^2}{24}, \quad b_{10} = s_1^{10}+\frac{5s_1^4s_3^2}{24}+\frac{s_5^2}{28},$$

b_r becoming more and more complicated as r increases. Since $s_n = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m^n}$ is tabulated for given values of n we can calculate b_r. The values of b_r for $r \leqslant 23$ are given in Table I.

For Table II, $C_0/(ak)$ was taken as the sum of the first twenty-four terms. The error for $\lambda < 0{\cdot}25$ does not exceed 0·0004 and for larger values of λ terms less than 0·0003 are neglected.

TABLE I.

n	b_n
1	0·6931472
2	0·4804530
3	0·3330247
4	0·2308350
5	0·1600027
6	0·1616728
7	0·1472722
8	0·1264589
9	0·1045614
10	0·0878869
11	0·0950424
12	0·0920363
13	0·0836623
14	0·0732056
15	0·0684944
16	0·0669417
17	0·0634795
18	0·0575255
19	0·0523108
20	0·0510056
21	0·0502608
22	0·0473677
23	0·0430889

TABLE II.

2λ	$C_0/(ak)$
0·05	1·036
0·10	1·074
0·15	1·116
0·20	1·161
0·25	1·210
0·30	1·263
0·40	1·384
0·50	1·532
0·60	1·719
0·70	1·968
0·80	2·329

Here I should like to thank Prof. G. B. Jeffery for suggesting the problem and for his help and advice throughout the writing of this paper.

NOTE ON NORMAL DIVISION ALGEBRAS*

By O. L. DAVIES.

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All known normal division algebras of order n^2 over a non-modular field F are of type R, *i.e.*, contain an element i satisfying an irreducible equation of degree n with coefficients in F, which has the distinct roots

$$\theta_0(i) = i, \quad \theta_1(i), \quad \theta_2(i), \quad \ldots, \quad \theta_{n-1}(i),$$

where the $\theta(i)$'s are polynomials in i with coefficients in F.

If $\theta_s(i) = \theta^s(i) = \theta[\theta^{s-1}(i)] = \theta[\theta\{\ldots\theta(i)\}]$ $(s = 1, 2, \ldots, n-1)$, then the algebra is *cyclic*.

L. E. Dickson† has constructed all normal division algebras of type R, but he does not show that these algebras are non-cyclic, *i.e.*, not equivalent to the much simpler cyclic algebras. The existence of non-cyclic algebras has not yet been demonstrated, and probably such algebras do not exist‡.

All normal division algebras of order less than or equal to 16 are cyclic§; and recently A. A. Albert‖ has shown that all normal division algebras of order 36 and type R are also cyclic.

He¶ also proved that if A is a cyclic normal division algebra of order r^2 over F, and if $r = p_1^{a_1} p_2^{a_2} \ldots p_t^{a_t}$, where the p_i are distinct primes, then A is a direct product of t cyclic normal division algebras of order $(p_i^{a_i})^2$, and conversely all such direct products are cyclic division algebras.

* A memoir by H. Hasse on Cyclic Algebras appears in *Trans. Amer. Math. Soc.*, 34 (1932), 171–214.

† *Trans. Amer. Math. Soc.*, 28 (1926), 207–234, and 32 (1930), 319–334.

‡ [*Added* 17 *April*, 1932. This statement now needs alteration. Recently (April, 1932), A. A. Albert, in a paper read to the American Mathematical Society, gives a construction of non-cyclic normal division algebras. In a second paper A. A. Albert and H. Hasse determine all normal division algebras over an algebraic field.]

§ Wedderburn, *Trans. Amer. Math. Soc.*, 22 (1921), 129–135. Albert, *Trans. Amer. Math. Soc.*, 32 (1930), 171–195.

‖ *Trans. Amer. Math. Soc.*, 33 (1931), 235–243.

¶ *Trans. Amer. Math. Soc.*, 33 (1931), 219–234.

R. Brauer* has proved that if A is a normal division algebra of order r^2 over F, and if $r = p_1^{a_1} p_2^{a_2} \dots p_t^{a_t}$, where the p_i are distinct primes, then A is a direct product of t normal division algebras of order $(p_i^{a_i})^2$.

From the above results it follows that

1. *All normal division algebras of order* 36 *are cyclic.*

And from the fact that all normal division algebras of type R and of order p^2, p a prime, are necessarily cyclic, it follows that

2. *All normal division algebras of type R and of order* m^2 *where* $m = p_1 p_2 \dots p_r$, *the* p_i *being distinct primes, are cyclic.*

By means of the above result of Brauer on direct products of normal division algebras, the determination of all division algebras is reduced to the determination of those whose orders are powers of primes†.

* *Math. Zeitschrift* 30 (1929), 79–107.

† See foot-note ‡ on previous page.

THEORY OF THE THETA-FUCHSIAN FUNCTIONS

By D. P. DALZELL.

[Received 12 May, 1931.—Read 14 May, 1931.]

1. *Introduction.*

The genus of a Fuchsian group is commonly defined by the use of Euler's theorem, by means of Riemann's surfaces, or by borrowing the results of the theory of algebraic functions*. None of these methods belongs properly to the theory of Fuchsian automorphic functions.

The most important transcendental functions associated with Fuchsian groups were discovered by Poincaré and he called them Theta-Fuchsian functions. We shall, in this paper, develop their theory and show that it yields the definition and the fundamental properties of the genus without an appeal to topology or to a previous theory of algebraic functions. The general theory of automorphic functions, as invented and contrived by Poincaré, is reproduced in the second volume of his collected works; his method consists largely in the use of an absolutely convergent series for the construction of the functions associated with a Fuchsian group. We adopt a procedure similar to that of Poincaré, except in the construction of a new function, the Eta-Fuchsian function, which requires the selection of convergent sequences from normal families of analytic functions†. Apart from the Eta-Fuchsian function, which itself could be obtained easily by assuming the existence of normal integrals of the third kind, the functions discussed in this paper are not essentially new, but the methods employed appear to have some new features.

When the existence of the integrals of the first and second kinds has been demonstrated, the arguments employed thereafter for the completion

* Appell and Goursat, 2, 140. Ford, 227.

† Montel, 21.

of the scheme of the paper are, naturally, similar to those known and employed in the theory of algebraic functions. The part of the paper devoted to the theory of functions associated with Fuchsian groups includes a mention of Abel's theorem, and can be taken as a statement, in terms of automorphic forms alone, of the basis of the theory of algebraic functions.

The paper terminates with a theorem in the theory of a pair of functions meromorphic in the unit circle and connected by an algebraic relation.

2. *Synopsis of fundamental theorems.*

We now give a summary of the theory required as the basis of the main part of the paper.

A Fuchsian transformation is defined to be a transformation of the interior of the unit circle $|z|<1$ into itself. Thus, if Tz be a Fuchsian transform of z, then Tz is a rational function of z, of degree unity, has one zero (say $z=\alpha$) in the unit circle, and is of modulus unity on the circumference of the unit circle. The following equations and inequalities can, then, easily be verified:

$$\frac{1-|\alpha|}{2} \leqslant \left|\frac{dTz}{dz}\right| = \frac{1-|Tz|^2}{1-|z|^2} = \frac{1-|\alpha|^2}{|1-\bar{\alpha}z|^2} \leqslant \frac{2(1-|\alpha|)}{(1-|z|)^2}, \qquad (1)$$

where $T\alpha=0$, $|\alpha|<1$, and $\bar{\alpha}$ is the complex number conjugate to α. Homographic transformations are of four types, called elliptic, hyperbolic, parabolic, and loxodromic, but a Fuchsian transformation cannot be loxodromic*.

A Fuchsian group is defined as an enumerable set of Fuchsian transformations $T_0\equiv 1$, T_1, T_2, ..., T_ν, ..., such that the product of two transformations and the inverse of any transformation belong also to the set. This is in accordance with the usual definition of a group.

The group is properly discontinuous at the point $z=\zeta$ if a positive number, say ρ, exists such that

$$|\zeta - T_\nu z| > \rho$$

for all except, possibly, a finite number of the transformations of the group. The fundamental theorem in the discontinuity of Fuchsian groups is that a sufficient condition for the group to be properly discontinuous everywhere in the unit circle is that it should be properly discontinuous

* Appell and Goursat, 2, 7 and 26. Ford, 23 and 67.

at a single point in the unit circle*. We shall be concerned only with those Fuchsian groups which contain neither elliptic nor parabolic transformations, and are composed therefore wholly of hyperbolic transformations. A Fuchsian hyperbolic transformation does not transform any interior point of the unit circle into itself, so that, given any point $z = \zeta$, there is, in this case, a positive number $\rho(\zeta)$ such that

$$|\zeta - T_\nu \zeta| > \rho(\zeta)$$

for all transformations $T_\nu \neq 1$, of the properly discontinuous group.

Given a set E of points z in the unit circle, the set $T_\nu E$ of points $T_\nu z$ is said to be congruent with E when $T_\nu \neq 1$ is a transformation of the group. The group property implies that the relation "congruent or identical with" is symmetrical and transitive.

A Fuchsian group has a fundamental region R with the property that no two points of R are congruent and every point in the unit circle is congruent with some point of R. Thus none of the congruent regions R and $R_\nu = T_\nu R$ have common points. The fundamental region is commonly defined in such a manner that, if z be an interior point of R, then z satisfies the inequality

$$|z| < |T_\nu z|$$

where T_ν is any transformation of the group†. In this case R is simply connex, bounded by arcs of circles orthogonal to the unit circle, and any two interior points can be joined by a circular arc wholly interior to R and which, when produced, is orthogonal to the unit circle. The fundamental region does not necessarily possess any specially convenient arrangement of sides, nor do we attempt to choose a fundamental region with such a property, for the procedure would be equivalent to choosing a specially convenient set of cuts in a Riemann's surface, and we wish to avoid topological discussions.

At this point we must mention another limitation of our analysis. We have stated above that we only consider groups of hyperbolic transformations. We shall, further, only consider groups whose fundamental region R lies wholly within a circle of radius less than unity, and is, consequently, bounded only by a finite number of sides. We define the term canonical group to mean a Fuchsian group of hyperbolic transformations whose fundamental region lies wholly within a circle of radius less than unity; and we shall confine our attention henceforward only to canonical groups.

* Appell and Goursat, 2, 83. Giraud, 13.

† Appell and Goursat, 2, 104. Ford, 71.

Let now $T_0 \equiv 1$, T_1, T_2, ..., T_ν, ... be the transformations of a canonical group, and let $T_\nu a_\nu = 0$, $|a_\nu| < 1$. We require

LEMMA I. *The series* $\sum_0^\infty (1-|a_\nu|)^{1+\delta}$ *is convergent when* δ *is a fixed positive real number.*

Demonstration. We observe that

$$\int \frac{d\sigma}{(1-r^2)^{1-\delta}} = \frac{\pi}{\delta} \quad (\delta > 0, \quad r = |z|),$$

where the integral is a surface integral over the unit circle $|z| < 1$. The fundamental region R and its transforms R_ν are distinct and exhaust the circle, whence

$$\Sigma \int_{R_\nu} \frac{d\sigma}{(1-r^2)^{1-\delta}} = \frac{\pi}{\delta}.$$

An integral in R_ν can be expressed as an integral in R by a change of variable which gives

$$\sum_0^\infty \int_R \frac{|T_\nu' z|^2 d\sigma}{(1-|T_\nu z|^2)^{1-\delta}} = \frac{\pi}{\delta}.$$

By (1) above, this may be written in the form

$$\sum_0^\infty \int_R |T_\nu' z|^{1+\delta} \frac{d\sigma}{(1-r^2)^{1-\delta}} = \frac{\pi}{\delta}.$$

Then, finally, by applying the inequality $2|T_\nu' z| \geqslant 1-|a_\nu|$, given in (1) above, we get

$$\{\Sigma(1-|a_\nu|)^{1+\delta}\} \int_R \frac{d\sigma}{(1-r^2)^{1-\delta}} \leqslant \frac{\pi}{\delta} 2^{1+\delta},$$

which verifies the lemma.

This result, for the case $\delta \geqslant 1$, was obtained by Poincaré* and we have, substantially, used his method in the demonstration of the lemma. When $\delta = 0$ the series diverges, but we do not need to use this fact.

A Theta-Fuchsian function $\Theta(m, z)$ of rank m, where m is a positive integer, is defined to be meromorphic in the unit circle, and to have the transformation property

$$\Theta(m, T_\nu z)\{T_\nu' z\}^m = \Theta(m, z), \tag{2}$$

where T_ν is any transformation of the group. A Theta-Fuchsian function which is regular in the unit circle is called an integral Theta-Fuchsian function.

* Appell and Goursat, 2, 247. Ford, 104.

Poincaré established the existence of Theta-Fuchsian functions by means of Theta-Fuchsian series, which are defined to be of the form

$$Q(z)+\sum_{1}^{\infty} Q(T_\nu z)\{T_\nu' z\}^m \quad (m \geqslant 2), \tag{3}$$

where the positive integer m is the rank of the series, and $Q(z)$ is a rational function regular on the circumference of the unit circle. In these circumstances the series converges uniformly and absolutely in any fixed circle of radius less than unity to a meromorphic sum which is a Theta-Fuchsian function of rank m, or zero*. When $Q(z)$ has $N(\infty)$ non-congruent poles in the unit circle the sum of the series (3) is a meromorphic Theta-Fuchsian function of rank m with $N(\infty)$ poles in the fundamental region R. When $z=0$ and $Q(z)$ is constant and equal to unity the sum of the series (3) tends to unity as $m\to\infty$, so that integral Theta-Fuchsian functions exist for all sufficiently large values of m.

We shall, however, enlarge slightly the definition of a Theta-Fuchsian series. We shall say that the series

$$\psi(z)+\sum_{1}^{\infty} \psi(T_\nu z)\{T_\nu' z\}^m \quad (m \geqslant 1) \tag{4}$$

is a Theta-Fuchsian series of rank m, where m is a positive integer, when $\psi(z)$ is not merely rational, but meromorphic in the unit circle; then convergence is not inherent in the form of the series, and we have to include in the definition the requirement that the series converges uniformly and absolutely, except near an enumerable set of points, to a sum which is everywhere meromorphic in the unit circle and is, then, necessarily a Theta-Fuchsian function of rank m.

There is an integer P, characteristic of the canonical group, and such that, if $\Theta(m, z)$ has $N(0)$ zeros and $N(\infty)$ poles in the fundamental region R, then

$$N(0)-N(\infty)=mP>0. \tag{5}$$

Poincaré evaluated P in terms of the angles of the fundamental region by integrating the logarithmic derivate of a Theta-Fuchsian function round the boundary of R†, but this procedure is not absolutely necessary in order to establish merely the existence of P‡. The genus p is related to P by the equation $P=2p-2$, but as we wish to avoid any assumptions

* Appell and Goursat, 2, 258. Ford, 103.

† Ford, 114.

‡ Appell and Goursat, 2, 346.

concerning the properties of the genus, we shall commence by taking P as the fundamental integer and call it the class of the group.

In connection with the construction of certain functions associated with canonical groups we shall require

LEMMA II. *Let $M(z)$ be a real, positive and continuous function of z in the unit circle, and let*

$$M(T_\nu z)|T_\nu' z|^m = M(z)$$

for all the transformations T_ν of a canonical group; then $\lambda > 0$, independent of z, exists such that

$$M(z) < \frac{\lambda}{(1-r)^m} \quad (r = |z|).$$

Demonstration. The function $(1-|z|^2)^m M(z)$ is, by (1), invariant for the transformations of the group, and it is continuous in the unit circle; it has, therefore, an upper bound λ in R, and retains this upper bound in the whole of the unit circle. This proves the lemma.

In particular, $M(z)$ may be taken to be the modulus of an integral Theta-Fuchsian function.

To conclude our preliminary account of the Theta-Fuchsian function we require the important and very simple theorem of zero residue.

The sum of the residues at the poles of a Theta-Fuchsian function of unit rank, in the fundamental region, is zero.

Demonstration. It is easily seen that, if $\Theta(1, z)$ is a Theta-Fuchsian function of unit rank, then the integral

$$\int \Theta(1, z)\, dz$$

taken along the boundary* of R consists of pairs of cancelling terms arising from conjugate sides†, and is zero. This is the theorem.

The theorem of zero residue is familiar in the theory of elliptic functions, in the form which states that the sum of the residues at the poles of an elliptic function in a fundamental period parallelogram is zero.

A deduction from the theorem of zero residue is that no Theta-Fuchsian function of unit rank has only a single pole in R. The proof is obvious.

Let now $\Psi(z)$ be meromorphic in the unit circle, and suppose that

$$\Psi(T_\nu z)\{T_\nu' z\}^m = \theta_\nu \Psi(z),$$

* Indentation is used when poles of the integrand lie on the boundary of R, in the same manner as in the theory of elliptic functions.

† Appell and Goursat, 2, 121. Ford, 70.

where θ_ν is independent of z but depends on the transformation T_ν. Then the number of zeros of $\Psi(z)$ exceeds the number of poles of $\Psi(z)$, in the fundamental region, by mP. Thus the rule (5), which applies to the Theta-Fuchsian functions, applies also to functions of the type of $\Psi(z)$. We easily verify this statement by observing that

$$\frac{\Theta'(m, z)}{\Theta(m, z)} - \frac{\Psi'(z)}{\Psi(z)}$$

is a Theta-Fuchsian function of unit rank. The theorem of zero residue then effects the required verification.

An automorphic function $f(z)$ is defined to be meromorphic and to be invariant for transformations of the group, that is, $f(T_\nu z) = f(z)$. It is a well-known fact that the number $w(f)$ of zeros of $f(z) - \xi$ in the fundamental region is independent of ξ*, and we can deduce it from the theorem of zero residue by considering the function

$$\frac{f'(z)}{f(z) - \xi},$$

which is a Theta-Fuchsian function of unit rank. We call $w(f)$ the order of the automorphic function.

There are no automorphic functions of unit order associated with a canonical group; for $f'(z)$ is a Theta-Fuchsian function of unit rank with at least P zeros by (5), each one of which yields a multiple value of $f(z)$, or $w(f) \geqslant 2$.

When there is an automorphic function of the second order associated with a canonical group, the group is said to be hyperelliptic.

3. *Linearly independent Theta-Fuchsian functions.*

A finite set of integral Theta-Fuchsian functions $\Theta_\nu(m, z)$ of rank m is said to be a set of linearly independent functions when the identity

$$\sum_\nu \lambda_\nu \Theta_\nu(m, z) = 0 \quad (|z| < 1)$$

implies that all the constants λ_ν are zero.

An estimate of the number of linearly independent Theta-Fuchsian functions of given rank is required. We have remarked in the synopsis that integral Theta-Fuchsian functions of sufficiently high rank exist.

* Appell and Goursat, 2, 238. Ford, 94.

Let $\Theta(m, z)$ be one of them, and let its zeros, in R, be $z = a_1, a_2, \ldots, a_n$, of orders $\nu_1, \nu_2, \ldots, \nu_n$ so that, by (5), $\nu_1+\nu_2+\ldots\nu_n = mP$. Write

$$\Phi(z, r, a) = \frac{1}{(a-z)^r} + \sum_1^\infty \frac{\{T_\nu' z\}^2}{(a-T_\nu z)^r} \quad (r > 0).$$

Then $\Phi(z, r, a)$ is a Theta-Fuchsian function of rank 2 with one pole, in R, of order r at $z = a$. The functions

$$\Theta(m, z)\Phi(z, r, a_s) \quad (s = 1, 2, \ldots, n, \ r \leqslant \nu_s)$$

are Theta-Fuchsian functions of rank $m+2$, and form a set of mP linearly independent functions. Hence for all sufficiently large values of m there are at least $(m-2)P$ linearly independent integral Theta-Fuchsian functions. This result implies that, given an arbitrary point $z = a$ in the unit circle, and any sufficiently large integer m, a Theta-Fuchsian function of rank m exists with a zero of order not less than $\{(m-2)P-1\}$ at $z = a$.

4. *The Eta-Fuchsian function.*

Let $z = a$ be any point of R. We know that if m is sufficiently large there is an integral Theta-Fuchsian function of rank m with a zero at $z = a$ of order not less than $\{(m-2)P-1\}$. Let $\Theta_m(z)$ be the Theta-Fuchsian function of rank m with maximum modulus unity in R and with a zero at $z = a$ of the greatest possible order. Then the number of its zeros in R other than at $z = a$ is not greater than

$$mP-\{(m-2)P-1\} = 2P+1.$$

Since $2P+1$ is fixed, there is an infinite sequence of values of m, say $m = m_1(\nu)$, such that the zero of $\Theta_m(z)$ at $z = a$ is of order

$$(m-2)P-1+s, \quad 0 \leqslant s \leqslant 2P+1,$$

the number of the remaining zeros in R being fixed and equal to $2P+1-s$.

There is, then, a sequence of values $m_2(\nu)$ of m such that the zeros in R of $\Theta_m(z)$ tend to fixed positions as $m = m_2(\nu) \to \infty$. Then all the zeros of $\Theta_m(z)$ tend to fixed positions as $m = m_2(\nu) \to \infty$. Let E be the set of points towards which the zeros of $\Theta_m(z)$ converge. Surround the points of E in R by small circles of radius ρ and the other points of E by the transforms of these circles by the transformations of the group. Let $E(\rho)$ be the set of all these circles; there is only a finite number of circles of $E(\rho)$ with points in R. Write

$$F_m(z) = |\Theta_m(z)|^{1/m} \quad [m = m_2(\nu)].$$

Then $F_m(z)$ attains the value unity on the boundary of R, is the modulus of a function regular, for all sufficiently large values of $m = m_2(\nu)$, at every point in the unit circle and outside $E(\rho)$, and by Lemma II

$$F_m(z) < \frac{1}{1-|z|} \quad [m = m_2(\nu)].$$

Then there is a sequence of values $m_3(\nu)$ of m such that $F_m(z)$ converges as $m = m_3(\nu) \to \infty$ uniformly in any region inside any fixed circle of radius less than unity and outside any $E(\rho)$*.

Suppose then that

$$F_m(z) \to F(z) \quad [|z| < 1, \ z \text{ not in } E, \ m = m_3(\nu) \to \infty].$$

Then $F(z)$ is not identically zero. If it were, the sequence $F_m(z)$ $[m = m_3(\nu)]$ would converge uniformly in every circle of radius less than unity and then $F_m(z)$ would not attain unity on the boundary of R for any sufficiently large values of $m = m_3(\nu)$.

Now let us examine the zeros of $F_m(z)$. Let $z = a_1(m)$, $a_2(m), \ldots, a_r(m), \ldots$, be the zeros in R of $\Theta_m(z)$, $m = m_3(\nu)$, distinct from $z = a$. Let $\varpi_m(z)$ be the polynomial whose zeros are $z = a_1(m), \ldots, a_r(m), \ldots$, that is

$$\varpi_m(z) = \prod_{r=1}^{2P+1-s} \{z - a_r(m)\}.$$

The convergence of the sequence $a_r(m)$, $m = m_3(\nu)$, to some point of E implies the convergence of $\varpi_m(z)$ to a limit as $m = m_3(\nu) \to \infty$, and, except in E,

$$|\varpi_m(z)|^{1/m} \to 1 \quad [m = m_3(\nu) \to \infty].$$

Then, except for a finite number of points in R,

$$G_m(z) = \frac{F_m(z)}{|\varpi_m(z)|^{1/m}} |z-a|^{(2P+1-s)/m} \to F(z)$$
$$[z \text{ in } R, \ m = m_3(\nu) \to \infty].$$

$G_m(z)$ is, however, the modulus of a function regular in R with a zero at $z = a$ of order P. Hence $F(z)$ is the modulus of a function regular in R, and therefore, everywhere in the unit circle, with a zero of order not less than P at $z = a$.

Thus, except in E,

$$|\Theta_m(z)|^{1/m} \to F(z) \quad [|z| < 1, \ m = m_3(\nu)],$$

so that

$$F(T_\nu z)|T_\nu' z| = F(T_\nu z),$$

* Montel, 21.

and then, by the rule (5), $F(z)$ is the modulus of a function regular in the unit circle with zeros of order P at $z=a$ and points congruent with $z=a$, and none elsewhere. Hence again $F(z)^{1/P}$ is the modulus of a function which is regular everywhere in the unit circle, and has only one zero in R. We denote this regular function by $H(z, a)$ and call it the Eta-Fuchsian function. We may suppose that $(z-a)^{-1}H(z, a)\to 1$ as $z\to a$. We then have

THEOREM I. *Given any point $z=a$ in the fundamental region R of a canonical group, there is a function $H(z, a)$, regular in the unit circle, with a single zero in R at $z=a$, and with the transformation property*

$$|H(T_\nu z, a)(T_\nu' z)^{1/P}| = |H(z, a)|,$$

where P is the class of the group and T_ν is any transformation belonging to it.

By Lemma II, the maximum modulus of the Eta-Fuchsian function is limited by an inequality

$$|H(z, a)| < \frac{\lambda}{(1-|z|)^{1/P}} \quad [|z|<1,\ |H'(a, a)|=1].$$

We omit the proof that $\lambda>0$ is independent of a when a is in R.

We observe also that $H(z, a)$ is not an analytic function of the parameter a.

All automorphic functions and Theta-Fuchsian functions can be expressed as ratios of products of Eta-Fuchsian functions. We omit the proof.

5. *Existence of periods.*

Consider the function $\Theta(z)$ defined as the sum of the series

$$\Theta(z) = \sum_0^\infty \frac{T_\nu' z}{H(T_\nu z, a)(a-T_\nu z)},$$

where $z=a$ is a point in R. Then, by Theorem I,

$$\left|\frac{T_\nu' z}{H(T_\nu z, a)(a-T_\nu z)}\right| = \frac{|T_\nu' z|^{1+(1/P)}}{|H(z, a)(a-T_\nu z)|},$$

and also, since $|T_\nu z|\to 1$ as $\nu\to\infty$, we have, for all sufficiently large values of ν when z is in any fixed circle interior to the unit circle,

$$|a-T_\nu z| \geqslant \tfrac{1}{2}(1-|a|).$$

Hence, by Lemma I, the series converges and is Theta-Fuchsian in the sense of the extended definition given in the synopsis, and the function $\Theta(z)$ is a meromorphic Theta-Fuchsian function of unit rank.

Only one term of the series has a pole of the second order at $z = a$, so that $\Theta(z)$ has a pole of the second order at $z = a$, and no others in R. By the theorem of zero residue, the residue at this pole is zero. Any integral, say $F(z)$, of $\Theta(z)$ is, then, single-valued in the unit circle, and has a single pole of the first order at $z = a$. Also $F(z)$ has the transformation property

$$F(T_\nu z) = F(z)+k(T_\nu),$$

where $k(T_\nu)$ is a number depending on the transformation T_ν and independent of z. The numbers $k(T_\nu)$ are not all zero, for if they were, $F(z)$ would be an automorphic function of unit order, and we have, in the synopsis, already excluded this possibility. By considering any two transformations, T_μ and T_ν, we find that

$$k(T_\mu T_\nu) = k(T_\nu T_\mu) = k(T_\mu)+k(T_\nu).$$

We shall call a set of numbers $k(T_\nu)$ with this property a system of periods. We have, then, demonstrated the existence of at least one system of periods.

Given a system of periods $k(T_\nu)$, the exponentials $\theta(T_\nu)$ of these periods have the property

$$\theta(T_\mu T_\nu) = \theta(T_\mu)\,\theta(T_\nu),$$

and are called a system of factors.

We now require a lemma on the magnitude of the periods.

Let $k(T_\nu)$ be a system of real periods, and write

$$\begin{aligned} \phi(z) &= (1-\delta-|z|)^2 \quad (|z| < 1-\delta) \\ &= 0 \qquad\qquad\quad (|z| \geqslant 1-\delta), \end{aligned}$$

where δ is sufficiently small for the circle $|z| < 1-2\delta$ to contain R. Let

$$f(x, y) = -\frac{\sum_{\nu=0}^{\infty} k(T_\nu)\,\phi(T_\nu z)}{\sum_{\nu=0}^{\infty} \phi(T_\nu z)} \qquad (z = x+iy);$$

then the two sums in the definition of $f(x, y)$ are finite because $|T_\nu z| \to 1$ as $\nu \to \infty$, and $f(x, y)$ is continuous, differentiable, and

$$f(x, y)-f(0, 0) = \int_{(0,0)}^{(x,y)} \left\{ \frac{\partial f(\xi, \eta)}{\partial \xi}\, d\xi + \frac{\partial f(\xi, \eta)}{\partial \eta}\, d\eta \right\}.$$

We easily verify that

$$f(x_\nu y_\nu) = f(x, y)+k(T_\nu), \quad x_\nu+iy_\nu = T_\nu(x+iy).$$

Let $$M(z) = \left\{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right\}^{\frac{1}{2}} > 0.$$

Then $M(z)$ is a continuous function of z and

$$M(T_\nu z)|T_\nu' z| = M(z),$$

whence, by Lemma II, λ exists such that

$$M(z) < \frac{\lambda}{1-r} \quad (r = |z| < 1).$$

By integrating this inequality along a radius of the circle $|z| < 1$, we find

$$|f(x, y) - f(0, 0)| < \lambda \log \frac{1}{1-r} \quad (r = |x+iy| < 1),$$

and then, for the particular case $x+iy = T_\nu 0$, $|T_\nu 0| = |\alpha_\nu|$,

$$|k(T_\nu)| < \lambda \log \frac{1}{1-|T_\nu 0|} = \lambda \log \frac{1}{1-|\alpha_\nu|}.$$

This is the required inequality. We can, then, enunciate

LEMMA III. *When $k(T_\nu)$ is a system of periods associated with a canonical group of transformations T_ν, then a constant λ exists such that*

$$|k(T_\nu)| < \lambda \log \frac{1}{1-|T_\nu 0|},$$

and likewise, when $\theta(T_\nu)$ is a system of factors, λ exists such that

$$\max\left[\theta(T_\nu), \frac{1}{\theta(T_\nu)}\right] < \frac{1}{(1-|T_\nu 0|)^\lambda}.$$

6. *Existence of integral Theta-Fuchsian functions of unit rank.*

Consider the function $\Psi(m, z)$ defined by the series

$$\Psi(m, z) = \sum_0^\infty \frac{(T_\nu' z)^m}{\theta(T_\nu)} \quad (|z| < 1),$$

where m is an integer and $\theta(T_\nu)$ is a system of real positive factors. Then, by (1) and Lemma III,

$$\left|\frac{(T_\nu' z)^m}{\theta(T_\nu)}\right| \leqslant \frac{2^m (1-|\alpha_\nu|)^{m-\lambda}}{(1-|z|)^{2m}},$$

and, therefore, by Lemma I the defining series converges absolutely to a function regular in the unit circle if $m > 1+\lambda$. Also $\Psi(m, 0) \to 1$ as $m \to \infty$, and therefore $\Psi(m, z)$ differs from zero for all sufficiently large

values of m. We easily see that

$$\Psi(m,\ T_\nu z)(T_\nu' z)^m = \theta(T_\nu)\Psi(m,\ z),$$

so that, by (5), $\Psi(m, z)$ has mP zeros in R. Let these zeros be at $z = a_r$ $(r = 1, 2, \ldots, mP)$. Then, by Theorem I, the function $F(z)$ defined by the equation

$$F(z) = \frac{\Psi(m,\ z)}{\prod_{\nu=1}^{mP} H(z,\ a_r)}$$

is regular in the unit circle, is without zeros, and has the transformation property

$$|F(T_\nu z)| = \theta(T_\nu)|F(z)|.$$

Let $F(z) = \exp U(z)$, where $U(z)$ is singled-valued and regular in $|z| < 1$ when $U(0)$ has been assigned. Then

$$U(T_\nu z) = U(z) + k(T_\nu) + i\eta(T_\nu),$$

where $\theta(T_\nu) = \exp k(T_\nu)$, and $\eta(T_\nu)$ is a system of real periods not identically zero. $U(z)$ is an integral of the first kind.

Hence there exists an integral of the first kind whose system of periods has any assignable system of real periods for real part. This theorem is familiar in the Riemannian theory of algebraic functions.

The derivate of $U(z)$ is an integral Theta-Fuchsian function of unit rank; hence at least one such function exists.

At this point we introduce the genus by the definition:

The genus p of a canonical group is the maximum number of linearly independent integral Theta-Fuchsian functions of unit rank associated with that group.

A complete set of linearly independent integral Theta-Fuchsian functions of unit rank is such that any integral Theta-Fuchsian function of unit rank can be expressed linearly in terms of the functions of the set. Thus a complete set of linearly independent integral Theta-Fuchsian functions of unit rank contains exactly p members.

Similarly, a complete set of linearly independent systems of real periods contains exactly $2p$ systems of periods.

7. *Integrals of the second kind.*

The integrals of the second kind are defined as functions meromorphic in the unit circle and with a period associated with each transformation

of the group. The periods may, in particular, be all zero and then the integral of the second kind is an automorphic function. We first construct a Theta-Fuchsian series of unit rank. Consider the series

$$\sum_0^\infty \frac{T_\nu' z}{H(T_\nu z, a)(a-T_\nu z)^n} \quad (|z|<1,\ n\geqslant 1),$$

where $z=a$ is any point of R. Then by (1) and Theorem I, we have, for all sufficiently large values of ν,

$$\left|\frac{T_\nu' z}{H(T_\nu z, a)(a-T_\nu z)^n}\right| \leqslant \frac{1}{|H(z, a)|}\,\frac{2^{n+1+(1/P)}(1-|a_\nu|)^{1+(1/P)}}{(1-|z|)^{2+(2/P)}(1-|a|)^n},$$

when $|z|$ is less than any fixed positive number less than unity. Hence by Lemma I the series converges uniformly to a meromorphic sum, which is a Theta-Fuchsian function of unit rank with a single pole of order $n+1$ at $z=a$ in the fundamental region.

By the theorem of zero residue the residues at the isolated multiple poles in each of R, R_ν are zero, and then the integral of the sum of the series is single-valued. This integral is an integral of the second kind with a pole of order $n\geqslant 1$ at $z=a$ in R.

In order to reduce the integrals of the second kind to some standard form, we may combine those with poles of orders not greater than n at a single common point, $z=a$ in R, in such a manner that the principal part is reduced to the single term $(z-a)^{-n}$. An integral of the first kind can be constructed to have an arbitrary real period system for the real part of its period system, so that by adding a suitable integral of the first kind the period system of an integral of the second kind can be made purely imaginary. We shall then denote the canonical integral of the second kind with a pole at the single point $z=a$ in R by $Z(n, z, a)$, where

$$Z(n, z, a)=\frac{1}{(z-a)^n}+O(1) \quad (n\geqslant 1),$$

and for any transformation T_ν the period

$$Z(n, T_\nu z, a)-Z(n, z, a)$$

is imaginary. Given a and n, this function is unique except for a constant term. In order to be definite we shall call the constant term in the Laurent expansion about $z=a$ the constant of integration, but shall postpone the discussion of its choice. The constant of integration may, of course, be a function of the parameter a.

We shall suppose, in general, that the real parts of the periods of any integral of the second kind have been reduced to zero.

Let $Z_r(z)$ $(r=1, 2, \ldots, p+1)$ be a set of $p+1$ integrals of the second kind with systems of periods $ik_r(T)$, and suppose that no linear combination of some or all of them is regular. Let $U_r(z)$, $1 \leqslant r \leqslant p+1$, be the set of integrals of the first kind, constructed with systems of periods $k_r(T)+i\eta_r(T)$, the $\eta_r(T)$ being determined by the $k_r(T)$. Consider the function

$$Z(z) = \sum_{r=1}^{p+1} (A_r+iB_r)\, Z_r(z) + \sum_{r=1}^{p+1} B_r\, U_r(z).$$

Then the system of periods $iK(T)$ of $Z(z)$ is imaginary, is given by the equation

$$K(T) = \sum_{r=1}^{p+1} \{A_r k_r(T) + B_r \eta_r(T)\},$$

and is compounded of $2p+2$ systems of real periods. The maximum number of linearly independent systems of real periods is only $2p$, so that there exists a set of real numbers A_r, B_r not all zero, such that $K(T)=0$ for all the transformations $T=T_\nu$ of the group. Then $Z(z)$ is an automorphic function. Hence, given any $p+1$ integrals of the second kind, of which no linear combination is regular, an automorphic function can be constructed by combining them linearly with integrals of the first kind*.

In particular, the conclusion of the preceding paragraph applies to the functions $Z(n, z, a)$, $1 \leqslant n \leqslant p+1$, so that, given any point $z=a$ in R an automorphic function exists with a pole of order not greater than $p+1$ at the single point $z=a$. We shall, provisionally, denote by $f(z, a)$ the automorphic function with a pole of the lowest possible order $\Omega(a) \leqslant 1+p$ at the single point $z=a$ in R.

We notice that $p \geqslant 2$, for we know that $p>0$, and if $p=1$, then $\Omega(a)=2$ everywhere. Let then $\Theta(1, z)$ be the integral Theta-Fuchsian function of unit rank, and let $z=a$ be one of its zeros. Then $f(z, a)\,\Theta(1, z)$ must be another integral Theta-Fuchsian function of unit rank, by the theorem of zero residue. This contradicts $p=1$, hence $p \geqslant 2$.

8. *Relation between class and genus.*

Let $\Theta_r(1, z)$ $(r=1, 2, \ldots, p)$ be a complete set of linearly independent Theta-Fuchsian functions of rank unity. By combining these functions together linearly, a function $\Theta(1, z, a)$ with a zero of order not less than $p-1$ at an arbitrary point $z=a$ in R can be obtained. Let $\Phi(z)$ be the Theta-Fuchsian function of rank $\frac{1}{2}p(p+1)$ defined as the determinant of

* Appell and Goursat, 1, 331.

order p whose elements are $(d/dz)^s \Theta_r(1, z)$ $(s = 0, 1, \ldots, p-1\,;\ r = 1, 2, \ldots, p)$. Then $\Phi(z)$ is not identically zero. If $\Theta(1, z, a)$ has a zero of order greater than $p-1$ at $z = a$, then $\Phi(a) = 0$. Hence the order of the zero at $z = a$ of $\Theta(1, z, a)$ is $p-1$, except when a is one of a finite set of points whose number by (5) does not exceed $\frac{1}{2}Pp(p+1)$*. Hence, for nearly all points $z = a$ in R, a complete set of linearly independent integral Theta-Fuchsian functions $\Theta(1, z, r, a)$ exists such that

$$\Theta(1, z, r, a) = (z-a)^r + O(z-a)^p, \quad (0 \leqslant r \leqslant p-1).$$

Now suppose that $\Omega(a) < 1+p$. Then with the notation $f(z, a)$ of the last paragraph, we find that $f(z, a)\,\Theta(1, z, \Omega-1, a)$ is a Theta-Fuchsian function of unit rank and a single pole of the first order in R, which, by the theorem of zero residue, is impossible. Hence, except for a finite number of points in R, $\Omega(a) \geqslant 1+p$. But $\Omega(a) \leqslant 1+p$ always. Hence $\Omega(a) = 1+p$ when a is in R except for a finite number of points.

The function $\Theta(1, z, p-1, a)$ has $P-(p-1)$ zeros in R other than the multiple zero at $z = a$. Let these zeros be at $z = b_s$ and of orders ν_s. Then there exist exactly $P-(p-1)$ automorphic functions of the following form, where $U(z)$ is a suitable integral of the first kind,

$$f(z) = \sum_{n=1}^{P} A_n Z(n, z, a) + Z(n', z, b_s) + U(z), \quad (1 \leqslant n' \leqslant \nu_s),$$

and then the $P-(p-1)$ linearly independent Theta-Fuchsian functions of unit rank,

$$\Theta(1, z, p-1, a) f(z),$$

differ from zero and have each at most a single pole of order unity at $z = a$, and are therefore regular. Together with $\Theta(1, z, p-1, a)$, they form a set of $P-(p-2)$ linearly independent integral Theta-Fuchsian functions of unit rank, whose number cannot exceed p, whence $p \geqslant P-(p-2)$, or $2p-2 \leqslant P$. It is not difficult to see that every automorphic function of the form

$$\frac{\Theta(1, z)}{\Theta(1, z, p-1, a)}$$

can be expressed linearly in terms of the functions $f(z)$ defined above. Hence the set of linearly independent integral Theta-Fuchsian functions of unit rank constructed above is complete, so that $2p-2 = P$. This is the required relation between the class and the genus.

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* This argument is copied from Appell and Goursat, 1, 333, and Baker, 38.

9. *Complete sets of linearly independent integral Theta-Fuchsian functions of any rank.*

Let $\Theta(m, z)$, $m > 1$, be an integral Theta-Fuchsian function of rank m with zeros at $z = a_r$ in R, of orders ν_r, so that by (5) $\sum_r \nu_r = m(2p-2)$. We may suppose that $\Omega(a_1) = 1+p$ and that $\nu_1 \geqslant p$. For instance, $\Theta(m, z)$ may be taken to be $\{\Theta(1, z, p-1, a_1)\}^m$, which has a zero of order $m(p-1) = p+(m-2)(p-1)+(p-2) \geqslant p$ at $z = a_1$. Then, in the manner of the preceding paragraph, we can construct a set of $m(2p-2)-p$ automorphic functions $f(z)$ such that $\Theta(m, z)$ and $\Theta(m, z)f(z)$ form a set of $m(2p-2)-p+1$ linearly independent integral Theta-Fuchsian functions of rank m. Similarly, this set is complete. Hence the number of linearly independent integral Theta-Fuchsian functions of rank m is precisely $(2m-1)(p-1)$, for $m > 1$.

10. *Abel's theorem.*

Let $W(z)$ be a sum of integrals of the first and second kinds, and let $\Theta(m, z)$ denote the general integral Theta-Fuchsian function of rank m, so that

$$\Theta(m, z) = \sum_{\nu=1}^{(2m-1)(p-1)} \lambda_\nu \Theta_\nu(m, z),$$

where the functions $\Theta_\nu(m, z)$ are a complete set of integral Theta-Fuchsian functions of rank m when ν takes the values 1, 2, ..., $(2m-1)(p-1)$. Let Δ denote variation of the parameters λ_ν and consider the integral

$$S_m(W) = \frac{1}{2\pi i} \int W(z) \frac{\Theta'(m, z)}{\Theta(m, z)} dz,$$

where the contour integral is along the boundary of R. Then the consequences of Abel's theorem follow from the result

$$\Delta S_m(W) = 0,$$

which we now proceed to prove. We have

$$\Delta S_m(W) = \frac{1}{2\pi i} \int W(z) \left\{ \frac{\Delta\Theta'(m, z)}{\Theta(m, z)} - \frac{\Theta'(m, z)\,\Delta\Theta(m, z)}{\Theta^2(m, z)} \right\} dz$$

$$= \frac{1}{2\pi i} \int W(z) \frac{d}{dz} \frac{\Delta\Theta(m, z)}{\Theta(m, z)} dz.$$

Now integrate by parts, and observe that $W(z)$ is single-valued, then

$$\Delta S_m(W) = -\frac{1}{2\pi i} \int \frac{dW(z)}{dz} \frac{\Delta\Theta(m, z)}{\Theta(m, z)} dz.$$

The integrand is a Theta-Fuchsian function of unit rank so that, by the theorem of zero residue, we have

$$\Delta S_m(W) = 0$$

as stated. Hence, for all values of the parameters in a neighbourhood of any initial set of values, the integral $S_m(W)$ is constant.

In particular, the initial values of the parameters may be taken so that $\Theta(m, z)$ is equal to the m-th power of an integral Theta-Fuchsian function of unit rank, and then

$$S_m(W) = mS_1(W).$$

In the case of an integral $U(z)$ of the first kind, these results lead, in the usual manner*, to the conclusion that, given the zeros in R, $z = b_r$, where $r = 1, 2, \ldots, 2p-2$, of $\Theta(1, z)$, a set of non-congruent zeros $z = a_r$, where $r = 1, 2, \ldots, 2m(p-1)$, of $\Theta(m, z)$ exists such that

$$\sum_{r=1}^{2m(p-1)} U(a_\nu) = m \sum_{r=1}^{2p-2} U(b_\nu).$$

Now consider the integral of the second kind $Z(1, z, w)$, which contains a specific undetermined constant of integration, and let $\Theta(1, z)$ be an integral Theta-Fuchsian function of unit rank. Then the canonical choice of the constant of integration is that which secures $S_1(W) = 0$ for $W = Z(1, z, w)$, that is,

$$\int Z(1, z, w) \frac{\Theta'(1, z)}{\Theta(1, z)} dz = 0,$$

so that the constant of integration may depend on the parameter w. When the contour integral is expressed in terms of residues, this equation becomes

$$\sum_{r-1}^{2p-2} Z(1, b_\nu w) + \frac{\Theta'(1, w)}{\Theta(1, w)} = 0.$$

Thus, generally, we have

THEOREM II. *Given a canonical integral of the second kind* $Z(1, z, w)$, *and an integral Theta-Fuchsian function* $\Theta(m, z)$ *of rank* m, *a set* $z = a_\nu$, *where* $\nu = 1, 2, \ldots, 2m(p-1)$, *of non-congruent zeros of* $\Theta(m, z)$ *exists such that*

$$\sum_{r=1}^{2m(p-1)} Z(1, a_\nu, w) + \frac{\Theta'(m, w)}{\Theta(m, w)} = 0.$$

* Appell and Goursat, 1, 396.

It is, perhaps, worth noting that the individual terms $Z(1, a_\nu, w)$ are not, according to our definition of the canonical integrals of the second kind, analytic functions of the parameter w.

The canonical integral $Z(n, z, w)$ can be determined in a manner similar to that used for $Z(1, z, w)$.

We omit the discussion of integrals of the third kind.

11. *Application to the theory of functions meromorphic in the unit circle.*

Let $f_1(z)$ and $f_2(z)$ be meromorphic in $|z|<1$, and be connected by an irreducible algebraic relation of genus $p>1$. In this case we may quote the general theorem that a pair of canonical automorphic functions, $\phi_1(\zeta)$ and $\phi_2(\zeta)$, exists in the unit circle $|\zeta|<1$, and is such that the relations

$$\phi_1(\zeta)=f_1(z), \quad \phi_2(\zeta)=f_2(z)$$

define ζ as a regular function of z*. The canonical group may, if necessary, be transformed so that $\zeta=0$ when $z=0$, and then $|\zeta|\leqslant|z|$. The order of $\phi_1(\zeta)$ is n_2, say, and that of $\phi_2(\zeta)$ is n_1. By means of Eta-Fuchsian functions $\phi_1(\zeta)$, for example, may be written

$$\phi_1(\zeta)=\frac{F_1(\zeta)}{G_1(\zeta)},$$

where $F_1(\zeta)$ and $G_1(\zeta)$ are both regular in the unit circle $|\zeta|<1$, and their moduli are less than $M_1(r)$, $r=|\zeta|$, where

$$M_1(r)=(1-r)^{-n_2/(2p-2)}.$$

But also

$$f_1(z)=\frac{F_1\{\zeta(z)\}}{G_1\{\zeta(z)\}},$$

whence, since $|\zeta|\leqslant|z|$,

$$|F_1\{\zeta(z)\}|<(1-|\zeta|)^{-n_2/(2p-2)}\leqslant(1-|z|)^{-n_2/(2p-2)}.$$

Hence we can enunciate

THEOREM III. *When two functions $f_1(z)$ and $f_2(z)$, meromorphic in the unit circle, are connected by an irreducible algebraic relation of genus $p>1$, of degree n_1 in $f_1(z)$ and n_2 in $f_2(z)$, then $f_1(z)$ can be expressed*

* Appell and Goursat, 2, 445. Ford, 240. Montel, 252.

as the ratio of two regular functions whose moduli are less than $(1-|z|)^{-n_2/(2p-2)}$, *and* $f_2(z)$ *can be expressed as the ratio of two regular functions whose moduli are less than* $(1-|z|)^{-n_1/(2p-2)}$. *No better general result concerning the moduli in question can be obtained, since the functions* $\phi_1(z)$ *and* $\phi_2(z)$ *used in the proof satisfy the conditions of the theorem.*

12. *Conclusion.*

At the suggestion of a Referee, the paper has been revised in order to clarify the references to the theory of automorphic functions ; the revised form also differs from the original form by the inclusion of a paragraph on Abel's theorem. An indication of how this celebrated theorem is expressed in relation to canonical groups seemed desirable although, strictly, it is outside the general scheme of the paper, which is, substantially, the demonstration of the relation $P = 2p-2$ between the class P and the genus p of a canonical group.

List of references.

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THE LONDON MATHEMATICAL SOCIETY.

BALANCE SHEET, 31st October, 1931.

Liabilities.	£	s.	d.	£	s.	d.
Lieut.-Col. Campbell's Fund				499	7	5
Sir A. B. Kempe's Fund				200	0	0
Donations				366	19	0
Life Compositions Fund				3350	14	6
Reserve Fund in 1930	771	2	11			
Depreciation of Investments	468	7	0			
Reserve Fund				302	15	11
Sundry Creditors				31	6	2
Revenue Account, balance in hand				77	6	5
Petty Cash due to Treasurer					4	2
				£4828	13	7

Assets.	£	s.	d.
London, Midland & Scottish Railway £384 4 per Cent. Guaranteed Stock at 66¾ ...	256	6	4
India Stock £700. 7*s.* 1*d.* 3 per Cent. at 45	315	3	2
Funding Loan £3000 4 per Cent. at 86¾ ...	2602	10	0
Conversion Loan £1700 3½ per Cent. at 75¾	1287	15	0
The above-mentioned stocks are valued at the medium market prices of 31st Oct., 1931.			
Cash at Bank	366	19	1
	£4828	13	7

REVENUE ACCOUNT for the year 1st November, 1930, to 31st October, 1931.

	Year 1929-30							Year 1929-30.					
	£	s.	d.	£	s.	d.		£	s.	d.	£	s.	d.
Balance from last year	16	2	3	*253*	*19*	*1*	Printing and Circulating *Proceedings* ...	1248	1	9	*1349*	*12*	*9*
Income Tax returned	8	1	4	*33*	*3*	*6*	Ditto *Journal*	460	3	6	*463*	*13*	*6*
Dividends on—							General Printing	57	9	6	*93*	*18*	*6*
Lord Rayleigh's Fund	40	9	11	*40*	*11*	*5*	Purchases and Binding for Library ...	43	14	0	*74*	*18*	*8*
Lieut.-Col. A. Cunningham's Fund ...	46	2	8	*55*	*7*	*6*	Expenses of Meetings	30	0	0	*30*	*0*	*0*
The Society's Investments	216	14	11	*208*	*12*	*0*	Postages and Sundries	5	19	8	*11*	*7*	*4*
Entrance Fees	46	4	0	*73*	*10*	*0*	Balance in hand on 31st October, 1931 ...	77	6	5	*16*	*2*	*3*
Annual Subscriptions	556	3	9	*517*	*12*	*10*							
Sales of *Proceedings*	510	7	0	*456*	*18*	*4*							
Ditto *Journal*	100	19	0	*87*	*3*	*10*							
Advertisements revenue	31	10	0	*12*	*15*	*0*							
Grant from Royal Society	200	0	0	*150*	*0*	*0*							
Mr. H. T. J. Norton's Donation	150	0	0	*150*	*0*	*0*							
	£1922	14	10	*2039*	*13*	*0*		£1922	14	10	*2039*	*13*	*0*

REVENUE ACCOUNT of the DE MORGAN MEDAL FUND.

	£	s.	d.		£	s.	d.
Balance in hand 31st October, 1930 ...	3	0	3	Cash at Bank	6	0	3
Dividends received	2	6	6				
Income Tax returned		13	6				
	£6	0	3		£6	0	3

REVENUE ACCOUNT OF LIEUT.-COL. A. J. C. CUNNINGHAM'S TRUST FUND For Factorisation of Numbers.

	£	s.	d.		£	s.	d.
Dividends on £1538. 3*s.* 1*d.* Local Loans 3 per Cent. Stock	46	2	8	To Society's Revenue Account	46	2	8
	£46	2	8		£46	2	8

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12th November, 1931.

Signed on behalf of the Council:—

S. CHAPMAN, *President.*

A. E. WESTERN, *Treasurer.*

I report to the Members that I have obtained all the information and explanation I required as Auditor, and that I have examined the above Accounts with the books, and that, to the best of my information and of the explanations given to me, such Accounts are properly drawn up so as to exhibit a true and correct view of the state of the affairs of the Society as shown by the books of the Society.

A. E. JOLLIFFE, *Auditor.*

23rd November, 1931.

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CORRIGENDA.

THE STRONG AND WEAK CONVERGENCE OF FUNCTIONS OF GENERAL TYPE

[*Proc. London Math. Soc.* (2), 28 (1928), 493–500.]

J. C. BURKILL.

I am indebted to Drs. Birnbaum and Orlicz for pointing out that the hypothesis of Theorem 1 is insufficient. The function f is to be assumed to belong to class Q^1 instead of to the wider class Q. The remaining theorems are unaffected, since Theorem 1 is used only for functions of the class Q^1. For an interesting extension of the theorem see Birnbaum and Orlicz, "Über Approximation im Mittel", *Studia Math.*, 2 (1930), 197–206.

ON THE SUMMABILITY OF FOURIER SERIES

[*Proc. London Math. Soc.* (2), 31 (1930), 144–164.]

L. S. BOSANQUET.

In Theorem 4, *that*

$$\phi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)-2s\} \to 0 \quad (C, \beta)$$

as $t \to 0$ *whenever the Denjoy-Fourier series of* $f(x)$ *is summable* (C, α) *to* s, *where* $\beta > \alpha+1$ *and* $\alpha \geqslant -1$, my proof for the case $-1 \leqslant \alpha < 0$, $\beta < 1$, was incomplete*. The proof may be completed by the following lemma, valid also for $0 \leqslant \alpha \leqslant 1$.

LEMMA. *If* $\Sigma a_n \cos nt$ *is the Denjoy-Fourier series of* $\phi(t)$ *and* $a_n = o(n^\alpha)$, *then*

$$\phi_\beta(t) = \beta \Sigma a_n \gamma_\beta(nt),$$

for $t \neq 0$, *where* $\beta > \alpha+1$ *and* $-1 \leqslant \alpha < 0$.

Proof. Suppose that $0 < \beta < 2$. Then, writing

$$\gamma_{\beta,\epsilon}(x) = \int_0^{1-\epsilon} (1-u)^{\beta-1} \cos xu \, du,$$

we have†, since $(1-u)^{\beta-1}$ is of bounded variation in $(0, 1-\epsilon)$,

$$\int_0^{1-\epsilon} (1-u)^{\beta-1} \phi(tu) \, du = \Sigma a_n \gamma_{\beta,\epsilon}(nt).$$

Moreover, for a given $t \neq 0$, it is easily verified that $\gamma_{\beta,\epsilon}(nt) = O(\epsilon^\beta) + O(n^{-\beta})$, and hence

$$\Sigma a_n \{\gamma_\beta(nt) - \gamma_{\beta,\epsilon}(nt)\} = \sum_{n \leqslant 1/\epsilon} o(n^\alpha) O(\epsilon^\beta) + \sum_{n > 1/\epsilon} o(n^{\alpha-\beta}) = o(\epsilon^{\beta-\alpha-1}).$$

The result follows by letting $\epsilon \to 0$.

---

* The existence of $\phi_\beta(t) = \beta \int_0^1 (1-u)^{\beta-1} \phi(tu) \, du$ was in fact established only for *almost every* t.

† See Pollard, **15**, and Hobson, **13**, (2), 385.

ON HECKE'S MODULAR FUNCTIONS, ZETA FUNCTIONS, AND SOME OTHER ANALYTIC FUNCTIONS IN THE THEORY OF NUMBERS

[*Proc. London Math. Soc.* (2), 32 (1931), 501–556.]

L. J. MORDELL.

Prof. Siegel has been kind enough to point out to me the reason for the difference between my formula (6 . 23), p. 544, and the corresponding formula on p. 166 of Hecke's paper*, my formula containing a constant C_{11} which is absent from Hecke's formula. Hecke, in deducing his formula on p. 163, line 6 up, from the first formula on the same page (when $k = 1$) by deforming the path of integration, has overlooked the residue arising from the pole $s = 0$. The investigation of the simplest form and properties of C_{11} would be interesting.

I am also indebted to Prof. Siegel for the majority of the following corrections:

P. 505, formula 2 . 3. *For* $\mu \quad c \pmod{\mathfrak{a}}$ *read* $\mu \equiv c \pmod{\mathfrak{a}}$.

P. 534, l. 7 up. *For* $1, \epsilon, \epsilon^2, \epsilon^3, \ldots$ *read* $\pm 1, \pm\epsilon, \pm\epsilon^2, \pm\epsilon^3, \ldots$.

P. 536, l. 11. *In the exponent, for* $\lambda' e^{-t} c$ *read* $\lambda' e^{-t} c'$.

P. 537, l. 5 *For* a *read* α.

P. 537, l. 10. *For* $\alpha = \beta = 1$ *read* $\alpha = \beta = 0$.

P. 538, l. 11. *For* $-\dfrac{i\tau}{2\pi}$ *read* $-\dfrac{i\tau}{2\pi\Delta}$.

P. 538, l. 2 up. *For* $\eta + \tau \operatorname{sgn} \eta$ *read* $\eta + \tau \operatorname{sgn} \lambda$.

P. 540, formula (6 . 10). *For* $-2\pi i\tau \log \epsilon$ *read* $-2i\tau \log \epsilon$.

P. 541, foot. *For* $|\Delta|$ *read* $\dfrac{|\Delta|}{\sqrt{D}}$.

P. 542, l. 8. *For* $S(\lambda\rho) \equiv S(\sigma'\rho) \pmod{\mathfrak{a}QD}$ *read* $S(\lambda\rho) \equiv S(\sigma'\rho) \pmod{AQD}$.

P. 544, l. 4. *For* $c\tau$ *read* $i\tau$.

P. 544, l. 5. *For* $D(i\eta;\ \rho', \mathfrak{a}')$ *read* $D(i\eta', \sigma', \mathfrak{a}')$.

P. 548, l. 4. *In the exponent, for* $-\tau(\mu^2 t^2 + \mu'^2/t^2)$ *read* $-2\tau|\mu\mu'|$.

P. 551, l. 1. *For* P *read* η.

P. 551, l. 2. *For* ξ, η *read* ξ, P.

P. 553, l. 8. *For* $>$ *read* $\geqslant$.

ON SOME CLASSES OF FOURIER SERIES

[*Proc. London Math. Soc.* (2), 33 (1932), 287–327.]

S. VERBLUNSKY.

Equation (36) on p. 320 of this paper is false. A consequence of this is that the conditions obtained in §§ 11, 12, and which are stated to be necessary and sufficient, have only been shown to be sufficient. To obtain conditions which are both necessary and sufficient, we observe that, if $\{g_n(t)\}$ denote a sequence of integrable functions, then, in order that for every bounded $f(t)$ we have

$$\lim_{n\to\infty} \int_0^{2\pi} f(t)\, g_n(t)\, dt = 0, \tag{1}$$

it is necessary and sufficient that (i) there is a constant κ such that

$$\int_0^{2\pi} |g_n(t)|\, dt < \kappa, \tag{2}$$

---

* *Journal für Math.*, 157 (1927), 159–170.

and (ii), given a measurable set e in $(0, 2\pi)$,

$$\lim_{n\to\infty} \int_e g_n(t)\,dt = 0. \tag{3}$$

This is an easy congruence of Lemma 2, p. 293.

To obtain condition (2, 6), we must characterize the functions $g(t)$ for which equation (33), p. 320, is true. We amend the definition of class (b) as follows.

Amended definition of class (b). Let $g(t)$ be an integrable periodic function. To each n, let there correspond a number ν_n of non-overlapping intervals in $(0, 2\pi)$,

$$\{(\alpha_i^{(n)}, \beta_i^{(n)})\} \quad (i = 1, 2, \ldots, \nu_n), \tag{4}$$

such that

$$\lim_{n\to\infty} \sum_{i=1}^{\nu_n} (\beta_i^{(n)} - \alpha_i^{(n)}) = 0.$$

Let

$$g_n(t) = \sum_{i=1}^{\nu_n} \{g(t-\beta_i^{(n)}) - g(t-\alpha_i^{(n)})\}.$$

If there is a constant κ such that (2) holds, and if, given a measurable set e in $(0, 2\pi)$, (3) holds, then $g(t)$ is absolutely continuous in mean. The sequence $\{g^{(r)}(t)\}$ of functions is uniformly absolutely continuous in mean, if, with the notation

$$g_n^{(r)}(t) = \sum_{i=1}^{\nu_n} \{g_r(t-\beta_i^{(n)}) - g_r(t-\alpha_i^{(n)})\},$$

we have

$$\int_0^{2\pi} |g_n^{(r)}(t)|\,dt < \kappa; \quad \lim_{n\to\infty} \int_e g_n^{(r)}(t)\,dt = 0, \tag{5}$$

where κ is an absolute constant, and, for a given e, the limit is uniform in r.

The propositions enunciated in §11 become true with this amended definition. For §12, we must introduce an amended definition of R-integrability in mean.

Amended definition of R*-integrability in mean.* Let $g(t)$ be an integrable periodic function. To each n let there correspond a number ν_n of non-overlapping intervals (4) in $(0, 2\pi)$ such that

$$\lim_{n\to\infty} \operatorname*{Max}_{1\le i\le \nu_n} (\beta_i^{(n)} - \alpha_i^{(n)}) = 0.$$

Let $\gamma_i^{(n)}$, $\delta_i^{(n)}$ be arbitrarily chosen to satisfy $\alpha_i^{(n)} \leqslant \gamma_i^{(n)} < \delta_i^{(n)} \leqslant \beta_i^{(n)}$. Let

$$g_n(t) = \sum_{i=1}^{\nu_n} \{g(t-\delta_i^{(n)}) - g(t-\gamma_i^{(n)})\}\,(\beta_i^{(n)} - \alpha_i^{(n)}).$$

If there is a constant κ such that (2) holds, and if, given a measurable set e in $(0, 2\pi)$, (3) holds, then $g(t)$ is R-integrable in mean. The sequence $\{g^{(r)}(t)\}$ is uniformly R-integrable in mean, if, with the notation

$$g_n^{(r)}(t) = \sum_{i=1}^{\nu_n} \{g^{(r)}(t-\delta_i^{(n)}) - g^{(r)}(t-\gamma_i^{(n)})\}\,(\beta_i^{(n)} - \alpha_i^{(n)}),$$

the relations (5) hold, where κ is an absolute constant, and, for a given e, the limit is uniform in r.

With this amended definition Lemma 8 is true. This lemma gives us a form of the condition (2, 3). I am, however, unable to give an amended definition of a function of R-integrable variation in mean, and thus to give the condition (2, 3) by characterizing the function of which $\sum \lambda_n \sin nx/n$ is the Fourier series.

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